# Asymptotic Behavior for a Class of Nonclassical Parabolic Equations 

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This paper is devoted to the qualitative analysis of a class of nonclassical parabolic equations $u_{t}-\varepsilon \Delta u_{t}-\omega \Delta u+f(u)=g(x)$ with critical nonlinearity, where $\varepsilon \in[0,1]$ and $\omega>0$ are two parameters. Firstly, we establish some uniform decay estimates for the solutions of the problem for $g(x) \in H^{-1}(\Omega)$, which are independent of the parameter $\varepsilon$. Secondly, some uniformly (with respect to $\varepsilon \in[0,1]$ ) asymptotic regularity about the solutions has been established for $g(x) \in L^{2}(\Omega)$, which shows that the solutions are exponentially approaching a more regular, fixed subset uniformly (with respect to $\varepsilon \in[0,1]$ ). Finally, as an application of this regularity result, a family $\left\{\mathscr{C}_{\varepsilon}\right\}_{\varepsilon \in[0,1]}$ of finite dimensional exponential attractors has been constructed. Moreover, to characterize the relation with the reaction diffusion equation $(\varepsilon=0)$, the upper semicontinuity, at $\varepsilon=0$, of the global attractors has been proved.

## 1. Introduction

We study the long-time behavior of the following class of nonclassical parabolic equations:

$$
\begin{gather*}
u_{t}-\varepsilon \Delta u_{t}-\omega \Delta u+f(u)=g(x), \quad \text { in } \Omega \times \mathbb{R}^{+}, \\
u(x, 0)=u_{0}(x), \\
\left.u\right|_{\partial \Omega}=0
\end{gather*}
$$

where $\Omega \subset \mathbb{R}^{N}(N \geq 3)$ is a bounded domain with smooth boundary $\partial \Omega, \varepsilon \in[0,1]$ and $\omega>0$ are two parameters, the external force $g$ is time independent, and the nonlinearity $f$ satisfies some specified conditions later.

When $\varepsilon=0$ for the fixed constant $\omega(>0)$, equation ( $E_{0}$ ) is a usual reaction-diffusion equation, and its asymptotic behavior has been studied extensively in terms of attractors by many authors; see [1-5].

For each fixed $\varepsilon=\varepsilon_{0}>0$, equation $\left(E_{\varepsilon_{0}}\right)$ is a nonclassical reaction-diffusion equation, which arises as models to describe physical phenomena such as non-Newtonian flow, soil mechanics, heat conduction; see [6-8] and references therein. Aifantis in [6] provided a quite, general approach for obtaining these equations. The asymptotic behavior of the
solutions for this equation has been studied by many authors; see [9-16].

For the fixed constant $\omega(>0)$, any $\varepsilon \in[0,1]$, and the longtime behavior of the solutions of $\left(E_{\varepsilon}\right)$ has been considered by some researchers; see [10, 13]. In [10] the author proved the existence of a class of attractors in $H^{2} \cap H_{0}^{1}$ with initial data $u_{0} \in H^{2} \cap H_{0}^{1}$ and the upper semicontinuity of attractors in $H_{0}^{1}$ under subcritical assumptions and $g(x)=0$ in the case of $N \leq 3$. In [13] similar results have been shown when $N \geq 3$ and $g(x) \in H_{0}^{1}(\Omega)$.

In this paper, inspired by the ideas in $[17,18]$ and motivated by the dynamical results in [19-22], we study the uniform (with respect to the parameter $\varepsilon \in[0,1]$ ) qualitative analysis (a priori estimates) for the solutions of the nonclassical parabolic equations $\left(E_{\varepsilon}\right)$ and then give some information about the relation between the solutions of $\left(E_{0}\right)$ and those of $\left(E_{\varepsilon}\right)$. Our main difficulty comes from the critical nonlinearity and the uniformness with respect to $\varepsilon \in[0,1]$.

This paper is organized as follows. In Section 2, we introduce basic notations and state our main results. In Section 3, we recall some abstract results that we will use later. In Section 4, we present several dissipative estimates about the solution of $\left(E_{\varepsilon}\right)$ when $g(x) \in H^{-1}(\Omega)$, which hold uniformly with respect to $\varepsilon \in[0,1]$. The main results
are proved for $g(x) \in L^{2}(\Omega)$ in Section 5. Moreover, in Section 6, as an application, we construct a finite dimensional exponential attractor and prove the upper semicontinuity of the global attractor obtained in Section 5.

## 2. Main Results

Before presenting our main results, we first state the basic mathematical assumptions for considering the long-time behaviors of the nonclassical parabolic equations and then introduce some notations that we will use throughout this paper.
(i) $f \in \mathscr{C}^{1}(\mathbb{R})$ with $f(0)=0$ and satisfies the following conditions:

$$
\begin{gather*}
\left|f^{\prime}(s)\right| \leq C_{0}\left(1+|s|^{(N+2) /(N-2)-1}\right), \quad \forall s \in \mathbb{R},  \tag{1}\\
\liminf _{|s| \rightarrow \infty} \frac{f(s)}{s}>-\lambda_{1}, \tag{2}
\end{gather*}
$$

where $C_{0}$ is a positive constant and $\lambda_{1}$ is the first eigenvalue of $-\Delta$ on $H_{0}^{1}$. The number $(N+2) /(N-$ $2)-1$ is called the critical exponent. $f$ is not compact in this case, and this is one of the essential difficulties in studying the asymptotic regularity.
(ii) Assumption on the parameters $\varepsilon \in[0,1]$ and $\omega>0$. From the work in $[18,19]$, we know that a very large damping has the effect of freezing the system, if the damping acts only on the velocity $u_{t}$, and this prevents the squeezing of the component $u$. Therefore, the most dissipative situation occurs in between, that is, for a certain damping $\varepsilon_{*}$, which depends on the other coefficient of the equation. Therefore, in our frame, we choose $\omega>1$ such that $1 / \omega<\varepsilon$ as $\varepsilon \in[0,1]$ in order to obtain the uniformly (with respect to $\varepsilon \in[0,1]$ ) asymptotic regularity about the solutions of $\left(E_{\varepsilon}\right)$.
(iii) $A=-\Delta$ with domain $D(A)=H^{2} \cap H_{0}^{1}$, and consider the family of Hilbert space $D\left(A^{s / 2}\right), s \in \mathbb{R}$ with the standard inner products and norms, respectively,
$\langle\cdot \cdot \cdot\rangle_{D\left(A^{s / 2}\right)}=\left\langle A^{s / 2} \cdot, A^{s / 2} \cdot\right\rangle, \quad\|\cdot\|_{D\left(A^{s / 2}\right)}=\left\|A^{s / 2} \cdot\right\|$.
In particular, $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$ mean the $L^{2}(\Omega)$ inner product and norm, respectively.
(iv) $\mathscr{H}^{s}=D\left(A^{1+s / 2}\right), s \in[0,1]$ with the usual norm

$$
\begin{equation*}
\|u\|_{\mathscr{C}^{s}}^{2}=\left\|A^{(1+s) / 2} u\right\|^{2} \tag{4}
\end{equation*}
$$

In particular, we denote $\mathscr{H}=\mathscr{H}^{0}=H_{0}^{1}(\Omega)$ and $\|\cdot\|_{\mathscr{H}}=\|\cdot\|_{H_{0}^{1}}$.
(v) For each $u \in \mathscr{H}$, we define $\|\cdot\|_{\mathscr{H}_{\varepsilon}^{s}}(\varepsilon, s \in[0,1])$ as

$$
\begin{equation*}
\|u\|_{\mathscr{H}_{\varepsilon}^{s}}^{2}=\left\|A^{s / 2} u\right\|^{2}+\varepsilon\left\|A^{(s+1) / 2} u\right\|^{2} \tag{5}
\end{equation*}
$$

and define $\mathscr{H}_{\varepsilon}^{s}$ as

$$
\begin{equation*}
\mathscr{H}_{\varepsilon}^{s}=\mathrm{cl}_{\|\cdot\|_{\mathscr{E}_{\varepsilon}}}\left(\mathscr{H}^{1}\right) . \tag{6}
\end{equation*}
$$

Then $\left(\mathscr{H}_{\varepsilon}^{s},\|\cdot\|_{\mathscr{H}_{\varepsilon}}\right)$ is a Banach Space for every $\varepsilon, s \in$ $[0,1]$.

