Hindawi Publishing Corporation
International Journal of Analysis
Volume 2013, Article ID 126163, 7 pages
http://dx.doi.org/10.1155/2013/126163

Research Article

On a New I-Convergent Double-Sequence Space

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Received 26 November 2012; Revised 17 January 2013; Accepted 18 January 2013

Academic Editor: Wen Xiu Ma

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The sequence space BV_σ was introduced and studied by Mursaleen (1983) . In this article we introduce the sequence space ${}_2\mathrm{BV}_\sigma^I$ and study some of its properties and inclusion relations.

1. Introduction and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all natural, real, and complex numbers, respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{C}\}, \tag{1}$$

showing the space of all real or complex sequences.

Definition 1. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other [1]. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left|\left(x_{ij}\right) - a\right| < \epsilon, \quad \forall i, j \geq N, \tag{2}$$

(see [2]).

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let v denote the space of sequences of bounded variation; that is,

$$v = \left\{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\right\}, \tag{3}$$

where v is a Banach space normed by

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|, \tag{4}$$

(see [3]).

Definition 2. Let σ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k;

(ii) $\phi(e) = 1$, where $e = \{1, 1, 1, \ldots\}$;

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

In case σ is the translation mapping $n \to n + 1$, a σ-mean is often called a Banach limit (see [4]), and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences (see [5]).

If $x = (x_k)$, then $Tx = (Tx_k) = (x_{\sigma(k)})$. Then it can be shown that

$$V_\sigma = \left\{x = (x_k) : \sum_{m=1}^{\infty} t_{m,k}(x) = L \text{ uniformly in } k, \quad L = \sigma - \lim x\right\}, \tag{5}$$

where $m \geq 0$, $k > 0$. Consider

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \cdots + x_{\sigma^m(k)}}{m+1}, \quad t_{-1,k} = 0, \tag{6}$$

where $\sigma^m(k)$ denote the mth iterate of $\sigma(k)$ at k. The special case of (5) in which $\sigma(n) = n+1$ was given by Lorentz [5, Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c.

Theorem 3. *A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if, for all $k \geq 0$, $j \geq 1$, (see [3])*

$$\sigma^j(k) \neq k. \tag{7}$$

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{8}$$

assuming that $t_{-1,k} = 0$. A straight forward calculation shows (see [6]) that

$$\phi_{m,k}(x) = \begin{cases} \dfrac{1}{m(m+1)} \displaystyle\sum_{j=1}^{m} J\left(x_{\sigma^j(k)} - x_{\sigma^{j-1}(k)}\right) & (m \geq 1), \\ x_k, & (m = 0). \end{cases} \tag{9}$$

For any sequence x, y, and scalar λ, we have

$$\begin{gathered} \phi_{m,k}(x+y) = \phi_{m,k}(x) + \phi_{m,k}(y), \\ \phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x). \end{gathered} \tag{10}$$

Definition 4. A sequence $x \in l_\infty$ is of σ-bounded variation if and only if

(i) $\sum_{k=0}^{\infty} |\phi_{m,k}(x)|$ converges uniformly in n;

(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variation (see [7]):

$$\mathrm{BV}_\sigma = \left\{ x \in l_\infty : \sum_m \left|\phi_{m,k}(x)\right| < \infty, \text{uniformly in } n \right\}. \tag{11}$$

Theorem 5. BV_σ *is a Banach space normed by*

$$\|x\| = \sup_k \sum_{m=0}^{\infty} \left|\phi_{m,k}(x)\right|, \tag{12}$$

(see [8]).

Subsequently, invariant means have been studied by Ahmad and Mursaleen [9], Mursaleen et al. [3, 6, 8, 10–14], Raimi [15], Schaefer [16], Savas and Rhoades [17], Vakeel et al. [18–20], and many others [21–23]. For the first time, I-convergence was studied by Kostyrko et al. [24]. Later on, it was studied by Šalát et al. [25, 26], Tripathy and Hazarika [27], Ebadullah et al. [18–20, 28], and Vakeel et al. [1, 29].

