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Research Article

The Modified Simple Equation Method for Exact and Solitary Wave Solutions of Nonlinear Evolution Equation: The GZK-BBM Equation and Right-Handed Noncommutative Burgers Equations

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The modified simple equation method is significant for finding the exact traveling wave solutions of nonlinear evolution equations (NLEEs) in mathematical physics. In this paper, we bring in the modified simple equation (MSE) method for solving NLEEs via the Generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony (GZK-BBM) equation and the right-handed noncommutative Burgers' (nc-Burgers) equations and achieve the exact solutions involving parameters. When the parameters are taken as special values, the solitary wave solutions are originated from the traveling wave solutions. It is established that the MSE method offers a further influential mathematical tool for constructing the exact solutions of NLEEs in mathematical physics.

1. Introduction

The importance of nonlinear evolution equations (NLEEs) is now well established, since these equations arise in various areas of science and engineering, especially in fluid mechanics, biology, plasma physics, solid-state physics, optical fibers, biophysics and so on. As a key problem, finding their analytical solutions is of great importance and is actually executed through various efficient and powerful methods such as the Exp-function method [1–4], the tanh-function method [5, 6], the homogeneous balance method [7, 8], the (G'/G)-expansion method [9–16], the Hirota's bilinear transformation method [17, 18], the Backlund transformation method [19], the inverse scattering transformation [20], the Jacobi elliptic function method [21], the modified simple equation method [22–24] and so on.

The objective of this paper is to look for new study relating to the MSE method via the well-recognized GZK-BBM equation and right-handed nc-Burgers' equation and establish the originality and effectiveness of the method.

The paper is organized as follows: in Section 2, we give the description of the MSE method. In Section 3, we use this method to the nonlinear evolution equations pointed out above, and in Section 4 conclusions are given.

2. Description of the MSE Method

Suppose the nonlinear evolution equation is in the following form:

$$F\left(u, u_t, u_x, u_{xx}, u_{tt}, \ldots\right) = 0, \tag{1}$$

where F is a polynomial of $u(x,t)$ and its partial derivatives wherein the highest order derivatives and nonlinear terms are concerned. The main steps of the MSE method [22–24] are as follows.

Step 1. The traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x \pm \nu t \tag{2}$$

permits us to reduce (1) into the following ordinary differential equation (ODE):

$$P\left(u, u', u'', \ldots\right) = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, wherein $u'(\xi) = du/d\xi$.

Step 2. We suppose the solution of (3) is of the form

$$u(\xi) = \sum_{k=0}^{N} C_k \left[\frac{\Phi'(\xi)}{\Phi(\xi)}\right]^k, \tag{4}$$

where C_k $(k = 0, 1, 2, 3, \ldots)$ are arbitrary constants to be determined, such that $C_N \neq 0$, and $\Phi(\xi)$ is an unidentified function to be determined afterwards. In Exp-function method, (G'/G)-expansion method, tanh-function method, Jacobi elliptic function method, and so forth, the solution is offered in terms of some predefined functions, but in the MSE method, Φ is not predefined or not a solution of any predefined differential equation. Therefore, some fresh solutions might be found by this method. This is the merit of the MSE method.

Step 3. We determine the positive integer N come out in (4) by considering the homogeneous balance between the highest order derivatives and the highest order nonlinear terms occurring in (3).

Step 4. We compute all the required derivatives $u', u'', \ldots$, and substitute (4) and the derivatives into (3) and then we account for the function $\Phi(\xi)$. As a result of this substitution, we get a polynomial of $(\Phi'(\xi)/\Phi(\xi))$ and its derivatives. In this polynomial, we equate all the coefficients to zero. This procedure yields a system of equations whichever can be solved to find C_k and $\Phi(\xi)$.

3. Applications

3.1. The GZK-BBM Equation. In this subsection, we will use the MSE method to look for the exact solutions and then the solitary wave solutions to the GZK-BBM equation:

$$u_t + u_x + \alpha\left(u^3\right)_x + \beta\left(u_{xt} + u_{yy}\right)_x = 0, \tag{5}$$

where α and β are nonzero constants.

Using traveling wave transformation,

$$u = u(x,y,t), \qquad \xi = x + y - \nu t, \qquad u(x,y,t) = u(\xi), \tag{6}$$

(5) reduces to the following ODE:

$$-\nu u' + u' + \alpha\left(u^3\right)' + \beta\left(-\nu u'' + u''\right)' = 0. \tag{7}$$

Integrating (7) with respect to ξ, we obtain

$$(1-\nu)u + \alpha u^3 + \beta(1-\nu)u'' = 0. \tag{8}$$

Balancing the highest order derivative u'' and nonlinear term u^3, we obtain $N = 1$.