The global well-posedness of solutions and its asymptotic behavior for $\left(E_{\varepsilon_{0}}\right)$ have been studied extensively under assumptions (1)-(2) by many authors in [9-14] and references therein in fact note that $\left(\mathscr{H}_{\varepsilon_{0}}^{0},\|\cdot\|_{\mathscr{H}_{\varepsilon_{0}}}\right) \cong\left(\mathscr{H},\|\cdot\|_{\mathscr{H}}\right)$ for each fixed $\varepsilon_{0}$.

The main results of this paper are the following asymptotic regularity.

Theorem 1. Under assumptions (1), (2), and $\omega>1$, there exist a positive constant $v$, a bounded (in $\mathscr{H}^{1}$ ) subset $\mathbb{B} \subset \mathscr{H}^{1}$, and a continuous increasing function $Q(\cdot):[0, \infty) \rightarrow[0, \infty)$ such that, for any bounded (in $\mathscr{H}$ ) subset $B_{0} \subset \mathscr{H}$,

$$
\begin{array}{r}
\forall \varepsilon \in[0,1], \quad \operatorname{dist}_{\mathscr{H}}\left(S_{\varepsilon}(t) B_{0}, \mathbb{B}\right) \leq Q\left(\|B\|_{\mathscr{H}}\right) e^{-\nu t} \\
\forall t \geq 0, \tag{7}
\end{array}
$$

where $\mathbb{B}, v$, and $Q(\cdot)$ are all independent of $\varepsilon$, and $\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}$ is the semigroup generated by $\left(E_{\varepsilon}\right)$ in $\mathscr{H}_{\varepsilon}^{0}$.

This result says that asymptotically, for each $\left(E_{\varepsilon}\right)$, the solutions are exponentially approaching a more regular fixed subset $\mathbb{B}$ uniformly (with respect to $\in[0,1]$ ) for $\omega>1$. Moreover, it implies the following results.
(1) For each $\varepsilon \in[0,1],\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}$ has a global attractor $\mathscr{A}_{\varepsilon}$ in $\mathscr{H}$, and

$$
\begin{equation*}
\bigcup_{\varepsilon \in[0,1]} \mathscr{A}_{\varepsilon} \subset \mathrm{cl}_{\mathscr{H}^{1}}(\mathbb{B}) . \tag{8}
\end{equation*}
$$

(2) Based on Theorem 1, applying the abstract result devised in $[23,24]$, for each $\varepsilon \in[0,1]$ we can prove the existence of a finite dimensional exponential attractor $\mathscr{E}_{\varepsilon}$ in $\mathscr{H}$. Moreover, our attraction is uniform (with respect to $\varepsilon \in[0,1]$ ) under the $\mathscr{H}$-norm (not only with the $\mathscr{H}_{\varepsilon}^{0}$-norm); see Lemma 19.
(3) Since the global attractor $\mathscr{A}_{\varepsilon} \subset \mathscr{E}_{\varepsilon}$, it also implies that the fractal dimension of the global attractor $\mathscr{A}_{\varepsilon}$ is finite. Moreover, in line with Theorem 1, we prove the upper semicontinuity of $\mathscr{A}_{\varepsilon}$ at $\varepsilon=0$; see Lemma 20 .
For the proof of Theorem 1, the main difficulty comes from the critical nonlinearity and the uniformness with respect to $\varepsilon \in[0,1]$.

Hereafter, we will also use the following notation: denote by $\mathcal{J}$ the space of continuous increasing functions $J$ : $[0, \infty) \rightarrow[0, \infty)$ and by $\mathscr{D}$ the space of continuous decreasing functions $\beta:[0, \infty) \rightarrow[0, \infty)$ such that $\beta(\infty)<$ 1. Moreover, $C, C_{i}$, and $c_{i}$ are the generic constants, and $Q(\cdot)$, $Q_{i}(\cdot) \in \mathcal{F}$ are generic functions, which are all independent of $\varepsilon$; otherwise we will point out clearly.

## 3. Preliminaries

In this section, we recall some results used in the main part of the paper.

The first result comes from [17], which will be used to prove the asymptotic regularity for the case $g(x) \in L^{2}(\Omega)$.

Lemma 2 (see [17]). Let $X$ and $V$ be two Banach spaces and $\{T(t)\}_{t \geq 0}$ a $C^{0}$-semigroup on $X$ with a bounded absorbing set $\bar{B} \subset X$. For every $x \in \bar{B}$, assume that there exist two solution operators $V_{x}(t)$ on $X$ and $U_{x}(t)$ on $V$ satisfying the following properties.
(i) For any two vectors $y \in X$ and $z \in V$ satisfying $y+z=$ $x$,

$$
\begin{equation*}
T(t) x=V_{x}(T) y+U_{x}(t) z \quad \forall t \geq 0 \tag{9}
\end{equation*}
$$

(ii) There exists $\alpha \in \mathscr{D}$ such that

$$
\begin{equation*}
\sup _{x \in \bar{B}}\left\|V_{x}(t) y\right\|_{X} \leq \alpha(t)\|y\|_{X}, \quad \forall y \in \bar{B} \tag{10}
\end{equation*}
$$

(iii) There are $\beta \in \mathscr{D}$ and $J \in \mathscr{F}$ such that

$$
\begin{equation*}
\sup _{x \in \bar{B}}\left\|U_{x}(t) z\right\|_{V} \leq \beta(t)\|z\|_{V}+J(t), \quad \forall z \in V . \tag{11}
\end{equation*}
$$

Then, there exist positive constants $\rho, K$, and $\vartheta$ such that

$$
\begin{equation*}
\operatorname{dist}_{X}\left(T(t) \bar{B}, B_{V}(\rho)\right) \leq K e^{-9 t}, \quad \forall t \geq 0 \tag{12}
\end{equation*}
$$

where $B_{V}(\rho)=\left\{z \in V:\|z\|_{V} \leq \rho\right\}$.
Next, we recall a criterion for the upper semicontinuity of attractors.

Lemma 3 (see [25, 26]). Let $\left\{T_{\lambda}(t)\right\}_{t \geq 0}$ be a family of semigroups defined on the Banach space $X$, and for each $\lambda \in \Lambda$, let $\left\{T_{\lambda}(t)\right\}_{t \geq 0}$ have a global attractor $\mathscr{A}_{\lambda}$. Assume further that $\lambda_{0}$ is a nonisolated point of $\Lambda$ and that there exist $s>0, t_{0}>0$, and a compact set $K \subset X$ such that

$$
\begin{gather*}
\bigcup_{\lambda \in \mathcal{N}_{\lambda}\left(\lambda_{0}, s\right)} \mathscr{A}_{\lambda} \subset K,  \tag{13}\\
\text { if } \lambda_{n} \longrightarrow \lambda_{0} \text { and } x_{n} \longrightarrow x_{0} \quad\left(x_{n} \in \mathscr{A} \text { as } n \neq 0\right), \\
\text { then } T_{\lambda_{n}}\left(t_{0}\right) x_{n} \longrightarrow T_{\lambda_{0}}\left(t_{0}\right) x_{0} . \tag{14}
\end{gather*}
$$

Then the global attractors $\mathscr{A}_{\lambda}$ are upper semicontinuous on $\Lambda$ at $\lambda=\lambda_{0}$; that is,

$$
\begin{equation*}
\lim _{\Lambda \ni \lambda \rightarrow \lambda_{0}} \operatorname{dist}_{X}\left(\mathscr{A}_{\lambda}, \mathscr{A}_{\lambda_{0}}\right)=0 . \tag{15}
\end{equation*}
$$