Definition 6 (see [30, 31]). Let X be a nonempty set. Then, a family of sets $I \subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal in X if

(i) $\emptyset \in I$;

(ii) I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$;

(iii) I is hereditary that is, $A \in I$, $B \subseteq A \Rightarrow B \in I$;

An Ideal $I \subseteq 2^X$ is called nontrivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\pounds(I)$ corresponding to I. That is,

$$\pounds(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where } K^c = N - K. \tag{13}$$

Definition 7 (see [24, 31, 32]). A double sequence $(x_{ij}) \in \omega$ is said to be *I-convergent* to a number L if for every $\epsilon > 0$,

$$\left\{i, j \in \mathbb{N} : \left|x_{ij} - L\right| \geq \epsilon\right\} \in I. \tag{14}$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 8 (see [2]). A double sequence $(x_{ij}) \in \omega$ is said to be I-null if $L = 0$. In this case, we write

$$I - \lim x_{ij} = 0. \tag{15}$$

Definition 9. A double sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exist numbers $m = m(\epsilon)$, $n = n(\epsilon)$ such that

$$\left\{i, j \in \mathbb{N} : \left|x_{ij} - x_{mn}\right| \geq \epsilon\right\} \in I. \tag{16}$$

Definition 10. A double sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\left\{i, j \in \mathbb{N} : \left|x_{ij}\right| > M\right\}. \tag{17}$$

Definition 11. A double-sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $i, j \in \mathbb{N}$.

Definition 12 (see [24, 33]). A nonempty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if

(i) $\Phi \notin \pounds(I)$;

(ii) for $A, B \in \pounds(I)$, we have $A \cap B \in \pounds(I)$;

(iii) for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

Definition 13. Let X be a linear space. A function $g : X \to R$ is called a paranorm, if for all $x, y, z \in X$,

(i) $g(x) = 0$ if $x = \theta$;

(ii) $g(-x) = g(x)$;

(iii) $g(x+y) \leq g(x) + g(y)$;

(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

The concept of paranorm is closely related to that of linear metric spaces. It is a generalization of that of absolute value (see [34, 35]).

2. Main Results

In this paper, we introduce the sequence space

$$ {}_2\mathrm{BV}_\sigma^I := \left\{x = \left(x_{ij}\right) \in \omega : \left\{i, j \in \mathbb{N} : \left|\phi_{mnij}(x) - L\right| \geq \epsilon\right\} \in I \text{ for some } L \in \mathbb{C}\right\}. \tag{18} $$

Theorem 14. *${}_2\mathrm{BV}_\sigma^I$ is a linear space.*

Proof. Let $(x_{ij}), (y_{ij}) \in {}_2\mathrm{BV}_\sigma^I$ and α, β be two scalars in $\mathbb{C}$. Then for a given $\epsilon > 0$, we have

$$ \begin{aligned} &\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - L_1\right| \geq \frac{\epsilon}{2}\right\} \in I, \quad \text{for some } L_1 \in \mathbb{C}, \\ &\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(y) - L_2\right| \geq \frac{\epsilon}{2}\right\} \in I, \quad \text{for some } L_2 \in \mathbb{C}. \end{aligned} \tag{19} $$