Therefore, the solution (4) turns into the following form:

$$u(\xi) = C_0 + C_1\left(\frac{\Phi'}{\Phi}\right), \tag{9}$$

where C_0 and C_1 are constants such that $C_1 \neq 0$, and $\Phi(\xi)$ is an unidentified function to be determined. It is easy to find that

$$u' = C_1\left[\frac{\Phi''}{\Phi} - \left(\frac{\Phi'}{\Phi}\right)^2\right], \tag{10}$$

$$u'' = C_1\frac{\Phi'''}{\Phi} - 3C_1\frac{\Phi''\Phi'}{\Phi^2} + 2A_1\left(\frac{\Phi'}{\Phi}\right)^3, \tag{11}$$

$$u^3 = {C_1}^3\left(\frac{\Phi'}{\Phi}\right)^3 + 3{C_1}^2C_0\left(\frac{\Phi'}{\Phi}\right)^2 + 3C_1{C_0}^2\left(\frac{\Phi'}{\Phi}\right) + {C_0}^3. \tag{12}$$

Substituting the values of u, u'', and u^3 from (9)–(12) into (8) and then equating the coefficients of $\Phi^0, \Phi^{-1}, \Phi^{-2}$, and Φ^{-3} to zero, we obtain

$$\alpha{C_0}^3 + C_0 - \nu C_0 = 0, \tag{13}$$

$$\beta(\nu-1)\Phi''' - \left(3\alpha{C_0}^2 - \nu + 1\right)\Phi' = 0, \tag{14}$$

$$\beta(\nu-1)\Phi'' + \alpha C_0C_1\Phi' = 0, \tag{15}$$

$$\left(\alpha{C_1}^3 + 2\beta C_1 - 2\beta\nu C_1\right)\left(\Phi'\right)^3 = 0. \tag{16}$$

Equations (13) and (16), respectively, yield

$$C_0 = 0, \pm\sqrt{\frac{\nu-1}{\alpha}}, \qquad C_1 = \pm\sqrt{\frac{2\beta(\nu-1)}{\alpha}} \quad \text{since } C_1 \neq 0, \tag{17}$$

where $\nu \neq 1$.

From (14) and (15), we obtain

$$\frac{\Phi'''}{\Phi''} = -l, \tag{18}$$

where $l = (3\alpha{C_0}^2 - \nu + 1)/\alpha C_0C_1$.

Integrating (18), we obtain

$$\Phi''(\xi) = c_1e^{-l\xi}, \tag{19}$$

where c_1 is a constant of integration.

And from (15) and (19), we obtain

$$\Phi' = -me^{-l\xi}, \tag{20}$$

where $m = \beta(\nu-1)c_1/\alpha C_0C_1$

Integrating (20) with respect to ξ, we obtain

$$\Phi(\xi) = c_2 + \frac{m}{l} e^{-l\xi}, \tag{21}$$

where c_2 is a constant of integration.

Substituting the value of Φ and Φ' into solution (9) yields

$$u(\xi) = C_0 + C_1 \left(\frac{-lme^{-l\xi}}{c_2 l + me^{-l\xi}} \right). \tag{22}$$

Case 1. When $C_0 = 0$, solution (22) collapses, and hence this case is rejected.

Case 2. When $C_0 = \pm\sqrt{(\nu-1)/\alpha}$ and $C_1 = \pm\sqrt{2\beta(\nu-1)/\alpha}$, substituting the values of C_0 and C_1 into (22) and simplifying, we obtain the exact solutions:

$$u(x,y,t) = \pm\sqrt{\left(\frac{\nu-1}{\alpha}\right)} \times \left\{ 1 - \frac{2\beta c_1 \exp\left(\pm\sqrt{2/\beta}\,(x+y-\nu t)\right)}{\beta c_1 \exp\left(\pm\sqrt{2/\beta}\,(x+y-\nu t)\right) + 2c_2} \right\}. \tag{23}$$

Since c_1 and c_2 are arbitrary constants, therefore, if we set $c_2 = c_1/2$, the exact solution (23) turns out to the following solitary wave solutions

$$u(x,y,t) = \pm\sqrt{\left(\frac{\nu-1}{\alpha}\right)} \tanh\left(\sqrt{\frac{1}{2}}\,(x+y-\nu t)\right), \tag{24}$$

when $\beta = 1$. On the other hand, if $\beta = -1$, solution (23) turns into

$$u(x,y,t) = \pm\sqrt{\left(\frac{\nu-1}{\alpha}\right)} \coth\left(\sqrt{\frac{1}{2}}\,(x+y-\nu t)\right). \tag{25}$$

In particular, when $\alpha = 1$ and $\nu = 4$, solutions (24) and (25) convert to

$$u(x,y,t) = \pm\sqrt{3}\tanh\left(\sqrt{\frac{1}{2}}\,(x+y-4t)\right), \tag{26}$$

$$u(x,y,t) = \pm\sqrt{3}\coth\left(\sqrt{\frac{1}{2}}\,(x+y-4t)\right), \tag{27}$$

respectively.