Lemma 4 (see [27]). Let $\Phi$ be an absolutely continuous positive function on $\mathbb{R}^{+}$, which satisfies for some $\epsilon>0$ the differential inequality

$$
\begin{equation*}
\frac{d}{d t} \Phi(t)+2 \epsilon \Phi(t) \leq h_{1}(t) \Phi(t)+h_{2}(t) \tag{16}
\end{equation*}
$$

for almost every $t \in \mathbb{R}^{+}$, where $h_{1}$ and $h_{2}$ are functions on $\mathbb{R}^{+}$ such that

$$
\begin{equation*}
\int_{\tau}^{t}\left|h_{1}(y)\right| \leq m_{1}\left(1+(t-\tau)^{\mu}\right), \quad \forall t \geq \tau \geq 0 \tag{17}
\end{equation*}
$$

for some $m_{1} \geq 0$ and $\mu \in[0,1]$, and

$$
\begin{equation*}
\sup _{t \geq 0} \int_{t}^{t+1}\left|h_{2}(y)\right| d y \leq m_{2} \tag{18}
\end{equation*}
$$

for some $m_{2} \geq 0$. Then

$$
\begin{equation*}
\Phi(t) \leq \rho_{1} \Phi(0) e^{-\epsilon t}+\rho_{2}, \quad \forall t \in \mathbb{R}^{+}, \tag{19}
\end{equation*}
$$

for some $\rho_{1}=\rho_{1}\left(m_{1}, \mu\right) \geq 1$ and

$$
\begin{equation*}
\rho_{2}=\frac{\rho_{1} m_{2} e^{\epsilon}}{1-e^{-\epsilon}} . \tag{20}
\end{equation*}
$$

For the proof, we refer the reader to [27, Lemma 2.2].
A standard Gronwall-type lemma will also be needed.
Lemma 5. Let $\Psi$ be an absolutely continuous positive function on $\mathbb{R}^{+}$, which satisfies for some $\Psi>0$ the differential inequality

$$
\begin{equation*}
\frac{d}{d t} \Psi(t)+\epsilon \Phi(t) \leq \kappa e^{-t t} \Psi(t)+J(t) \tag{21}
\end{equation*}
$$

for some $\epsilon, \kappa, \iota>0$ and some $J \in \mathscr{G}$. Then,

$$
\begin{equation*}
\Psi(t) \leq e^{\kappa / \iota} e^{-\epsilon t} \Psi(0)+\epsilon^{-1} e^{\kappa / \iota} J(t) . \tag{22}
\end{equation*}
$$

## 4. Uniformly Decaying Estimates in $\mathscr{H}$

In this section, we always assume that (1), (2), and $\omega>1$ such that $1 / \omega<\varepsilon$ as $\varepsilon \in[0,1]$ hold and $g(x)$ only belongs to $H^{-1}(\Omega)$, so all results in this section certainly hold for the case $g(x) \in L^{2}(\Omega)$.

The main purpose of this section is to deduce some dissipative estimates about the semigroups $\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}(\varepsilon \in$ $[0,1])$ associated with $\left(E_{\varepsilon}\right)$ in $\mathscr{H}$. Here, using the method in $[19,20,22]$ for a strongly damped wave equation and a semilinear second order evolution equation, we will show that the radius of the absorbing set of $\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}$ associated with $\left(E_{\varepsilon}\right)$ in $\mathscr{H}$ can be chosen to be independent of $\varepsilon \in[0,1]$.

Lemma 6. There exists a positive constant $M$, which depends only on $\omega,\|g\|_{H^{-1}}$, and coefficients of (1)-(2), satisfying that for any $\varepsilon \in[0,1]$ and any bounded (in $\mathscr{H}_{\varepsilon}^{0}$ ) subset $B \in \mathscr{H}_{\varepsilon}^{0}$, there is a $t_{B}=t\left(\|B\|_{\mathscr{H} \ell}\right)>0$ (which depends only on the bound of $\left.\|B\|_{\mathscr{H}_{\varepsilon}^{0}}\right)$ such that

$$
\begin{equation*}
\left\|S_{\varepsilon}(t) u\right\|_{\mathscr{H}} \leq M, \quad \forall t \geq t_{B}, \forall u \in B \tag{23}
\end{equation*}
$$

where both $t_{B}$ and $M$ are independent of $\varepsilon \in[0,1]$.
Proof. Throughout the proof, the generic constants $C, C_{j}(j=$ $1,2, \ldots$ ) are independent of $\varepsilon$. For clarity, we separate the proof into three claims.

Claim 1. There exists an $M_{1}$ which depends on $\omega,|\Omega|,\|g\|_{H^{-1}}$ (but independent of $B$ and $\varepsilon$ ) such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) B\right\|_{\mathscr{H} e_{\varepsilon}^{0}}^{2} \leq M_{1}, \quad \text { as } t \geq T_{1 B} \tag{24}
\end{equation*}
$$

where $T_{1 B}=T_{1}\left(\|B\|_{\mathscr{H}_{\varepsilon}^{0}}\right)$ depends on $\|B\|_{\mathscr{H}_{\varepsilon}^{0}}$ but not on $\varepsilon$.

Multiplying $\left(E_{\varepsilon}\right)$ by $u$, we have
$\frac{d}{d t}\left(\|u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+2 \omega\|\nabla u\|^{2}+2\langle f(u), u\rangle=2\langle g, u\rangle$.

By virtue of (2), we conclude that there exists $0<\lambda<\lambda_{1}$, $C_{1}>0$ such that

$$
\begin{equation*}
2\langle f(u), u\rangle=2 \int_{\Omega} f(u) u d x \geq-2 \lambda\|u\|^{2}-2 C_{1} . \tag{26}
\end{equation*}
$$

At the same time, by the Hölder inequality, we get

$$
\begin{equation*}
2\langle g, u\rangle \leq \frac{1}{\omega}\|g\|_{H^{-1}}^{2}+\omega\|\nabla u\|^{2} . \tag{27}
\end{equation*}
$$

Substituting (26) and (27) into (25) and noticing $1 / \omega<\varepsilon$ as $\omega>1$, we obtain

$$
\begin{align*}
\frac{d}{d t} & \left(\|u\|^{2}+\varepsilon\|\nabla u\|^{2}\right) \\
\quad & +\varepsilon_{0}\left(\|u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)  \tag{28}\\
\leq & \zeta\left(\|u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+C\left(1+\|g\|_{H^{-1}}^{2}\right)
\end{align*}
$$

where $\zeta=\max \left\{2 \lambda-\varepsilon_{0}, \varepsilon_{0}\right\}$ and $\varepsilon_{0}$ is a small positive constant such that $\lambda>\varepsilon_{0} / 2$.

And then applying Lemma 4 to above inequality, it follows that

$$
\begin{align*}
& \|u\|^{2}+\varepsilon\|\nabla u\|^{2} \\
& \quad \leq \rho_{1}\left[\|u(0)\|^{2}+\varepsilon\|\nabla u(0)\|^{2}\right] e^{-\left(\varepsilon_{0} / 2\right) t}+\rho_{2} \tag{29}
\end{align*}
$$

where $\rho_{1}=\rho_{1}(\zeta), \rho_{2}=\rho_{1} C_{\|g\|_{H^{-1}}^{2}} e^{\varepsilon_{0} / 2} /\left(1-e^{-\left(\varepsilon_{0} / 2\right)}\right)$.
Then, Claim 1 follows from (29) immediately.
Claim 2. There exists an $M_{2}$ which depends on $\omega,|\Omega|$, and $\|g\|_{H^{-1}}$ (but is independent of $B$ and $\varepsilon$ ) such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad \int_{T_{1 B}}^{\infty}\|\nabla u(s)\| d s \leq M_{2} \tag{30}
\end{equation*}
$$

where $T_{1 B}$ is given in Claim 1.
Noting (25) and taking $\lambda=(1 / 2) \lambda_{1}$ in (26), it yields

$$
\begin{equation*}
\frac{d}{d t}\left(\|u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+(\omega-1)\|\nabla u\|^{2} \leq C\left(1+\|g\|_{H^{-1}}^{2}\right) . \tag{31}
\end{equation*}
$$

Then, for any $t \geq T_{1 B}$, integrating (31) over [ $\left.T_{1 B}, t\right]$ and using Claim 1, we can complete this claim immediately.

