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Research Article

Regularity in terms of Hyperideals

Kostaq Hila,[1] Bijan Davvaz,[2] Krisanthi Naka,[1] and Jani Dine[1]

[1] *Department of Mathematics and Computer Science, Faculty of Natural Sciences, University of Gjirokastra, Gjirokastra 6001, Albania*
[2] *Department of Mathematics, Yazd University, Yazd, Iran*

Correspondence should be addressed to Bijan Davvaz; davvaz@yazd.ac.ir

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This paper deals with the algebraic hypersystems. The notion of regularity of different type of algebraic systems has been introduced and characterized by different authors such as Iseki, Kovacs, and Lajos. We generalize this notion to algebraic hypersystems giving a unified generalization of the characterizations of Kovacs, Iseki, and Lajos. We generalize also the concept of ideal introducing the notion of j-hyperideal and hyperideal of an algebraic hypersystem. It turns out that the description of regularity in terms of hyperideals is intrinsic to associative hyperoperations in general. The main theorem generalizes to algebraic hypersystems some results on regular semigroups and regular rings and expresses a necessary and sufficient condition by means of principal hyperideals. Furthermore, two more theorems are obtained: one is concerned with a necessary and sufficient condition for an associative, commutative algebraic hypersystem to be regular and the other is concerned with nilpotent elements in the algebraic hypersystem.

1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, and coding theory.

Hyperstructure theory was introduced in 1934, when Marty [1] defined hypergroups based on the notion of hyperoperation, began to analyze their properties, and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on hyperstructure theory; see [2–5]. A recent book on hyperstructures [3] points out their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs, and hypergraphs. Another book [4] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

n-ary generalizations of algebraic structures are the most natural way for further development and deeper understanding of their fundamental properties [6, 7]. In [8], Davvaz and Vougiouklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. Also, we can consider n-ary hypergroups as a nice generalization of n-ary groups. Leoreanu-Fotea and Davvaz in [9] introduced and studied the notion of a partial n-ary hypergroupoid, associated with a binary relation. Some important results, concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-ary hypergroupoids. Davvaz et al. in [10, 11] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups, and n-ary semigroups. In this paper we deal with the algebraic hypersystems. The notion of regularity of different type of algebraic systems has been introduced and characterized by different authors such as Iseki [12], Kovacs [13], and Lajos [14]. We generalize this notion to algebraic

hypersystems giving a unified generalization of the characterizations of Kovacs, Iseki, and Lajos. We generalize also the concept of ideal introducing the notion of j-hyperideal and hyperideal of an algebraic hypersystem. It turns out that the description of regularity in terms of hyperideals is intrinsic to associative hyperoperations in general. The main theorem generalizes to algebraic hypersystems some results on regular semigroups and regular rings and expresses a necessary and sufficient condition by means of principal hyperideals. Furthermore, two more theorems are obtained: one is concerned with a necessary and sufficient condition for an associative, commutative algebraic hypersystem to be regular; another is concerned with nilpotent elements in the algebraic hypersystem.

2. Algebraic Hypersystems and m-Ary Hyperstructures

In this section we recall some known notions on what is meant by an algebraic hypersystem and m-ary hyperstructure.

Let H be a nonempty set and f a mapping $f : H \times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ denotes the set of all nonempty subsets of H. Then f is called a *binary (algebraic) hyperoperation* on H. In general, a mapping $f : H \times H \times \cdots \times H \to \mathscr{P}^*(H)$, where H appears m times, is called an *m-ary (algebraic) hyperoperation*, and m is called the *arity* of this hyperoperation. An algebraic system (H, f), where f is an m-ary hyperoperation defined on H, is called an *m-ary hypergroupoid* or an *m-ary hypersystem*. Since we identify the set $\{x\}$ with the element x, any m-ary (binary) groupoid is an m-ary (binary) hypergroupoid.

Let f an m-ary hyperoperation on H and $A_1, A_2, \ldots, A_m$ nonempty subsets of H. We define

$$\begin{aligned} &f(A_1, A_2, \ldots, A_m) \\ &\quad = \{f(x_1, x_2, \ldots, x_m) \mid x_i \in A_i, i = 1, 2, \ldots, m\}. \end{aligned} \tag{1}$$

We will use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. In this convention,

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_m) \tag{2}$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^m)$. In the case when $y_{i+1} = \cdots = y_j = y$, the last expression will be written in the form $f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m)$.