Now let,

$$ \begin{aligned} A_1 &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - L_1\right| < \frac{\epsilon}{2}\right\} \in I, \\ &\qquad \text{for some } L_1 \in \mathbb{C}, \\ A_2 &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(y) - L_2\right| < \frac{\epsilon}{2}\right\} \in I, \\ &\qquad \text{for some } L_2 \in \mathbb{C} \end{aligned} \tag{20} $$

be such that $A_1^c, A_2^c \in I$. Now consider

$$ \begin{aligned} &\left|\phi_{mn,ij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)\right| \\ &\quad = \left|\phi_{mn,ij}(\alpha x) + \phi_{mn,ij}(\beta y) - \alpha L_1 - \beta L_2\right| \\ &\quad = \left|\phi_{mn,ij}(\alpha x) - \alpha L_1 + \phi_{mn,ij}(\beta y) - \beta L_2\right| \\ &\quad \leq \left|\phi_{mn,ij}(\alpha x) - \alpha L_1\right| + \left|\phi_{mn,ij}(\beta y) - \beta L_2\right| \\ &\quad = |\alpha| \left|\phi_{mn,ij}(x) - L_1\right| + |\beta| \left|\phi_{mn,ij}(y) - L_2\right| \\ &\quad \leq |\alpha| \frac{\epsilon}{2} + |\beta| \frac{\epsilon}{2} \\ &\quad = (|\alpha| + |\beta|) \frac{\epsilon}{2} \\ &\quad \leq \epsilon' \quad (\text{say}), \end{aligned} \tag{21} $$

this implies that the sequence space

$$ A_3 = \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)\right| < \epsilon'\right\} \in I, \quad \text{for some } L_1, L_2 \in \mathbb{C}. \tag{22} $$

Hence, $(\alpha x + \beta y) \in {}_2\mathrm{BV}_\sigma^I$. Therefore, ${}_2\mathrm{BV}_\sigma^I$ is a linear space. □

Theorem 15. *The space ${}_2\mathrm{BV}_\sigma^I$ is a paranormed space, paranormed by*

$$ g\left(x_{ij}\right) = \sup_{ij} \left|\phi_{mn,ij}(x)\right|. \tag{23} $$

Proof. For $x = (x_{ij}) = 0$, $g(x_{ij}) = 0$ is trivial.

For $x = (x_{ij}) \neq 0$, $g(x_{ij}) \neq 0$, we have

(i) $g(x) = \sup_{ij}|\phi_{mn,ij}(x)| \geq 0$ for all $x \in {}_2\mathrm{BV}_\sigma^I$.

(ii) $g(-x) = \sup_{ij}|\phi_{mn,ij}(-x)| = \sup_{ij}|-\phi_{mn,ij}(x)| = \sup_{ij}|\phi_{mn,ij}(x)| = g(x)$ for all $x \in {}_2\mathrm{BV}_\sigma^I$.

(iii) $g(x + y) = \sup_{ij}|\phi_{mn,ij}(x + y)| \leq \sup_{ij}|\phi_{mn,ij}(x)| + \sup_{ij}|\phi_{mn,ij}(y)| = g(x) + g(y)$.

(iv) Let (λ_{ij}) be a sequence of scalars with $\lambda_{ij} \to \lambda$ $(ij \to \infty)$ and $(x) \in {}_2\mathrm{BV}_\sigma^I$ such that

$$ \phi_{mn,ij}(x) \longrightarrow L \quad (ij \longrightarrow \infty), \tag{24} $$

in the sense that

$$ g\left(\phi_{mn,ij}(x) - L\right) \longrightarrow 0 \quad (ij \longrightarrow \infty). \tag{25} $$

Therefore,

$$ \begin{aligned} &g\left(\lambda_{ij}\phi_{mn,ij}(x) - \lambda L\right) \\ &\quad \leq g\left(\lambda_{ij}\phi_{mn,ij}(x)\right) - g(\lambda L) \\ &\quad = \lambda_{ij} g\left(\phi_{mn,ij}(x)\right) - \lambda g(L) \longrightarrow 0 \quad \text{as } (ij \longrightarrow \infty). \end{aligned} \tag{26} $$

Hence, ${}_2\mathrm{BV}_\sigma^I$ is a paranormed space. □

Theorem 16. *${}_2\mathrm{BV}_\sigma^I$ is a closed subspace of ${}_2l_\infty^I$.*

Proof. Let $(x_{ij}^{(pq)})$ be a cauchy sequence in ${}_2\mathrm{BV}_\sigma^I$ such that $x^{(pq)} \to x$. We show that $x \in {}_2\mathrm{BV}_\sigma^I$. Since $(x_{ij}^{(pq)}) \in {}_2\mathrm{BV}_\sigma^I$, then there exists a_{pq} such that

$$ \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x^{(pq)}\right) - a_{pq}\right| \geq \epsilon\right\} \in I. \tag{27} $$

We need to show that

(i) (a_{pq}) converges to a.