Claim 3. Multiplying $\left(E_{\varepsilon}\right)$ by $u_{t}$, we find

$$
\begin{gather*}
\frac{d}{d t}\left(\|\nabla u\|^{2}+\frac{2}{\omega} \int_{\Omega} F(u) d x-\frac{2}{\omega}\langle g, u\rangle\right)  \tag{32}\\
+\frac{2}{\omega}\left\|u_{t}\right\|^{2}+\frac{2 \varepsilon}{\omega}\left\|\nabla u_{t}\right\|^{2}=0
\end{gather*}
$$

furthermore,

$$
\begin{equation*}
\frac{d}{d t}\left(\|\nabla u\|^{2}+\frac{2}{\omega} \int_{\Omega} F(u) d x-\frac{2}{\omega}\langle g, u\rangle\right) \leq 0 \tag{33}
\end{equation*}
$$

Then, from assumptions (1)-(2), Claim 1, and using Hölder inequality, there holds

$$
\begin{align*}
& \|\nabla u\|^{2}+\frac{2}{\omega} \int_{\Omega} F(u) d x-\frac{2}{\omega}\langle g, u\rangle \\
& \quad \geq \frac{1}{2}\|\nabla u\|^{2}-\frac{C_{1}}{\omega}\left(1+\|\nabla u\|^{2 N /(N-2)}+\|g\|_{H^{-1}}^{2}\right)  \tag{34}\\
& \quad \geq \frac{1}{2}\|\nabla u\|^{2}-\frac{C_{1}}{\omega^{2}}\left(1+M_{1}^{N /(N-2)}+\|g\|_{H^{-1}}^{2}\right) \\
& \|\nabla u\|^{2}+\frac{2}{\omega} \int_{\Omega} F(u) d x-\frac{2}{\omega}\langle g, u\rangle  \tag{35}\\
& \quad \leq 2\|\nabla u\|^{2}+\frac{C_{1}}{\omega^{2}}\left(1+M_{1}^{N /(N-2)}+\|g\|_{H^{-1}}^{2}\right)
\end{align*}
$$

On the other hand, from Claim 2 we know that for each $u_{0} \in B$ there is a time $t_{0} \in\left[T_{1 B}, T_{1 B}+1\right]$ such that

$$
\begin{equation*}
\left\|\nabla u\left(t_{0}\right)\right\| \leq M_{2} \tag{36}
\end{equation*}
$$

where $t_{0}$ depends on $u_{0}$.
When $t \geq T_{1 B}+1$, for each $u_{0} \in B$, integrating (33) over [ $\left.t_{0}, t\right]$ and applying (34)-(36), we obtain that

$$
\begin{equation*}
\|\nabla u\| \leq 4 M_{2}+\frac{2\left(C_{1}+C_{2}\right)}{\omega^{2}}\left(1+M^{N /(N-2)}+\|g\|_{H^{-1}}^{2}\right) . \tag{37}
\end{equation*}
$$

Now, taking $M=4 M_{2}+\left(2\left(C_{1}+C_{2}\right) / \omega^{2}\right)\left(1+M^{N /(N-2)}+\right.$ $\|g\|_{H^{-1}}^{2}$ ) (is independent of $\varepsilon \in[0,1]$ ), we can complete our proof.

Remark 7. Observing that above process of proof, we can also deduce that, for any $\varepsilon \in[0,1]$ and any $B \subset \mathscr{H}_{\varepsilon}^{0}$,

$$
\begin{equation*}
\left\|S_{\varepsilon}(t) B\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2} \leq Q\left(\omega,\|B\|_{\mathscr{H _ { \varepsilon } ^ { 0 }}}\right), \quad \forall t \geq 0, \tag{38}
\end{equation*}
$$

where $Q(\cdot) \in \mathscr{F}$ is independent of $B$ and $\varepsilon$.
Moreover, if $B$ is bounded in $\mathscr{H}$, then we can obtain

$$
\begin{equation*}
\varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) B\right\|_{\mathscr{H}}^{2} \leq C_{\omega,\|B\|_{\mathscr{H}}}, \quad \forall t \geq 0 \tag{39}
\end{equation*}
$$

for some constant $C_{\omega,\|B\|_{\mathscr{C}}}$ which depends on $\omega,\|B\|_{\mathscr{C}}$. Indeed, from the fact that there is a constant $c_{1}$ such that $\|\cdot\|_{\mathscr{H}_{\varepsilon}^{0}} \leq c_{1}\|\cdot\|_{\mathscr{H}}$ for any $\varepsilon \in[0,1]$, (39) can be obtained just by repeating the proof of Lemma 6 and taking $t_{0}=0$ in (35) since $B$ is bounded in $\mathscr{H}$.

On the other hand, from the proof of Claim 3 as follows, we can get further estimates about $u_{t}$

$$
\begin{array}{r}
\forall \varepsilon \in[0,1], \quad \int_{T_{1 B}+1}^{\infty}\left(\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}\right) \leq M  \tag{40}\\
\forall t \geq 0, u_{0} \in B
\end{array}
$$

Lemma 8. There exists a positive constant $M_{3}$ such that for any $\varepsilon \in[0,1]$ and any bounded (in $\mathscr{H}_{\varepsilon}^{0}$ ) subset $B \in \mathscr{H}_{\varepsilon}^{0}$,

$$
\begin{equation*}
\left\|u_{t}(t)\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}+\omega \int_{T_{1 B}+2}^{t}\left\|\nabla u_{t}(s)\right\|^{2} d s \leq M_{3} \tag{41}
\end{equation*}
$$

$$
\forall t \geq T_{1 B}+2
$$

where $u(t)=S_{\varepsilon}(t) u_{0}, u_{0} \in B, T_{1 B}$ is the time given in Claim 1, and $M_{3}$ only depends on $\omega$ but is independent of $B$ and $\varepsilon$.

Proof. By differentiation of $\left(E_{\varepsilon}\right)$, we can obtain the following equation:

$$
\begin{equation*}
u_{t t}-\varepsilon \Delta u_{t t}-\omega \Delta u_{t}+f^{\prime}(u) u_{t}=0 \tag{42}
\end{equation*}
$$

Multiplying (42) by $u_{t}$, we have

$$
\begin{equation*}
\frac{d}{d t}\left(\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}\right)+2 \omega\left\|\nabla u_{t}\right\|^{2}=-2\left\langle f^{\prime}(u) u_{t}, u_{t}\right\rangle . \tag{43}
\end{equation*}
$$

When $t \geq T_{1 B}+1$, using Lemma 6, there holds

$$
\begin{align*}
\left|-2\left\langle f^{\prime}(u) u_{t}, u_{t}\right\rangle\right| & \leq C\left(1+\|\nabla u\|^{4 /(N-2)}\right)\left\|\nabla u_{t}\right\|\left\|\nabla u_{t}\right\| \\
& \leq \frac{C}{4 \omega}\left(1+\|\nabla u\|^{8 /(N-2)}\right)\left\|\nabla u_{t}\right\|^{2}+\omega\left\|\nabla u_{t}\right\|^{2} \\
& \leq C_{M, \omega}\left(\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}\right)+\omega\left\|\nabla u_{t}\right\|^{2} . \tag{44}
\end{align*}
$$

So, we obtain

$$
\begin{align*}
& \frac{d}{d t}\left(\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}\right)+\omega\left\|\nabla u_{t}\right\|^{2}  \tag{45}\\
& \leq C_{M, \omega}\left(\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2}\right)
\end{align*}
$$

Therefore, as $t \geq T_{1 B}+2$, for $u_{0} \in B$, integrating (45) over $\left[T_{1 B}+2, t\right]$ and substituting (40), we can complete our proof at once.

For later applications, we present some Hölder continuity of $\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}$ in $\mathscr{H}_{\varepsilon}^{0}$.