Similarly, for subsets $A_1, A_2, \ldots, A_m$ of H we define

$$\begin{aligned} f(A_1^m) &= f(A_1, A_2, \ldots, A_m) \\ &= \{f(x_1^m) \mid x_i \in A_i, i = 1, \ldots, m\}. \end{aligned} \tag{3}$$

An m-ary hyperoperation f is called (i, j)-*associative* if

$$f\left(x_1^{i-1}, f\left(x_i^{m+i-1}\right), x_{m+i}^{2m-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{m+j-1}\right), x_{m+j}^{2m-1}\right) \tag{4}$$

holds for fixed $1 \le i < j \le m$ and all $x_1, x_2, \ldots, x_{2m-1} \in H$.

Note that (i, k)-associativity follows from (i, j)- and (j, k)-associativity.

If the above condition is satisfied for all $i, j \in \{1, 2, \ldots, m\}$, then we say that f is *associative*.

The m-ary hyperoperation f is called *commutative* if and only if for all $x_1, \ldots, x_m \in H$ and for all $\sigma \in \mathbb{S}_m$, $f(x_1, x_2, \ldots, x_m) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})$.

By an algebraic *hypersystem* $(H, f_1, f_2, \ldots, f_m)$ or simply H is meant a set H closed under a collection of m_i-ary hyperoperation f_i and often also satisfying a fixed set of laws, for instance, the associative law.

3. Regular Algebraic Hypersystems

Let H be an algebraic hypersystem. H is said to be *regular* with respect to the hyperoperation f if and only if for each $a \in H$ there exist $x_2, x_3, \ldots, x_m; y_1, y_3, \ldots, y_m; \ldots; z_1, z_2, \ldots, z_{m-1} \in H$ such that

$$\begin{aligned} a \in f\big(&f(a, x_2, \ldots, x_m), f(y_1, a, y_3, \ldots, y_m), \ldots, \\ &f(z_1, z_2, \ldots, z_{m-1}, a)\big). \end{aligned} \tag{5}$$

A subset S of H constitutes a *subhypersystem* if and only if S is closed under the same hyperoperations and satisfies the same fixed laws in H.

Let H be an algebraic hypersystem. A *j-hyperideal* $j = 1, 2, \ldots, m$ relative to the m-ary hyperoperation is defined to be a subhypersystem I_j such that, for any $x_1, x_2, \ldots, x_m \in H$, if $x_j \in I_j$, then $f(x_1, x_2, \ldots, x_m) \subseteq I_j$. The j-hyperideal relative to f generated by an element $a \in H$ (usually called a principal j-hyperideal) is denoted by

$$(a)_j = f\left(H, H, \ldots, \overset{j}{a}, \ldots, H\right) \cup \{a\}. \tag{6}$$

A subhypersystem I which is a j-hyperideal for each $j = 1, \ldots, m$ is simply called a *hyperideal*.

Theorem 1. *Let H be an algebraic hypersystem which is associative relative to an m-ary hyperoperation f. Then the following conditions are equivalent.*

(1) *H is regular relative to the hyperoperation f.*

(2) *$f(I_1, I_2, \ldots, I_m) = \bigcap_{j=1}^m I_j$ for any set of j-hyperideals I_j relative to the hyperoperation.*

(3) *$f((a_1)_1, (a_2)_2, \ldots, (a_m)_m) = \bigcap_{j=1}^m (a_j)_j$ for any set of elements $a_1, a_2, \ldots, a_m \in H$.*

(4) *$f((a)_1, (a)_2, \ldots, (a)_m) = \bigcap_{j=1}^m (a)_j$ for each element $a \in H$.*

Proof. "(1) $\Rightarrow$ (2)." Let H be regular relative to the m-ary hyperoperation f and let $a \in \bigcap_{j=1}^{m} I_j$ for any set of m j-hyperideals I_j relative to the hyperoperation. Then by regularity there exists $x_2, \ldots, x_m; y_1, y_3, \ldots, y_m, \ldots; z_1, \ldots, z_{m-1} \in H$ such that

$$a \in f\left(f\left(a, x_2, \ldots, x_m\right), f\left(y_1, a, \ldots, y_m\right), \ldots, f\left(z_1, \ldots, z_{m-1}, a\right)\right). \tag{7}$$

With I_j being a j-hyperideal for each $j = 1, \ldots, m$, we thus obtain $f(a, x_2, \ldots, x_m) \subseteq I_1$, $f(y_1, a, \ldots, y_m) \subseteq I_2, \ldots$, and $f(z_1, z_2, \ldots, z_{m-1}, a) \subseteq I_m$ and hence $\bigcap_{j=1}^{m} I_j \subseteq f(I_1, I_2, \ldots, I_m)$.