(ii) If $U = \{i, j \in \mathbb{N} : |x_{ij} - a| < \epsilon\}$, then $U^c \in I$.

Since $(x_{ij}^{(pq)})$ is a cauchy sequence in ${}_2\mathrm{BV}_\sigma^I$, then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{ij} \left|\phi_{mn,ij}\left(x_{ij}^{(pq)}\right) - \phi_{mn,ij}\left(x_{ij}^{(rs)}\right)\right| < \frac{\epsilon}{3}, \quad \forall p, q, r, s \geq k_0. \tag{28}$$

For a given $\epsilon > 0$, we have

$$\begin{aligned} B_{pqrs} &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(pq)}\right) - \phi_{mn,ij}\left(x_{ij}^{(rs)}\right)\right| < \frac{\epsilon}{3}\right\}, \\ B_{pq} &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(pq)}\right) - a_{pq}\right| < \frac{\epsilon}{3}\right\}, \\ B_{rs} &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(rs)}\right) - a_{rs}\right| < \frac{\epsilon}{3}\right\}. \end{aligned} \tag{29}$$

Then B_{pqrs}^c, B_{pq}^c, and $B_{rs}^c \in I$. Let $B^c = B_{pqrs}^c \cap B_{pq}^c \cap B_{rs}^c$, where $B = \{i, j \in \mathbb{N} : |a_{pq} - a_{rs}| < \epsilon\}$. Then $B^c \in I$. We choose $k_0 \in B^c$, then for each $p, q, r, s \geq k_0$, we have

$$\begin{aligned} &\left\{i, j \in \mathbb{N} : \left|a_{pq} - a_{rs}\right| < \epsilon\right\} \\ &\quad \supseteq \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(pq)}\right) - a_{pq}\right| < \frac{\epsilon}{3}\right\} \\ &\quad \cap \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(pq)}\right) - \phi_{mn,ij}\left(x_{ij}^{(rs)}\right)\right| < \frac{\epsilon}{3}\right\} \\ &\quad \cap \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(rs)}\right) - a_{rs}\right| < \frac{\epsilon}{3}\right\}. \end{aligned} \tag{30}$$

Then (a_{pq}) is a cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $(a_{pq}) \to a$, as $p, q \to \infty$.

For the next step, let $0 < \delta < 1$ be given. Then, we show that if

$$U = \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - a\right| < \delta\right\}, \tag{31}$$

then $U^c \in I$. Since $\phi_{mn,ij}(x^{(pq)}) \to \phi_{mn,ij}(x)$, then there exists $p_0, q_0 \in \mathbb{N}$ such that

$$P = \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(p_0q_0)}\right) - \phi_{mn,ij}(x)\right| < \frac{\delta}{3}\right\} \tag{32}$$

which implies that $P^c \in I$. The number p_0, q_0 can be so chosen that together with (32), we have

$$Q = \left\{i, j \in \mathbb{N} : \left|a_{p_0q_0} - a\right| < \frac{\delta}{3}\right\} \tag{33}$$

such that $Q^c \in I$. Since $\{i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{(p_0q_0)}) - a_{p_0q_0}| \geq \delta\} \in I$, then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(p_0q_0)}\right) - a_{p_0q_0}\right| < \frac{\delta}{3}\right\}. \tag{34}$$

Let $U^c = P^c \cap Q^c \cap S^c$, where $U = \{i, j \in \mathbb{N} : |\phi_{mn,ij}(x) - a| < \delta\}$.