Lemma 9. For any bounded $\left(\mathscr{H}_{\varepsilon}^{0}\right)$ subset $B \subset \mathscr{H}_{\varepsilon}^{0}$, there exists a constant $C_{\omega,\|B\|_{\mathscr{F}_{\varepsilon}^{0}}}$ which depends only on $\omega$ and $\|B\|_{\mathscr{H}_{\varepsilon}^{0}}$ such that

$$
\begin{array}{r}
\left\|S_{\varepsilon}(t) u_{1}-S_{\varepsilon}(t) u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \leq e^{C_{\omega,\|B\|} \|_{\varepsilon}} t\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}, \\
\forall t \geq 0, u_{i} \in B ;  \tag{53}\\
\left\|S_{\varepsilon}(t) u_{1}-S_{\varepsilon}(t) u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \leq e^{C_{\omega,\|B\|} \mathscr{P e}_{\varepsilon} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{\frac{1}{2}}, \\
\forall t \geq T_{1 B}+2, u_{i} \in B .
\end{array}
$$ Remark 7, noting now that $B_{0}$ is bounded in $\mathscr{H}$, we have

$$
\forall \varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) B_{0}\right\|_{\mathscr{H}}^{2} \leq C_{M}, \quad \forall t \geq 0
$$

## 5. Proof of the Main Results

Throughout this section, we always assume that (1), (2), and $\omega>1$ hold for $g(x) \in L^{2}(\Omega)$.
5.1. Decomposition of the Equation. For the nonlinear function $f$ satisfying (1)-(2), from $[12,17,19,22]$ for our situation we know that $f$ allows the following decomposition $f=$ $f_{0}+f_{1}$, where $f_{0}, f_{1} \in \mathscr{C}^{1}(\mathbb{R})$ and satisfy

$$
\begin{gather*}
\left|f_{0}(u)\right| \leq C|u|^{(N+2) /(N-2)}, \quad \forall u \in \mathbb{R}  \tag{54}\\
f_{0}(u) u \geq 0, \quad \forall u \in \mathbb{R} \tag{55}
\end{gather*}
$$

$\left|f_{1}(u)\right| \leq C\left(1+|u|^{\gamma}\right) \quad$ with some $\gamma<\frac{N+2}{N-2}, \forall u \in \mathbb{R}$,

$$
\begin{equation*}
\liminf _{|u| \rightarrow \infty} \frac{f_{1}(u)}{u}>-\lambda_{1} . \tag{56}
\end{equation*}
$$

Now, decomposing the solution $S_{\varepsilon}(t) u_{0}=u$ into the sum

$$
\begin{equation*}
S_{\varepsilon}(t) u_{0}=D_{\varepsilon}(t) u_{0}+K_{\varepsilon}(t) u_{0} \tag{58}
\end{equation*}
$$

for any $t \geq 0$ and any $u_{0} \in \mathscr{H}$, where $D_{\varepsilon}(t) u_{0}=v(t)$ and $K_{\varepsilon}(t) u_{0}=w(t)$ are the solutions of the following equations:

$$
\begin{gather*}
v_{t}-\varepsilon \Delta v_{t}-\omega \Delta v+f_{0}(v)=0 \quad \text { in } \Omega \times \mathbb{R}^{+}, \\
v(x, 0)=u_{0},  \tag{59}\\
\left.v\right|_{\partial \Omega}=0, \\
w_{t}-\varepsilon \Delta w_{t}-\omega \Delta w+f(u)-f_{0}(v)=g(x) \quad \text { in } \Omega \times \mathbb{R}^{+}, \\
w(x, 0)=0, \\
\left.w\right|_{\partial \Omega}=0 \tag{60}
\end{gather*}
$$

Applying the general results in $[9,12,14]$, we know that both (59) and (60) are global well-posed in $\mathscr{H}$, and $\left\{D_{\varepsilon}(t)\right\}_{t \geq 0}$ also forms a semigroup.

Moreover, as in Section 4, we can deduce a similar estimate for $\left\{D_{\varepsilon}(t)\right\}_{t \geq 0}$ in $\mathscr{H}$, and so $\left\{K_{\varepsilon}(t)\right\}_{t \geq 0}$. There exist constants $C_{M}$ ( $M$ is given in Lemma 6) and $\Lambda_{1}$ such that for any $\varepsilon \in[0,1]$ and any $u_{0} \in B_{0}$,

$$
\begin{align*}
& \left\|D_{\varepsilon}(t) B_{0}\right\|_{\mathscr{H}}^{2}+\left\|v_{t}(t)\right\|^{2}+\varepsilon\left\|\nabla v_{t}\right\|^{2} \\
& \quad+\omega \int_{T_{1 B}+2}^{t}\left\|\nabla v_{t}(s)\right\|^{2} d s \leq \bar{M}, \quad \forall t \geq \Lambda_{1}, \tag{61}
\end{align*}
$$

$\forall \varepsilon \in[0,1], \quad\left\|D_{\varepsilon}(t) B_{0}\right\|_{\mathscr{C}}^{2}+\left\|K_{\varepsilon}(t) B_{0}\right\|_{\mathscr{C}}^{2} \leq C_{M}$,

$$
\begin{equation*}
\forall t \geq 0 \tag{62}
\end{equation*}
$$

5.2. The First A Priori Estimate. We begin with the decay estimates for the solution of (59).

Lemma 10. There exists a constant $k>0$ and $Q(\cdot) \in \mathscr{J}$ such that

$$
\begin{array}{r}
\left\|D_{\varepsilon}(t) B_{0}\right\|_{\mathscr{H}}^{2} \leq C_{\bar{M}, \omega} Q\left(\left\|B_{0}\right\|_{\mathscr{H}}\right) e^{-k t},  \tag{63}\\
\forall t \geq 0 \text { and any } \varepsilon \in[0,1],
\end{array}
$$

where both $k$ and $Q(\cdot)$ are independent of $\varepsilon \in[0,1]$.

Proof. Multiplying (59) by $v$, we have

$$
\begin{equation*}
\frac{d}{d t}\left(\|v\|^{2}+\varepsilon\|\nabla v\|^{2}\right)+2 \omega\|\nabla v\|^{2}+2\left\langle f_{0}(v), v\right\rangle=0 \tag{64}
\end{equation*}
$$

By means of (55), it follows that $(d / d t)\left(\|v\|^{2}+\varepsilon\|\nabla v\|^{2}\right) \leq 0$. Therefore, there exists $k_{1}>0$ such that

$$
\begin{align*}
\|v(t)\|^{2}+\varepsilon\|\nabla v(t)\|^{2} & \leq\|v(0)\|^{2}+\varepsilon\|\nabla v(0)\|^{2} \\
& \leq Q\left(\left\|B_{0}\right\|_{\mathscr{H}}\right) e^{-k_{1} t}, \tag{65}
\end{align*}
$$

for all $t \geq 0$ and any $\varepsilon \in[0,1]$.
As a result, we multiply (59) by $v$ and obtain

$$
\begin{equation*}
\omega\|\nabla v\|^{2} \leq\left\|v_{t}\right\|\|v\|+\varepsilon\left\|\nabla v_{t}\right\|\|\nabla v\|-\left\langle f_{0}(v), v\right\rangle . \tag{66}
\end{equation*}
$$

Then integrating with (55), (61), (62), and (65), we conclude

$$
\begin{equation*}
\omega\|\nabla v\|^{2} \leq \sqrt{\bar{M}}(\|v\|+\varepsilon\|\nabla v\|) \leq \sqrt{\bar{M}} Q\left(\left\|B_{0}\right\|_{\mathscr{H}}\right) e^{-k_{1} t} . \tag{67}
\end{equation*}
$$

Thus, using the following Lemma 11 with (67), allows us to complete our proof by taking $k=k_{1} / 2$ and some increasing function $Q(\cdot)$.

Lemma 11. Let $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on the Banach space $X$, satisfying

$$
\begin{align*}
& \|S(t) B\|_{X} \leq Q_{1}\left(\|B\|_{X}\right) e^{-\mu t}, \quad \forall t \geq t_{0}  \tag{68}\\
& \|\{S(t) B: t \geq 0\}\|_{X} \leq Q_{2}\left(\|B\|_{X}\right)
\end{align*}
$$

Then

$$
\begin{equation*}
\|S(t) B\|_{X} \leq Q_{3}\left(\|B\|_{X}\right) e^{-\mu t}, \quad \forall t \geq 0 \tag{69}
\end{equation*}
$$

Its proof is obvious and we omit it here.
The next estimate is about the solution of (60).
Lemma 12. For every (given) $T>0$ and any $\varepsilon \in[0,1]$, there is a positive constant $J$ which only depends on $T, \omega$, $\|g\|$, and $\left\|u_{0}\right\|_{\mathscr{H}}$ such that the solutions of (60) satisfy

$$
\begin{equation*}
\left\|K_{\varepsilon}(t) B_{0}\right\|_{\mathscr{C _ { \varepsilon } ^ { \sigma }}}^{2} \leq J \tag{70}
\end{equation*}
$$

where both $J$ are independent of $\varepsilon \in[0,1]$, and $\sigma=\min \{1 / 4$, $(N+2-(N-2) \gamma) / 2\}$.