Conversely, if $a \in f(I_1, I_2, \ldots, I_m)$, then $a \in f(i_1, i_2, \ldots, i_m)$ for $i_j \in I_j$, $j = 1, \ldots, m$, and therefore $a \in I_j$ for each $j = 1, \ldots, m$. Hence, (2) is proved.

"(2) $\Rightarrow$ (3) $\Rightarrow$ (4)" are obvious.

"(4) $\Rightarrow$ (1)." Let $f((a)_1, (a)_2, \ldots, (a)_n) = \bigcap_{j=1}^{m} (a)_j$ for each $a \in H$. Since for each $a \in H$, $a \in \bigcap_{j=1}^{m} (a)_j$, then $a \in f(b_1, b_2, \ldots, b_m)$, where either $b_k = a$ or $b_k \in f(c_1, c_2, \ldots, c_m)$ with $c_k = a$. Thus, we have in any case the following:

$$\begin{aligned} a &\in f\left(b_1, b_2, \ldots, b_m\right) \\ &= f\left(f\left(a, x_2, \ldots, x_m\right), f\left(y_1, a, \ldots, y_m\right), \ldots, f\left(z_1, z_2, \ldots, z_{m-1}, a\right)\right) \end{aligned} \tag{8}$$

for some $x_2, \ldots, x_m; y_1, y_3, \ldots, y_m; \ldots; z_1, \ldots z_{m-1} \in H$. This shows that H is regular with respect to the hyperoperation. □

Theorem 2. *An algebraic hypersystem H which is associative and commutative relative to an m-ary hyperoperation f is regular with respect to the same hyperoperation if and only if every hyperideal I of H is idempotent; that is, $f(I, I, \ldots, I) = I$.*

Proof. If H is commutative relative to f, then $f(a, H, \ldots, H) = f(H, a, \ldots, H) = \cdots = f(H, H, \ldots, a)$ and hence every j-hyperideal is also a k-hyperideal for all $j, k = 1, \ldots, m$. Hence, by regularity

$$f(I, I, \ldots, I) = I \cap I \cap I \cap \cdots \cap I = I \quad \forall \text{ hyperideal } I \text{ in } H. \tag{9}$$

Conversely, suppose that every hyperideal in H is idempotent. If $I_1, I_2, \ldots, I_m$ are hyperideals of H, then $\bigcap_{j=1}^{m} I_j$ is also an hyperideal and therefore

$$\bigcap_{j=1}^{m} I_j = f\left(\bigcap_{j=1}^{m} I_j, \bigcap_{j=1}^{m} I_j, \ldots, \bigcap_{j=1}^{m} I_j\right) \subseteq f\left(I_1, I_2, \ldots, I_m\right) \tag{10}$$

inasmuch as I_j contains the intersection for each j. Furthermore, since each I_j, $j = 1, \ldots, m$ is also a j-hyperideal, then $f(I_1, I_2, \ldots, I_m) \subseteq \bigcap_{j=1}^{m} I_j$. Hence, the conclusion follows.

By what we mentioned in the beginning of the section, note that in case the m-ary hyperoperation f is an associative m-ary hyperoperation in H one may conveniently abbreviate

$$\begin{gathered} f(a, a, \ldots, a) = f\left(a^m\right), \\ f\left(f\left(a^m\right), a, \ldots, a\right) = f\left(a^{2m-1}\right), \\ f\left(f\left(a^m\right), f\left(a^m\right), \ldots, a\right) = f\left(a^{3m-2}\right), \\ \vdots \\ f\left(f\left(a^m\right), f\left(a^m\right), \ldots, f\left(a^m\right)\right) = f\left(a^{m^2}\right) = f\left(a^{(m+1)m-m}\right). \end{gathered} \tag{11}$$

Thus, the admissible exponents of compositions of rank at most 2 are each of the form $km - k + 1$ for some integer. Proceeding inductively, suppose that $k_1 m - k_1 + 1, k_2 m - k_2 + 1, \ldots, k_m m - k_m + 1$ are previously known admissible exponents; then the exponent

$$\sum_{i=1}^{m} \left(k_i m - k_i + 1\right) = \left(\sum_{i=1}^{m} k_i - 1\right) m - \sum_{i=1}^{m} k_i \tag{12}$$

of $f(f(a^{k_1 m - k_1 + 1}), f(a^{k_2 m - k_2 + 1}), \ldots, f(a^{k_m m - k_m + 1}))$ is evidently also of the same form. Hence, every admissible exponent of an m-ary hyperoperation is of the form $km - k + 1$.