Therefore, for each $i, j \in U^c$, we have

$$\begin{aligned} &\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - a\right| < \delta\right\} \\ &\quad \supseteq \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(p_0q_0)}\right) - \phi_{mn,ij}(x)\right| < \frac{\delta}{3}\right\} \\ &\quad \cap \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}\left(x_{ij}^{(p_0q_0)}\right) - a_{p_0q_0}\right| < \frac{\delta}{3}\right\} \\ &\quad \cap \left\{i, j \in \mathbb{N} : \left|a_{p_0q_0} - a\right| < \frac{\delta}{3}\right\}. \end{aligned} \tag{35}$$

Hence, the result ${}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$ follows. □

Theorem 17. *The space ${}_2\mathrm{BV}_\sigma^I$ is nowhere dense subsets of ${}_2l_\infty^I$.*

Proof. Proof of the result follows from the previous theorem. □

Theorem 18. *The space ${}_2\mathrm{BV}_\sigma^I$ is solid and monotone.*

Proof. Let $(x_{ij}) \in {}_2\mathrm{BV}_\sigma^I$ and α_{ij} be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then, we have

$$\begin{aligned} \left|\alpha_{ij}\phi_{mn,ij}(x)\right| &\leq \left|\alpha_{ij}\right|\left|\phi_{mn,ij}(x)\right| \\ &\leq \left|\phi_{mn,ij}(x)\right|, \quad \forall i, j \in \mathbb{N}. \end{aligned} \tag{36}$$

The space ${}_2\mathrm{BV}_\sigma^I$ is solid follows from the following inclusion relation:

$$\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x)\right| \geq \epsilon\right\} \supseteq \left\{i, j \in \mathbb{N} : \left|\alpha_{ij}\phi_{mn,ij}(x)\right| \geq \epsilon\right\}. \tag{37}$$

Also a sequence space is solid implies monotone. Hence, the space ${}_2\mathrm{BV}_\sigma^I$ is monotone. □

Theorem 19. *${}_2c_0^I \subset {}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$ and the inclusions are proper.*

Proof. Let $x = (x_{ij}) \in {}_2c_0^I$. Then, we have $\{i, j \in \mathbb{N} : |x_{ij}| \geq \epsilon\} \in I$. Since ${}_2c_0 \subset {}_2\mathrm{BV}_\sigma$, $x = (x_{ij}) \in {}_2\mathrm{BV}_\sigma^I$ implies

$$\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x)\right| \geq \epsilon\right\} \in I. \tag{38}$$

Now let,

$$\begin{aligned} A_1 &= \left\{i, j \in \mathbb{N} : \left|x_{ij}\right| < \epsilon\right\}, \\ A_2 &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x)\right| < \epsilon\right\} \end{aligned} \tag{39}$$

be such that $A_1^c, A_2^c \in I$. As $l_\infty = \{x = (x_{ij}) : \sup_{ij}|x_{ij}| < \infty\}$, taking supremum over i, j we get $A_1^c \subset A_2^c$. Hence, ${}_2c_0^I \subset {}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$.

Next we show that the inclusion is proper

(i) First for ${}_2c_0^I \subset {}_2\mathrm{BV}_\sigma^I$. Consider $x \in {}_2\mathrm{BV}_\sigma^I$, then by the definition

$$\begin{aligned} {}_2\mathrm{BV}_\sigma^I := \Big\{x = \left(x_{ij}\right) \in \omega : \\ \left\{i, j \in \mathbb{N} : \left|\phi_{mnij}(x) - L\right| \geq \epsilon\right\} \in I \\ \text{for some } L \in \mathbb{C}\Big\}, \end{aligned} \tag{40}$$

we have

$$\phi_{mn,ij}(x) = t_{mn,ij}(x) - t_{(m-1)(n-1),ij}(x), \tag{41}$$

where

$$t_{mn,ij}(x) = \frac{x_{ij} + x_{\sigma(ij)} + x_{\sigma^2(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn}. \tag{42}$$

