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Research Article

On the Domination Polynomial of Some Graph Operations

Saeid Alikhani[1,2]

[1] *Department of Mathematics, Yazd University, Yazd 89195-741, Iran*
[2] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

Correspondence should be addressed to Saeid Alikhani; alikhani206@gmail.com

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Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,\lambda) = \sum_{i=0}^{n} d(G,i)\lambda^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Every root of $D(G,\lambda)$ is called the domination root of G. In this paper, we study the domination polynomial of some graph operations.

1. Introduction

Let G be a simple graph. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. An *i-subset* of $V(G)$ is a subset of $V(G)$ of cardinality i. Let $\mathscr{D}(G,i)$ be the family of dominating sets of G which are i-subsets and let $d(G,i) = |\mathscr{D}(G,i)|$. The polynomial $D(G,x) = \sum_{i=0}^{|V(G)|} d(G,i)x^i$ is defined as *domination polynomial* of G [1]. This polynomial has been introduced by the author in his Ph.D. thesis in 2009 [2]. A root of $D(G,x)$ is called a *domination root* of G. More recently, domination polynomial has found application in network reliability [3]. For more information and motivation of domination polynomial and domination roots refer to [1, 2].

The join $G = G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. The corona of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2 [4]. the Cartesian product of two graphs G and H is denoted by $G\square H$, is the graph with vertex set $V(G)\cup V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

In this paper, we study the domination polynomials of some graph operations.

2. Main Results

As is the case with other graph polynomials, such as chromatic polynomials and independence polynomials, it is natural to consider the domination polynomial of composition of two graphs. It is not hard to see that the formula for domination polynomial of join of two graphs is obtained as follows.

Theorem 1 (see [1]). *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then*

$$D(G_1 + G_2, x) = ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + D(G_1, x) + D(G_2, x). \tag{1}$$

It is obvious that this operation of graphs is commutative. Using this product, one is able to construct a connected graph G with the number of dominating sets n, where n is an arbitrary odd natural number; see [5].

Let to consider the corona of two graphs. The following theorem gives us the domination polynomial of graphs of the form $H \circ K_1$ which is the first result for domination polynomial of specific corona of two graphs.

Theorem 2 (see [1]). *Let G be a graph. Then $D(G,x) = x^n(x+2)^n$ if and only if $G = H \circ K_1$ for some graph H of order n.*

It is easy to see that the corona operation of two graphs does not have the commutative property. The following theorem gives us the domination polynomial of $K_1 \circ H$.

Theorem 3. *For every graph H of order m, $D(K_1 \circ H, x) = x(1+x)^m + D(H, x)$.*

Proof. In each graph of the form $K_1 \circ H$, we have two cases for a dominating set S.

Case 1. S includes u (the vertex originally in K_1) and an arbitrary subset of the m vertices from the copy of H. The generating function for the number of dominating sets of graph in this case is $x(1+x)^m$.

Case 2. S does not include u and it is exactly a dominating set of H. In this case $D(H, x)$ is the generating function.

By addition principle, we have $D(K_1 \circ H, x) = x(1+x)^m + D(H, x)$. □

The following theorem gives a formula for domination polynomial of corona products of two graphs.

Theorem 4. *Let $G = (V, E)$ and $H = (W, F)$ be nonempty graphs of order n and m, respectively. Then*

$$D(G \circ H, x) = \left(x(1+x)^m + D(H, x)\right)^n. \tag{2}$$

Proof. By Theorem 3, it suffices to prove that $D(G \circ H, x) = (D(K_1 \circ H, x))^n$. In the corona of two graphs G and H, every vertex $u \in V$ of G is adjacent to all vertices of the corresponding copy of H. So, we can delete all edges in E in the corona. Therefore, the arising graph is the disjoint union of $|V|$ copies of the corona $K_1 \circ H$. Therefore, $D(G \circ H, x) = (D(K_1 \circ H, x))^n$. □

As a consequence of the above theorem, we have the following corollary.

Corollary 5. *(i) Let G be a connected graph of order n. Then, $G = H \circ \overline{K_2}$, for some graph H, if and only if $D(G, x) = x^{(n/3)}(x^2 + 3x + 1)^{(n/3)}$ (see [1]).*

(ii) Let H be a graph of order n and $G = H \circ \overline{K_m}$. Then $D(G, x) = (x(1+x)^m + x^m)^n$.

It is interesting that for the classification of graphs with exactly two, three, and four domination roots, we must consider some kinds of corona of two graphs. For more information, see [1].

To study more we need the following theorem.

Theorem 6 (see [2]). *If G has t connected components $G_1, \ldots, G_t$, then $D(G, x) = \prod_{i=1}^{t} D(G_i, x)$.*

Now we will consider the Cartesian product of two graphs. First we prove the following easy result.

Theorem 7. *If G has t connected components $G_1, \ldots, G_t$, then*

$$D(G \Box H, x) = \prod_{i=1}^{t} D(G_i \Box H, x). \tag{3}$$

Proof. Since we have $\bigcup_{i=1}^{t} G_i \Box H = \bigcup_{i=1}^{t} (G_i \Box H)$, we have the result by Theorem 6. □

Despite the above property, it is difficult to determine the domination polynomial of this product, even in such simple cases as the grid graphs $P_n \Box P_m$.