An element $a \in H$ such that

$$\begin{aligned} f\left(a, x_1, \ldots, x_{m-1}\right) &= f\left(x_1, a, \ldots, x_{m-1}\right) \\ &= f\left(x_1, \ldots, x_{m-1}, a\right) = \{a\}, \end{aligned} \tag{13}$$

for all $x_1, x_2, \ldots, x_{m-1} \in H$, is called *zero element*. The zero element is denoted by 0. A *nilpotent* element $a \in H$ is one which satisfies $f(a^{km-k+1}) = \{0\}$ for some integer k greater than 0. □

Theorem 3. *An algebraic hypersystem H which is commutative, associative, regular, and has a 0 with respect to an m-ary hyperoperation f possesses no nilpotent element other than 0.*

Proof. For all $0 \neq a \in H$, let $[a]$ denote the subhypersystem of H generated by a, which may be inductively defined as follows:

(1) $a \in [a]$

(2) $f(a^m) \subseteq [a]$

(3) whenever $f(a^{n_1}), \ldots, f(a^{n_m}) \subseteq [a]$, then also $f(a^{n_1 + \cdots + n_m}) \subseteq [a]$.

In order to prove the theorem it suffices to show that $a \notin [a]$. We proceed inductively as follows.

(1) $a \neq 0$ by assumption.

(2) Consider $f(a^m) \neq \{0\}$. For, if $f(a^m) = \{0\}$, then by virtue of the associativity, commutativity, and regularity of the given hyperoperation, there exist

$x_1, \ldots, x_{m-1} \in H$ such that $a \in f(a, x_1, a, \ldots, f(\ldots, x_{m-1}, a)) = f(f(a, a, \ldots, a), x_1, \ldots, x_{m-1}) = f(f(a^m), x_1, \ldots, x_{m-1}) = f(0, x_1, \ldots, x_{m-1}) = \{0\}$ contrary to (1).

(3) We now show that if $f(a^{n_1}), \ldots, f(a^{n_m})$ are all nonzero elements of $[a]$, then $f(f(a^{n_1}), f(a^{n_2}), \ldots, f(a^{n_m})) = f(a^{n_1+n_2+\cdots+n_m}) \neq \{0\}$. Suppose $f(a^{n_1+n_2+\cdots+n_m}) = \{0\}$. Then by the above remark, we have

$$n_i = k_i m - k_i + 1 \quad \text{for } i = 1, 2, \ldots, m. \tag{14}$$

Since f is commutative, it may be assumed without loss of generality that $n_1 = \max_i n_i$. Then

$$\begin{aligned} mn_i &= \sum_{i=1}^{m} n_i + \left(mn_1 - \sum_{i=1}^{m} n_i \right) = \sum_{i=1}^{m} n_i + \sum_{i=1}^{m} (n_i - n_i) \\ &= \sum_{i=1}^{m} n_i + \sum_{i=1}^{m} \left[(k_1 - k_i) m - (k_1 - k_i) \right] \\ &= \sum_{i=1}^{m} n_i + \left(\sum_{i=1}^{m} (k_1 - k_i) m \right. \\ &\qquad \left. - \sum_{i=1}^{m} (k_1 - k_i) - m + 2 \right) + (m - 2) \\ &= \sum_{i=1}^{m} n_i + \left\{ \left[\sum_{i=1}^{m} (k_1 - k_i) - 1 \right] m \right. \\ &\qquad \left. - \left[\sum_{i=1}^{m} (k_1 - k_i) - 2 \right] \right\} + (m + 2) \\ &= \sum_{i=1}^{m} n_i + p + (m - 2), \end{aligned} \tag{15}$$

where p is an admissible exponent. Hence, by associativity, commutativity, and regularity of the hyperoperation f, there exist $x_1, x_2, \ldots, x_{m-1} \in H$ such that

$$\begin{aligned} \{0\} &\neq f(a^{n_1}) \\ &= f(f(a^{n_1}), x_1, f(a^{n_1}), \ldots, f(\ldots, x_{m-1}, f(a^{n_1}))) \\ &= f(f(a^{mn_1}), x_1, \ldots, x_{m-1}) \\ &= f\left(f\left(a^{n_1+n_2+\cdots+n_m+p+(m-2)}\right), x_1, \ldots, x_{m-1}\right) \\ &= f(f(a^{n_1+n_2+\cdots+n_m}), \\ &\qquad f(f(a^p), a, a, \ldots, a, x_1), x_2, \ldots, x_{m-1}) \\ &= f(0, f(f(a^p), a, a, \ldots, a, x_1), x_2, \ldots, x_{m-1}) = \{0\}, \end{aligned} \tag{16}$$

a contradiction. Thus, every element of $[a]$ is nonzero and the conclusion follows. □

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