Therefore,

$$\begin{aligned} t_{mn,ij}(x) &- t_{(m-1)(n-1),ij}(x) \\ &= \frac{x_{ij} + x_{\sigma(ij)} + x_{\sigma^2(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn} \\ &\quad - \frac{x_{ij} + x_{\sigma(ij)} + x_{\sigma^2(ij)} + \cdots + x_{\sigma^{(m-1)(n-1)}(ij)}}{(m-1)(n-1)} \\ &= \frac{(m-1)(n-1)\left(x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)}\right)}{mn(m-1)(n-1)} \\ &\quad - \frac{mn\left(x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{(m-1)(n-1)}(ij)}\right)}{mn(m-1)(n-1)}. \end{aligned} \tag{43}$$

On solving, we get

$$\begin{aligned} \phi_{mn,ij}(x) &= \frac{mnx_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)} \\ &\quad + \frac{(1-m-n)\left(x_{ij} + x_{\sigma(ij)} + x_{\sigma^2(ij)} + \cdots + x_{\sigma^{mn}(ij)}\right)}{mn(m-1)(n-1)}. \end{aligned} \tag{44}$$

As σ is a translation map, that is, $\sigma(n) = n+1$, we have

$$\begin{aligned} \phi_{mn,ij}(x) &= \frac{mnx_{(i+m)(j+n)}}{mn(m-1)(n-1)} \\ &\quad + \frac{(1-m-n)\left(x_{ij} + x_{(i+1)(j+1)} + \cdots + x_{(i+m)(j+n)}\right)}{mn(m-1)(n-1)}. \end{aligned} \tag{45}$$

Taking $\lim i, j \to \infty$, we have

$$\begin{aligned} &\lim_{(i,j)\to\infty} \phi_{mn,ij}(x) \\ &\quad = \lim_{(i,j)\to\infty} \Big[\Big(mnx_{(i+m)(j+n)} + (1-m-n) \\ &\qquad \times \left(x_{ij} + x_{(i+1)(j+1)} + \cdots + x_{(i+m)(j+n)}\right)\Big) \\ &\qquad \times (mn(m-1)(n-1))^{-1}\Big], \\ &L(mn(m-1)(n-1)) \\ &\quad = \lim_{(i,j)\to\infty} \Big[mnx_{(i+m)(j+n)} + (1-m-n) \\ &\qquad \times \left(x_{ij} + x_{(i+1)(j+1)} + \cdots + x_{(i+m)(j+n)}\right)\Big]. \end{aligned} \tag{46}$$

Since $m, n, L \neq 0$, therefore $\lim_{i,j\to\infty}\phi_{mn,ij}(x) \neq 0$ which implies that $x \notin ({}_2c_0^I)$. Hence, we get that the inclusion is proper.

(ii) Second for ${}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$.

The result of this part follows from the proof of Theorem 18. □

Theorem 20. ${}_2c^I \subset {}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$ *and the inclusions are proper.*

Proof. Let $x = (x_{ij}) \in {}_2c^I$. Then, we have

$$\left\{i, j \in \mathbb{N} : \left|x_{ij} - L\right| \geq \epsilon\right\} \in I. \tag{47}$$

Since ${}_2c \subset {}_2\mathrm{BV}_\sigma \subset {}_2l_\infty$, which implies $x = (x_{ij}) \in {}_2\mathrm{BV}_\sigma^I$ implies

$$\left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - L\right| \geq \epsilon\right\} \in I. \tag{48}$$

Now let,

$$\begin{aligned} B_1 &= \left\{i, j \in \mathbb{N} : \left|\phi_{i,j} - L\right| < \epsilon\right\}, \\ B_2 &= \left\{i, j \in \mathbb{N} : \left|\phi_{mn,ij}(x) - L\right| < \epsilon\right\} \end{aligned} \tag{49}$$

be such that $B_1^c, B_2^c \in I$. As

$${}_2l_\infty = \left\{x = \left(x_{ij}\right) : \sup_{ij}\left|x_{ij}\right| < \infty\right\}, \tag{50}$$

taking lim sup over i, j, we get $B_1^c \subset B_2^c$. Hence, ${}_2c^I \subset {}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$.