Proof. Multiplying (60) by $A^{\sigma} w(t)$ and integrating over $\Omega$, Then the proof is completely similar to that in [12, Lemma 3.4], so, we omit it.

Based on Lemmas 10 and 12, following the idea in Zelik [21], we can now decompose $u(t)$ as follows.

Lemma 13. Let $u(t)$ be the solution of $\left(E_{\varepsilon}\right)$ corresponding to the initial data $u_{0} \in B_{0}$. Then, for any $\eta>0$, we can decompose $u(t)=S_{\varepsilon}(t) u_{0}$ as

$$
\begin{equation*}
u(t)=v_{1}(t)+w_{1}(t), \quad \forall t \geq 0 \tag{71}
\end{equation*}
$$

where $v_{1}(t)$ and $w_{1}(t)$ satisfy the following estimates:

$$
\begin{gather*}
\int_{s}^{t}\left\|\nabla v_{1}(\tau)\right\|^{2} d \tau \leq \eta(t-s)+C_{\eta}, \quad \forall t \geq s \geq 0  \tag{72}\\
\left\|A^{(1+\sigma) / 2} w_{1}(t)\right\|^{2} \leq K_{\eta}, \quad \forall t \geq 0
\end{gather*}
$$

with the constants $C_{\eta}$ and $K_{\eta}$ depending on $\eta, \omega,\left\|B_{0}\right\|_{\mathscr{H}}$, and $\|g\|$, but both independent of $\varepsilon \in[0,1]$.

Proof. The proof is completely similar to that of [12, Lemma 4.6] and [22, Lemma 5.4], since the estimates in Lemmas 10 and 12 hold uniformly with respect to $\varepsilon \in$ $[0,1]$.

Note that in the above decomposition in Lemma 13, we can require further that $v_{1}(t)$ satisfies the following: there is a constant $M_{5}$ which depends only on $\omega,\left\|u_{0}\right\|_{\mathscr{H}}$ such that

$$
\begin{equation*}
\left\|\nabla v_{1}(t)\right\|^{2} \leq Q\left(\left\|B_{0}\right\|_{\mathscr{H}}\right):=M_{4}, \quad \forall t \geq 0, u_{0} \in B_{0} \tag{73}
\end{equation*}
$$

5.3. The Second A Priori Estimate. The main purpose of this subsection is to deduce some uniformly asymptotic (with respect to $\varepsilon \in[0,1]$ and $t$ ) the a priori estimates about the solution of $\left(E_{\varepsilon}\right)$.

Lemma 14. There exists positive constants $\bar{\nu}, \bar{R}>0$, and $Q(\cdot) \in$ $\mathscr{F}$ such that for each $\varepsilon \in[0,1]$, there is a subset $\bar{B}_{\varepsilon} \subset \mathscr{H}_{\varepsilon}^{1}$ satisfying

$$
\begin{equation*}
\left\|\bar{B}_{\varepsilon}\right\|_{\mathscr{H _ { \varepsilon } ^ { 1 }}}^{2}=\sup _{u \in \bar{B}_{\varepsilon}}\left\{\|\nabla u\|^{2}+\varepsilon\|\Delta u\|^{2}\right\} \leq \bar{R} \tag{74}
\end{equation*}
$$

and the exponential attraction

$$
\begin{equation*}
\operatorname{dist}_{\mathscr{H}_{\varepsilon}^{0}}\left(S_{\varepsilon}(t) B_{0}, \bar{B}_{\varepsilon}\right) \leq Q_{1}\left(\left\|B_{0}\right\|_{\mathscr{H}}\right) e^{-\bar{v} t}, \quad \forall t \geq 0 \tag{75}
\end{equation*}
$$

where all $\bar{\nu}, \bar{R}$, and $Q_{1}(\cdot)$ are independent of $\varepsilon \in[0,1]$, and dist $\left.\mathscr{H e}_{\varepsilon}^{( } \cdot, \cdot\right)$ denotes the Hausdorff semidistance with respect to the $\mathscr{H}_{\varepsilon}^{0}$-norm.

Proof. It is convenient to separate our proof into three steps. We emphasize, especially, that all the generic constants in the proof are independent of $\varepsilon \in[0,1]$.

Step 1. We first claim that (recall $\sigma=\min \{1 / 4,(N+2-(N-$ 2) $\gamma$ )/2\}): $\exists v_{\sigma}, R_{\sigma}>0$ and $Q_{\sigma}(\cdot) \in \mathscr{F}$ such that for each $\varepsilon \in$ $[0,1]$, there is a subset $\bar{B}_{\sigma, \varepsilon} \subset \mathscr{H}_{\varepsilon}^{\sigma}$ satisfying

$$
\begin{equation*}
\left\|\bar{B}_{\sigma, \varepsilon}\right\|_{\mathscr{H}_{\varepsilon}^{\sigma}}^{2}=\sup _{u \in \bar{B}_{\varepsilon}}\left\{\left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2}\right\} \leq R_{\sigma} \tag{76}
\end{equation*}
$$

and the exponential attraction

$$
\begin{equation*}
\operatorname{dist}_{\mathscr{H}_{\varepsilon}^{0}}\left(S_{\varepsilon}(t) B_{0}, \bar{B}_{\sigma, \varepsilon}\right) \leq Q_{\sigma}\left(\left\|B_{0}\right\|_{\mathscr{H}}\right) e^{-v_{\sigma} t}, \quad \forall t \geq 0 \tag{77}
\end{equation*}
$$

We will apply Lemma 2 with $X=\mathscr{H}_{\varepsilon}^{0}$ and $V=\mathscr{H}_{\varepsilon}^{\sigma}$ (note that $B_{0} \subset \mathscr{H} \subset \mathscr{H}_{\varepsilon}^{0}$ for any $\left.\varepsilon \in[0,1]\right)$. From (54), we can write

$$
\begin{equation*}
f_{0}(s)=s \varphi(s) \quad \text { with }|\varphi(s)| \leq C|s|^{4 /(N-2)} \tag{78}
\end{equation*}
$$

For any $x \in B_{0}$ and $y \in \mathscr{H}_{\varepsilon}^{0}, z \in \mathscr{H}_{\varepsilon}^{\sigma}$ satisfying $x=y+z$, we decompose the solution of $\left(E_{\varepsilon}\right)$ as $S_{\varepsilon}(t)(x)=V_{x}^{\varepsilon}(t) y+$ $U_{x}^{\varepsilon}(t) z$, where

$$
\begin{equation*}
V_{x}^{\varepsilon}(t) y=\bar{v}(t), \quad U_{x}^{\varepsilon}(t) z=\bar{w}(t) \tag{79}
\end{equation*}
$$

which uniquely solves the following equations, respectively:

$$
\begin{gather*}
\bar{v}_{t}-\varepsilon \Delta \bar{v}_{t}-\omega \Delta \bar{v}=h_{1} \\
\bar{v}(x, 0)=y  \tag{80}\\
\left.\bar{v}\right|_{\partial \Omega}=0 \\
\bar{w}_{t}-\varepsilon \Delta \bar{w}_{t}-\omega \Delta \bar{w}=h_{2} \\
\bar{w}(x, 0)=z  \tag{81}\\
\left.\bar{w}\right|_{\partial \Omega}=0
\end{gather*}
$$

with $h_{1}=-\bar{v} \varphi(v)$ and $h_{2}=g(x)-f(u)+\bar{v} \varphi(v)$, and $v(t)$ is the solution of (59) corresponding to the initial data $x$.

