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Research Article

Asymptotic Solution for a Water Quality Model in a Uniform Stream

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We employ approximate analytical method, namely, Optimal Homotopy Asymptotic Method (OHAM), to investigate a one-dimensional steady advection-diffusion-reaction equation with variable inputs arises in the mathematical modeling of dispersion of pollutants in water is proposed. Numerical values are obtained via *Runge-Kutta-Fehlberg fourth-fifth* order method for comparison purpose. It was found that OHAM solution agrees well with the numerical solution. An example is included to demonstrate the efficiency, accuracy, and simplicity of the proposed method.

1. Introduction

Differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, fluid mechanics, biology, and engineering. Consequently, considerable attention has been given to the solutions of higher order ordinary differential equations, integral equations, and fractional order partial differential equations of physical interest. Number of literatures concerning the application of higher order differential equations in nonlinear dynamics has grown rapidly in the recent years [1–5]. Several numerical and semianalytical methods have been developed for solving high order boundary value problems [6–9].

A mathematical model for the dispersion of pollutants in a river is presented. The optimal homotopy asymptotic method for assessment of the chemical oxygen demand (COD) concentration in a river is considered. Pochai and Tangmanee [10] have provided a mathematical model of water pollution with the help of numerical method. Furthermore, Pochai and coworkers [11–14] have used numerical methods for the solution of hydrodynamic model with constant coefficients in the uniform reservoir and stream.

The optimal homotopy asymptotic method is an approximate analytical tool that is simple and straightforward and does not require the existence of any small or large parameter as does traditional perturbation method. Optimal Homotopy Asymptotic Method (OHAM) has been successfully applied to a number of nonlinear problems arising in fluid mechanics and heat transfer by various researchers [15–19].

This paper is organized as follows. First in Section 2, advection-diffusion-reaction equation is presented. In Section 3 we described the basic principles of OHAM. The OHAM solution of the problem is given in Section 4. Section 5 is devoted for the concluding remarks.

2. Dispersion in a Stream

The dispersion of chemical oxygen demand (COD) is described by the advection-diffusion-reaction equation (ADRE) [11] in the domain $[a, b]$:

$$-D_x \frac{d^2c}{dx^2} + u\frac{dc}{dx} + Rc - Q = 0, \tag{1}$$

where $c(x)$ is the concentration of COD at the point $x \in [a, b]$ (kg/m^3), $u = u(x)$ is the flow velocity in the x direction (m/s), $D_x = D(x)$ is the diffusion coefficient (m^2/s), $R = R(x)$ is the substance decay rate (s^{-1}), and $Q = Q(x)$ is the rate of change of substance concentration due to a source (kg/m^3 s).

The boundary conditions are

$$c = c_0 \quad \text{at } x = a,$$
$$\frac{dc}{dx} = T_0 \quad \text{at } x = b. \tag{2}$$

3. Basic Principles of OHAM

We review the basic principles of OHAM as illustrated in [3] and other works.

(i) Consider the following differential equation:

$$A[v(x)] + a(x) = 0, \quad x \in \Omega, \tag{3}$$

where Ω is problem domain, $A(v) = L(v) + N(v)$, where L, N are linear and nonlinear operators, $v(x)$ is an unknown function, and $a(x)$ is a known function.

(ii) Construct an optimal homotopy equation as

$$(1-p)[L(\phi(x;p)) + a(x)] - H(p)[A(\phi(x;p)) + a(x)] = 0, \tag{4}$$

where $0 \le p \le 1$ is an embedding parameter and $H(p) = \sum_{i=1}^{m} p^i K_i$ is auxiliary function on which the convergence of the solution greatly dependents. The auxiliary function $H(p)$ also adjusts the convergence domain and controls the convergence region.

(iii) Expand $\phi(x; p, K_j)$ in Taylor's series about p; one has an approximate solution:

$$\phi(x;p,K_j) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,K_j)\, p^k, \quad j = 1,2,3,\ldots. \tag{5}$$

Many researchers have observed that the convergence of the series in (5) depends upon K_j ($j = 1,2,\ldots,m$); if it is convergent then, we obtain

$$\tilde{v} = v_0(x) + \sum_{k=1}^{m} v_k(x;K_j). \tag{6}$$

(iv) Substitute (6) in (3); we have the following residual:

$$R(x;K_j) = L(\tilde{v}(x;K_j)) + a(x) + N(\tilde{v}(x;K_j)). \tag{7}$$

If $R(x;K_j) = 0$, then $\tilde{v}$ will be the exact solution. For nonlinear problems, generally, this will not be the case. For determining K_j ($j = 1,2,\ldots,m$), collocation method, Ritz method, or the method of least squares can be used.

(v) Finally, substitute these constants in (6) and one can get the approximate solution.

4. Application of OHAM

Consider the advection-dispersion-reaction equation (1) in the form

$$c'' = p(x)c' + q(x)c + r(x). \tag{8}$$

We assume that there is a plant which discharges waste water into the channel at the starting point 0.0 km and that the COD concentrations of the waste water are 1.2500 kg/m^3. Let the physical parameter values be diffusion coefficient 2, flow velocity $u(x) = 5 - x$ m/s, where $x \in [0,2]$, substance decay rate 3 s^{-1}, and rate of change of substance concentration due to the source 1 kg/m^3 s; we can obtain variable coefficients of convection-diffusion equation (8) as

$$p(x) = \frac{5-x}{D_x} = \frac{5-x}{2},$$
$$q(x) = \frac{R}{D_x} = \frac{3}{2}, \tag{9}$$
$$r(x) = \frac{-Q}{D_x} = \frac{-1}{2}.$$

Equation (8) becomes

$$c'' = \frac{5-x}{2}c' + \frac{3}{2}c - \frac{1}{2}, \tag{10}$$

subject to the boundary conditions:

$$c \longrightarrow 1.25 \quad \text{at } x = 0,$$
$$c' \longrightarrow 0 \quad \text{at } x = 2, \tag{11}$$

where primes denote differentiation with respect to x.