Now we consider another operation of two graphs. Let G and H be graphs, with $V(G) = \{v_1, \ldots, v_n\}$. The graph $G \diamond H$ formed by substituting a copy of H for every vertex of G is formally defined by taking a disjoint copy of H, H_v, for every vertex v of G and joining every vertex in H_u to every vertex in H_v if and only if u is adjacent to v in G.

The following result is also proven in [6, Lemma 3].

Theorem 8. *For any graph G, $D(G \diamond K_t, x) = D(G, (1+x)^t - 1)$.*

Proof. Note that the closed neighborhood of the vertex (u, v) of graph $G \diamond K_t$ is $(N_G[u], K_t)$. To make a dominating set of $G \diamond K_t$, suppose that $S \subset V(G)$ is a dominating set for G. It is easy to see that $\bigcup_{s \in S} \{(s, v) \mid v \in A_s\}$ is a dominating set of $G \diamond K_t$, where $\{A_s \mid s \in S\}$ is a family of arbitrary nonempty subsets of $V(K_t)$. Therefore, every $v \in S$ corresponds to all nonempty subsets of $G \diamond K_t$ which have the generating function $(1+x)^t - 1$. So we have the result. □

We would like to obtain some corollaries. We recall the following theorems.

Theorem 9 (see [7]). *(i) For every $n \geq 4$,*

$$\begin{aligned} D(P_n, x) = x\,[&D(P_{n-1}, x) \\ &+ D(P_{n-2}, x) + D(P_{n-3}, x)], \end{aligned} \tag{4}$$

with the initial values $D(P_1, x) = x$, $D(P_2, x) = x^2 + 2x$, $D(P_3, x) = x^3 + 3x^2 + x$.

(ii) For every $n \geq 4$,

$$\begin{aligned} D(C_n, x) = x\,[&D(C_{n-1}, x) \\ &+ D(C_{n-2}, x) + D(C_{n-3}, x)], \end{aligned} \tag{5}$$

with the initial values $D(C_1, x) = x$, $D(C_2, x) = x^2 + 2x$, $D(C_3, x) = x^3 + 3x^2 + 3x$.

Here we consider the graphs obtained by selecting one vertex in each of n triangles and identifying them. Some call them Dutch Windmill graphs [8, 9] and friendship graphs.

Theorem 10. *For every $n \in \mathbb{N}$,*

$$D(G_3^n, x) = \left(2x + x^2\right)^n + x(1+x)^{2n}. \tag{6}$$

Proof. It is easy to see that G_3^n is join of K_1 and nK_2. Now by Theorem 1, we have

$$\begin{aligned} D(G_3^n, x) &= D(K_1, x) + D(nK_2, x) + \left((1+x)^{2n} - 1\right)x \\ &= \left(2x + x^2\right)^n + x(1+x)^{2n}. \end{aligned} \tag{7}$$

□

Theorem 8 can be used to generalize recurrence relations for the domination polynomial of some families of graphs. For example, we state and prove the following theorem.

Theorem 11. *(i) Suppose that $G_{n,t} = P_n \diamond K_t$. Then*

$$D(G_{n,t}, x) = ((x+1)^t - 1) \times [D(G_{n-1,t}, x) + D(G_{n-2,t}, x) + D(G_{n-3,t}, x)]. \tag{8}$$

(ii) Suppose that $H_{n,t} = C_n \diamond K_t$. Then

$$D(H_{n,t}, x) = ((x+1)^t - 1) \times [D(H_{n-1,t}, x) + D(H_{n-2,t}, x) + D(H_{n-3,t}, x)]. \tag{9}$$

(iii)

$$D(G_3^n \diamond K_t, x) = ((1+x)^{2t} - 1)^n + ((1+x)^t - 1)(1+x)^{2nt}. \tag{10}$$

Proof. (i) From Theorem 8, we have $D(G_{n,t}, x) = D(P_n, (1+x)^t - 1)$. Now by Part (i) of Theorem 9, we have the result.

(ii) From Theorem 8, we have $D(H_{n,t}, x) = D(C_n, (1+x)^t - 1)$. Now by Part (ii) of Theorem 9, we have the result.

(iii) From Theorem 8, we have $D(G_3^n \diamond K_t, x) = D(G_3^n, (1+x)^t - 1)$. Now by Theorem 10, we have the result. □

3. Conclusion

In this paper, we studied the domination polynomials of some graph operations. There are some open problems which are interesting to consider.

(i) What is the basic formula for the domination polynomial of the Cartesian product of two graphs?

For two graphs G and H, let $G[H]$ be the graph with vertex set $V(G) \times V(H)$ and such that vertex (a, x) is adjacent to vertex (b, y) if and only if a is adjacent to b (in G) or $a = b$ and x is adjacent to y (in H). The graph $G[H]$ is the lexicographic product (or composition) of G and H and can be thought of as the graph arising from G and H by substituting a copy of H for every vertex of G. There is a main problem.

(ii) How can compute the domination polynomial of Lexicographic product of the two graphs?

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