For (80), from (54), (56), (78), and Lemmas 10 and 12, we can directly calculate that

$$
\begin{equation*}
\left\|h_{1}\right\|_{L^{2 N /(N+2)}} \leq C\|\nabla \bar{v}\|\|\nabla v\|^{4 /(N-2)} \leq C_{\bar{M}, \omega, N} e^{-k^{\prime} t}\|\nabla \bar{v}\| \tag{82}
\end{equation*}
$$

where $k^{\prime}=(2 /(N-2)) k, k$ is given in Lemma 10.
Multiplying $\bar{v}$ by (80), we have

$$
\begin{align*}
\frac{d}{d t}\left(\|\bar{v}\|^{2}+\varepsilon\|\nabla \bar{v}\|^{2}\right)+2 \omega\|\nabla \bar{v}\|^{2} & =2\left\langle h_{1}, \bar{v}\right\rangle \\
\leq & 2\left\|h_{1}\right\|_{L^{2 N /(N+2)}}\|\bar{v}\|_{L^{2 N /(N-2)}} \\
\leq & C_{\bar{M}, \omega, N} e^{-k^{\prime} t}\|\nabla \bar{v}\|^{2} \\
\leq & C_{M, \omega, N} e^{-2 k^{\prime} t} \varepsilon\|\nabla \bar{v}\|^{2} \\
& +\omega\|\nabla \bar{v}\|^{2} \tag{83}
\end{align*}
$$

Furthermore, using the similar estimates of Lemma 6, we get

$$
\begin{gather*}
\frac{d}{d t}\left(\|\bar{v}\|^{2}+\varepsilon\|\nabla \bar{v}\|^{2}\right)+\varepsilon_{1}\left(\|\bar{v}\|^{2}+\varepsilon\|\nabla \bar{v}\|^{2}\right)  \tag{84}\\
\quad \leq 2 C_{\bar{M}, \omega, N} e^{-2 k^{\prime} t}\left(\|\bar{v}\|^{2}+\varepsilon\|\nabla \bar{v}\|^{2}\right)
\end{gather*}
$$

where $\varepsilon_{1}$ is a small positive constant such that $\varepsilon_{1} \leq$ $\min \left\{C_{\bar{M}, \omega, N} e^{-2 k^{\prime} t}, \lambda_{1} \omega\right\}$ for all $t \geq 0$.

And then applying Lemma 5 to above inequality, there holds

$$
\begin{equation*}
\left\|V_{x}^{\varepsilon}(t) y\right\|_{\mathscr{C _ { \varepsilon } ^ { 0 }}}^{2} \leq e^{C_{M, \omega, N} / k^{\prime}} e^{-\varepsilon_{1} t}\|y\|_{\mathscr{H} \ell_{\varepsilon}}^{2} \tag{85}
\end{equation*}
$$

For (81), since

$$
\begin{equation*}
h_{2}=g(x)-f(u)+f(v)-f_{1}(v)+w \varphi(v)-\bar{w} \varphi(v), \tag{86}
\end{equation*}
$$

then

$$
\begin{align*}
\left|h_{2}\right| \leq & |g|+C|w|\left(1+|u|^{4 /(N-2)}+|v|^{4 /(N-2)}\right)  \tag{87}\\
& +C|\bar{w}||v|^{4 /(N-2)}+C\left(1+|v|^{\gamma}\right) .
\end{align*}
$$

Using Hölder inequality we get

$$
\begin{align*}
& \left\|h_{2}\right\|_{L^{2 N /(N+2(1-\sigma))}} \leq 2 C|\Omega|^{(2-2 \sigma) / 2 N}\|g\| \\
& \quad+2 C\|w\|_{L^{2 N /(N-2(1+\sigma))}} \\
& \quad \times\left(1+\|u\|_{L^{2 N /(N-2)}}^{4 /(N-2)}+\|v\|_{L^{2 N /(N-2)}}^{4 /(N-2)}\right) \\
& \quad+2 C\|v\|_{L^{2 N /(N-2)}}^{4 /(N-2)}\|\bar{w}\|_{L^{2 N /(N-2(1-\sigma))}} \\
& \quad+2 C\left(1+\|v\|_{L^{2 N /(N-2)}}^{4 /(N-2)}\right) \\
& \leq C_{|\Omega|}\|g\|^{2}+C\left\|A^{(1+\sigma) / 2} w\right\|  \tag{88}\\
& \quad \times\left(1+\|\nabla u\|^{4 /(N-2)}+\|\nabla v\|^{4 /(N-2)}\right) \\
& \quad+C\|\nabla v\|^{4 /(N-2)}\left\|A^{(1+\sigma) / 2} \bar{w}\right\| \\
& \quad+C\left(1+\|\nabla v\|^{4 /(N-2)}\right) \\
& \leq \\
& \\
& \quad C_{\bar{M}, \omega, N} e^{-k^{\prime} t}\left\|A^{(1+\sigma) / 2} \bar{w}\right\| \\
& \quad+C_{\bar{M},\|g\|^{2},|\Omega|} J(t), \quad \forall t \geq 0
\end{align*}
$$

where we used (53), (62), and Lemmas 10 and 12.
Hence, multiplying $A^{\sigma} \bar{w}$ by (83), we have

$$
\begin{align*}
\frac{d}{d t} & \left(\left\|A^{\sigma / 2} \bar{w}\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2}\right)+2 \omega\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2} \\
& =2\left\langle h_{2}, \bar{w}\right\rangle \\
\leq & 2\left\|h_{2}\right\|_{L^{2 N /(N+2(1-\sigma))}}\left\|A^{\sigma} \bar{w}\right\|_{L^{2 N /(N-2(1-\sigma))}}  \tag{89}\\
\leq & 2 C_{\bar{M}, \omega, N} e^{-2 k^{\prime} t} \varepsilon\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2}+2 C_{\bar{M},\|g\|^{2},|\Omega|} J(t) \\
& +\omega\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2} .
\end{align*}
$$

Furthermore, we have

$$
\begin{align*}
& \frac{d}{d t}\left(\left\|A^{\sigma / 2} \bar{w}\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2}\right) \\
& \quad+\varepsilon_{1}\left(\left\|A^{\sigma / 2} \bar{w}\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2}\right)  \tag{90}\\
& \leq \\
& \quad 2 C_{\bar{M}, \omega, N} e^{-2 k^{\prime} t}\left(\left\|A^{\sigma / 2} \bar{w}\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} \bar{w}\right\|^{2}\right) \\
& \quad+C_{\bar{M},\|g\|^{2}, \Omega \mid} J(t),
\end{align*}
$$

where $\varepsilon_{1}$ is a small positive constant given in (84).
Then, using Lemma 5 we obtain

$$
\begin{equation*}
\left\|U_{x}^{\varepsilon}(t) z\right\|_{\mathscr{C _ { \varepsilon } ^ { \sigma }}}^{2} \leq e^{\bar{M}, \omega, N / k^{\prime}} e^{-\varepsilon_{1} t}\|z\|_{\mathscr{\ell _ { \varepsilon } ^ { \sigma }}}^{2}+\frac{1}{\varepsilon_{1}} e^{\varepsilon_{1}} C_{\bar{M},\|g\|^{2},|\Omega|} J(t) . \tag{91}
\end{equation*}
$$

Therefore, combining (85) and (91), we can verify that all the conditions of Lemma 2 are satisfied for the cases $X=\mathscr{H}_{\varepsilon}^{0}$, $V=\mathscr{H}_{\varepsilon}^{\sigma}$, and $T(t)=S_{\varepsilon}(t)$. Moreover, since there is a $c_{1}>0$ (independent of $\varepsilon$ ) such that $c_{1}\left\|B_{0}\right\|_{\mathscr{L}} \geq\left\|B_{0}\right\|_{\mathscr{H}_{\varepsilon}^{0}}$ for any $\varepsilon \in$ $[0,1]$ and the constants in our estimates are all independent of $\varepsilon$; consequently, $v_{\sigma}, R_{\sigma}$, and $Q_{\sigma}(\cdot)$ are all independent of $\varepsilon \in[0,1]$, and then we can deduce our claim.