According to OHAM, we have

$$L(\phi(x,p)) = \frac{d^2}{dx^2}c(x,p) - \frac{5}{2}\frac{d}{dx}c(x,p) - \frac{3}{2}c(x,p),$$
$$a(x) = \frac{1}{2}. \tag{12}$$

Zeroth order problem is

$$c_0'' - \frac{5}{2}c_0' - \frac{3}{2}c_0 + \frac{1}{2} = 0, \tag{13}$$

with boundary conditions:

$$c_0(0) = 1.25, \qquad c_0'(2) = 0. \tag{14}$$

The solution of (13) with boundary conditions (14) is

$$c_0(x) = \frac{77e^{3x}}{12(6e^7+1)} + \frac{1}{3}. \tag{15}$$

First order problem is

$$\frac{-5}{2}c_1''(x,K_1) + c_1''(x,K_1) - \frac{3}{2}c_1(x,K_1) - \frac{11xK_1}{8}\left(\frac{e^{3x} - e^{(14-x)/2}}{6e^7+1}\right) = 0, \tag{16}$$

Table 1: Comparison of $c(x)$ via OHAM and numerical method.

x	RKF45	OHAM
0.0	1.2500	1.2500
0.1	1.2031	1.2031
0.2	1.1580	1.1580
0.3	1.1147	1.1146
0.4	1.0731	1.0731
0.5	1.0333	1.0332
0.6	0.9952	0.9951
0.7	0.9588	0.9588
0.8	0.9242	0.9242
0.9	0.8914	0.8913
1.0	0.8604	0.8602
1.1	0.8313	0.8310
1.2	0.8042	0.8037
1.3	0.7792	0.7784
1.4	0.7564	0.7553
1.5	0.7362	0.7347
1.6	0.7187	0.7167
1.7	0.7042	0.7016
1.8	0.6832	0.6899
1.9	0.6862	0.6821

with boundary conditions:

$$c_1(0) = 0, \qquad c_1'(2) = 0. \tag{17}$$

The solution of (16) with boundary conditions (17) is

$$\begin{aligned} c_1(x, K_1) = {} & \frac{-11K_1\left(e^{3x+7} - e^{-x/2}\right)}{343\left(6e^7 + 1\right)} \\ & + \frac{1}{\left(16464e^7 + 2744\right)} \\ & \times \left(539K_1\left(x^2 - \frac{4x}{7} + \frac{8}{49}\right)e^{3x} \right. \\ & \left. + \left(\frac{4x}{7} + \frac{8}{49} + x^2\right)e^{(14-x)/2}\right). \end{aligned} \tag{18}$$

The terms of second order problem and its solution are too large to be written above; therefore the final three-term solution via OHAM for $p = 1$ is

$$\tilde{c}(x, K_1, K_2) = c_0(x) + c_1(x, K_1) + c_2(x, K_1, K_2). \tag{19}$$

We use the method of least squares to obtain K_1 and K_2, the unknown convergent constants in $\tilde{c}$. The values of the convergent constants are $K_1 = 0.001881235287$, $K_2 = -1.103025681$.

By substituting the values of K_1 and K_2 in (19) and after simplification, we obtain the second order approximate solution via OHAM. To check the accuracy of the OHAM solution, a comparison between the solutions obtained by OHAM and numerical method was made and is presented in Table 1. Graphical representation of the solution using OHAM and Runge-Kutta-Fehlberg-fourth fifth order method is shown in Figure 1; an excellent agreement can be observed.

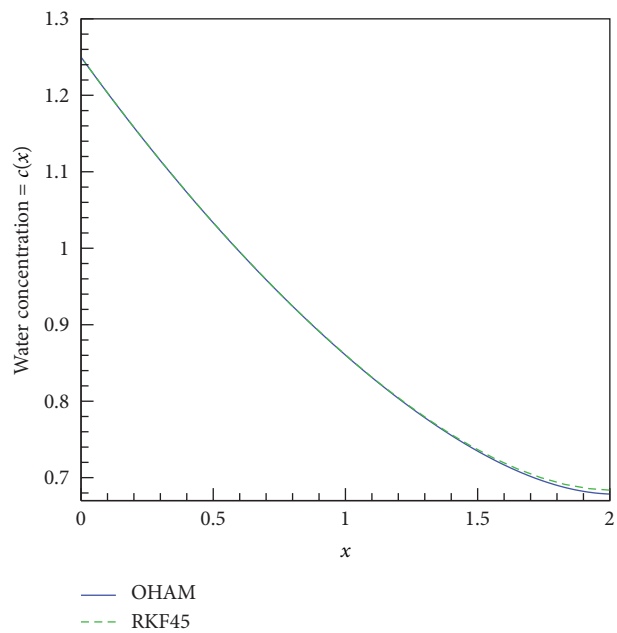

Figure 1: Comparison of two methods.

5. Concluding Remarks

In this paper, we have presented the solution of the one-dimensional steady advection-diffusion-reaction equation with variable inputs using homotopy approach and Runge-Kutta-Fehlberg fourth-fifth order method. Both approximate analytical and numerical results are obtained for the given problem. The validity of the proposed procedure, called the Optimal Homotopy Asymptotic Method (OHAM), was demonstrated on an example, and very good agreement was found between the approximate analytic results and numerical simulation results. The proposed scheme provides us with a simple and accurate way to optimally control and adjust the convergence of a solution and can give very good approximations in a few terms.

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