Next, we show that the inclusion is proper

(i) First for ${}_2c^I \subset {}_2\mathrm{BV}_\sigma^I$. We show that ${}_2c^I \subsetneq {}_2\mathrm{BV}_\sigma^I$.

Let $x = (x_{ij}) \in {}_2\mathrm{BV}_\sigma^I$, then by the definition

$$\begin{aligned} {}_2\mathrm{BV}_\sigma^I := \Big\{x = \left(x_{ij}\right) \in \omega : \\ \left\{i, j \in \mathbb{N} : \left|\phi_{mnij}(x) - L\right| \geq \epsilon\right\} \in I \\ \text{for some } L \in \mathbb{C}\Big\}. \end{aligned} \tag{51}$$

We have, $|\phi_{mnij}(x) - L| \geq \epsilon$. We say that the $I - \lim(\phi_{mnij}(x)) = L$.

Now considering the case when $|\phi_{mnij}(x) - L| < \epsilon$, then

$$\left|t_{mn,ij}(x) - t_{(m-1)(n-1),ij}(x) - L\right| < \epsilon, \tag{52}$$

when $m, n = 0$, then we have $\phi_{mnij}(x) = t_{ij}(x) = x_{ij}$. Therefore we get,

$$\left|x_{ij} - L\right| < \epsilon \quad \forall i, j \in \mathbb{N}. \tag{53}$$

Hence, $x \notin {}_2c^I = \{i, j \in \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I$. Hence, the inclusion is proper.

(ii) Second for ${}_2\mathrm{BV}_\sigma^I \subset {}_2l_\infty^I$.

The result follows from the proof of Theorem 18. □

Acknowledgment

The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

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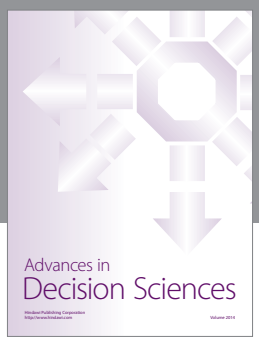

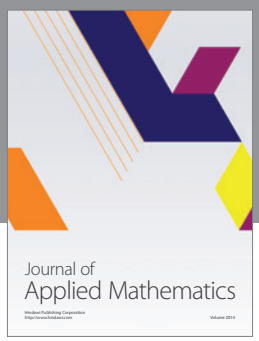

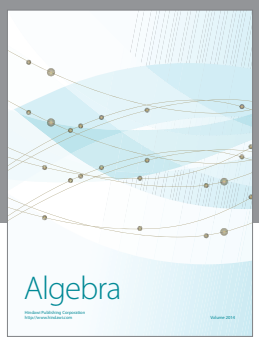

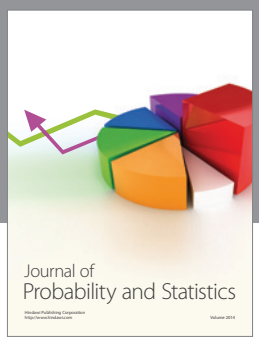

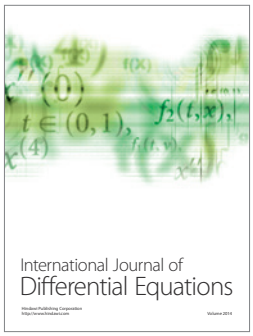

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