For $y = 0$, the solution $u(x,y,t)$ presented in (26) is sketched in Figure 1.

Again for $y = 0$, the solution $u(x,y,t)$ presented in (27) is sketched in Figure 2.

The MSE method is applied to investigate solitary wave solutions to the GZK-BBM equation and obtained solutions with free parameters involving the known solutions in the open literature. Obviously we might choose the values of the arbitrary constants c_1 and c_2 equal to other values, resulting in diverse solitary shapes. The free parameters imply some physical meaningful results in gravity water waves in the long-wave regime.

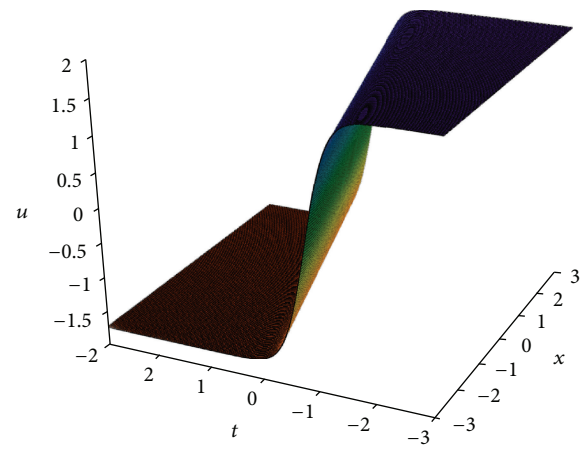

Figure 1

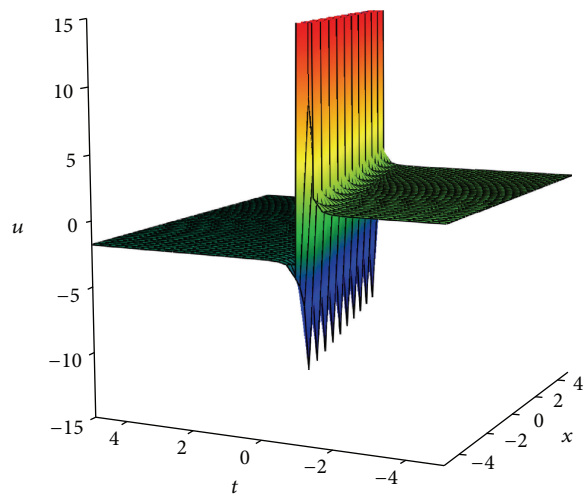

Figure 2

3.2. The Right-Handed nc-Burgers' Equation. In this subsection, we will bring to bear the MSE method to find the traveling wave solutions and then the solitary wave solutions to the right-handed nc-Burgers' equation:

$$u_t = u_{xx} + 2uu_x. \tag{28}$$

Using traveling wave transformation (2), (28) is reduced to the following ODE:

$$u'' + 2uu' + \nu u' = 0. \tag{29}$$

Integrating (29) with respect to ξ and setting the constant of integration to zero, we obtain

$$u' + u^2 + \nu u = 0. \tag{30}$$

Balancing the highest order derivative and nonlinear term, we obtain $N = 1$.

Therefore, solution (4) becomes

$$u(\xi) = C_0 + C_1 \left(\frac{\Phi'}{\Phi} \right). \tag{31}$$

Executing the parallel course of action which is described in Section 3.1, we obtain

$$\nu C_0 + {C_0}^2 = 0, \tag{32}$$

$$\Phi'' + (2C_0 + \nu)\Phi' = 0, \tag{33}$$

$${C_1}^2 - C_1 = 0. \tag{34}$$

Solving (32) and (34), we obtain $C_0 = 0, -\nu$ and $C_1 = 1$, since $C_1 \neq 0$, respectively.

Case 1. When $C_0 = 0$ and $C_1 = 1$ and solving (33), we receive the value of Φ, and substituting the value of Φ into (31), we obtain the following exact solution:

$$u(x,t) = \frac{\nu\ c_1 \exp(-\nu\ (x - \nu\ t))}{\nu\ c_2\ - c_1\ \exp(-\nu\ (x - \nu\ t))}, \tag{35}$$

where c_1 and c_2 are constants of integration. Therefore, we can make choices at random the parameters c_1 and c_2; if we choose $c_1 = \nu$ and $c_2 = 1$, the exact solution (35) turns into the under determined solitary wave solution

$$u(x,t) = \frac{-\nu}{2}\left\{1 - \coth\left(\nu\frac{(x - \nu t)}{2}\right)\right\}. \tag{36}$$

And if $c_1 = -\nu$ and $c_2 = 1$, the solution (35) turn into,

$$u(x,t) = \frac{-\nu}{2}\left\{1 - \tanh\left(\nu\frac{(x - \nu t)}{2}\right)\right\}. \tag{37}$$

Case 2. When $C_0 = -\nu$ and solving (33), we get the value of Φ, and substituting this value into (31), we obtain the subsequent exact solution:

$$u(x,t) = -\nu + \frac{\nu c_1 \exp(\nu(x - \nu t))}{c_1 \exp(\nu(x - \nu t)) + \nu c_2}. \tag{38}$$

We can arbitrarily pick the parameters c_1 and c_2. Therefore, exact solution (38) turns into the following solitary wave solutions:

$$u(x,t) = -\nu + \frac{\nu}{2}\left\{1 + \tanh\left(\nu\frac{(x - \nu t)}{2}\right)\right\}, \tag{39}$$

when $c_1 = \nu$ and $c_2 = 1$, and

$$u(x,t) = -\nu + \frac{\nu}{2}\left\{1 + \coth\left(\frac{\nu(x - \nu t)}{2}\right)\right\}, \tag{40}$$

when $c_1 = -\nu$ and $c_2 = 1$.

The solution $u(x,t)$ given in (39) is presented in Figure 3.
The solution $u(x,t)$ given in (40) is presented in Figure 4.

For specific values of the parameters in the generalized exact solutions (35) and (38), we obtain the solitary wave shape solutions to the right-handed nc-Burgers' equation which are shown in Figures 3 and 4. Of course we might choose other values of the arbitrary constants c_1 and c_2, resulting in diverse solitary wave shapes. The free parameters may imply some physical meaningful results in fluid mechanics, gas dynamics, and traffic flow.

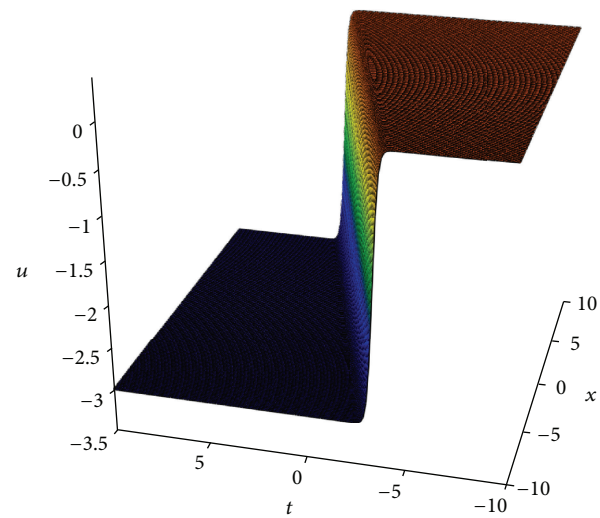

Figure 3

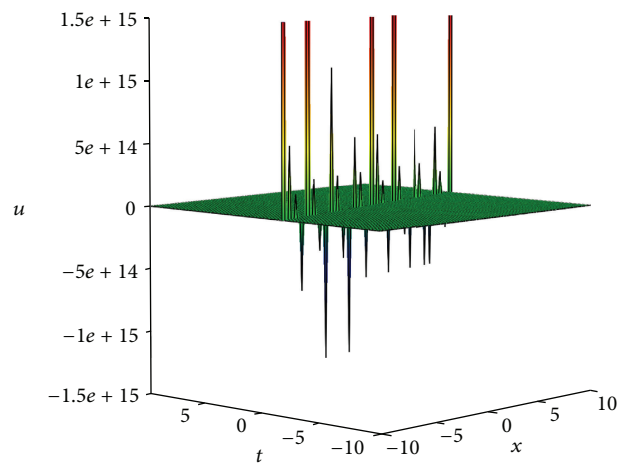

Figure 4

4. Conclusions

The modified simple equation method presented in this paper has been successfully implemented to find the exact and the solitary wave solutions for NLEEs via the GZK-BBM and right-handed nc-Burgers' equation. The method offers solutions with free parameters that might be important to explain some intricate physical phenomena. Some special solutions including the known solitary wave solution are originated by setting appropriate values for the parameters. Compared to the currently proposed method with other methods, such as the (G'/G)-expansion method, the Exp-function method and the tanh-function method, we might conclude that the exact solutions to (5) and (28) can be investigated using these methods with the help of the symbolic computational software such as Mathematica and Maple to facilitate the complex algebraic computations. On the other hand, via the proposed method, the exact and solitary wave solutions to these equations have been achieved without using any

symbolic computation software because the method is very simple and has easy computations.

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