Step 2. We claim that there exists a constant $\bar{R}_{\sigma}>0$ which depends only on $R_{\sigma}$ such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) \bar{B}_{\sigma, \varepsilon}\right\|_{\mathscr{H}_{\varepsilon}^{\sigma}}^{2} \leq \bar{R}_{\sigma}, \quad \forall t \geq 0 \tag{92}
\end{equation*}
$$

Multiplying $\left(E_{\varepsilon}\right)$ by $A^{\sigma} u(t)$, we only need to note the following:

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} & \left(\left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2}\right)+\omega\left\|A^{(1+\sigma) / 2} u\right\|^{2}  \tag{93}\\
& =\left\langle g, A^{\sigma} u\right\rangle-\left\langle f(u), A^{\sigma} u\right\rangle
\end{align*}
$$

First, since $\sigma<1$, we have $\sigma<(1+\sigma) / 2$ and then

$$
\begin{align*}
\left|\left\langle g, A^{\sigma} u(t)\right\rangle\right| \leq & \|g\|\left\|A^{\sigma} u\right\| \\
\leq & \|g\|\left\|A^{(1+\sigma) / 2} u\right\|  \tag{94}\\
\leq & 4 \omega\|g\|^{2}+\frac{\omega}{16}\left\|A^{(1+\sigma) / 2} u\right\|^{2}, \\
\left\langle f(u), A^{\sigma} u\right\rangle \leq & C \int_{\Omega}\left(1+|u|^{(N+2) /(N-2)}\right)\left|A^{\sigma} u\right| d x \\
\leq & 4 \omega|\Omega|^{2}+\frac{\omega}{16}\left\|A^{(1+\sigma) / 2} u\right\|^{2}  \tag{95}\\
& +C \int_{\Omega}|u|^{4 /(N-2)}|u|\left|A^{\sigma} u\right| d x,
\end{align*}
$$

while

$$
\begin{align*}
& \int_{\Omega}|u|^{4 /(N-2)}|u|\left|A^{\sigma} u\right| d x \\
& \quad \leq \int_{\Omega}\left(\left|v_{1}\right|^{4 /(N-2)}+\left|w_{1}\right|^{4 /(N-2)}\right)|u|\left|A^{\sigma} u\right| d x \\
& \int_{\Omega}\left|v_{1}\right|^{4 /(N-2)}|u|\left|A^{\sigma} u\right| d x  \tag{96}\\
& \leq C\left\|\nabla v_{1}\right\|^{4 /(N-2)}\left\|A^{(1+\sigma) / 2}\right\|^{2} \\
& \leq \frac{\omega}{16}\left\|A^{(1+\sigma) / 2} u\right\|^{2}+C M_{4}\left\|\nabla v_{1}\right\|\left\|A^{(1+\sigma) / 2} u\right\|^{2}
\end{align*}
$$

where we used (73).

Moreover, since $\sigma \leq 1 / 4$, we have $2 \leq 2 N(N-2) /(N(N-$ $4-2 \sigma)+4(1+3 \sigma)) \leq 2 N /(N-2-2 \sigma)$ and then

$$
\begin{align*}
& \int_{\Omega}\left|w_{1}\right|^{4 /(N-2)}|u|\left|A^{\sigma} u\right| d x \\
& \leq C\left\|\nabla w_{1}\right\|_{L^{2 N /(N-2-2 \sigma)}}^{4 /(N-2)}\|u\|_{L^{2 N(N-2) /(N(N-4-2 \sigma)+4(1+3 \sigma))}\left\|A^{\sigma} u\right\|} \\
& \leq C\left\|A^{(1+\sigma) / 2} w_{1}\right\|^{4 /(N-2)} \\
& \times\|u\|_{L^{2 N(N-2) /(N(N-4-2 \sigma)+4(1+3 \sigma))}}\left\|A^{(1+\sigma) / 2} u\right\| \\
& \leq C K_{\eta}^{4 /(N-2)}\|u\|_{L^{2 N(N-2) /(N(N-4-2 \sigma)+4(1+3 \sigma))}} \\
& \times\left\|A^{(1+\sigma) / 2} u\right\|  \tag{97}\\
& \leq C K_{\eta}^{4 /(N-2)}\left(C_{M}+\frac{\omega}{32 C K_{\eta}^{4 /(N-2)}}\left\|A^{(1+\sigma) / 2} u\right\|\right) \\
& \times\left\|A^{(1+\sigma) / 2} u\right\| \\
& \leq \frac{\omega}{32}\left\|A^{(1+\sigma) / 2} u\right\|^{2}+C_{M} K_{\eta}^{4 /(N-2)}\left\|A^{(1+\sigma) / 2} u\right\| \\
& \leq C_{M} K_{\eta}^{8 /(N-2)}+\frac{\omega}{16}\left\|A^{(1+\sigma) / 2} u\right\|^{2},
\end{align*}
$$

where $K_{\eta}$ is given in Lemma 13.
Hence, substituting the above estimates into (93), applying the Poincaré inequality we have

$$
\begin{align*}
\frac{d}{d t}( & \left.\left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2}\right) \\
& +\left(C-C M_{4}\left\|\nabla v_{1}\right\|^{2}\right)\left(\left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2}\right) \\
\leq & 8 \omega\|g\|^{2}+C_{M}\left(1+K_{\eta}^{8 /(N-2)}\right) \tag{98}
\end{align*}
$$

Then using the Gronwall inequality and integrating over $[0, t]$ (from Lemma 12), we obtain

$$
\begin{aligned}
& \left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2} \\
& \leq e^{-\int_{0}^{t}\left(C-C M_{4}\left\|\nabla v_{1}(s)\right\|^{2}\right) d s} \\
& \quad \times\left(\left\|A^{\sigma / 2} u(0)\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u(0)\right\|^{2}\right) \\
& \quad+C_{M}\left(1+K_{\eta}^{8 /(N-2)}\right) \\
& \quad \times \int_{0}^{t} e^{-\int_{t}^{s}\left(C-C M_{4}\left\|\nabla v_{1}(y)\right\|^{2}\right) d y} d s \\
& \quad+8 \omega \int_{0}^{t}\|g\|^{2} e^{-\int_{t}^{s}\left(C-C M_{4}\left\|\nabla v_{1}(y)\right\|^{2}\right) d y} d s .
\end{aligned}
$$

Taking $\eta$ (in Lemma 13) small enough such that $\eta<$ $C / 2 C M_{4}$, we have

$$
\begin{align*}
& \int_{0}^{t} e^{-\int_{t}^{s}\left(C-C M_{4}\left\|\nabla v_{1}(y)\right\|^{2}\right) d y} d s \\
& \quad=\int_{0}^{t} e^{C(s-t)} e^{\left.\int_{s}^{t} C M_{4}\left\|\nabla v_{1}(y)\right\|^{2}\right) d y} d s \\
& \quad \leq \int_{0}^{t} e^{C(s-t)} e^{C M_{4} \eta(t-s)+C M_{4} C_{\eta}} d s  \tag{100}\\
& \quad \leq e^{C M_{4} C_{\eta}} \int_{0}^{t} e^{C(s-t) / 2} d s \leq \frac{2 e^{C M_{4} C_{\eta}}}{C}, \\
& e^{-\int_{0}^{t}\left(C-C M_{4}\left\|\nabla v_{1}(s)\right\|^{2}\right) d s} \leq e^{-C t / 2} e^{C M_{4} C_{\eta}} . \tag{101}
\end{align*}
$$

Thus,
$\int_{0}^{t}\|g\|^{2} e^{-\int_{t}^{s}\left(C-C M_{5}\left\|\nabla v_{1}(y)\right\|^{2}\right) d y} d s \leq \frac{e^{C M_{4} C_{\eta}}}{1-e^{-(C / 2)}}\|g\|^{2}$.
Substituting above (100) and (102) into (99), we get that for all $t \geq 0$

$$
\begin{align*}
& \left\|A^{\sigma / 2} u\right\|^{2}+\varepsilon\left\|A^{(1+\sigma) / 2} u\right\|^{2} \\
& \quad \leq e^{C M_{5} C_{\eta}} R_{\sigma} \\
& \quad+\frac{2 C_{\omega}\left(1+K_{\eta}^{8 /(N-2)}\right)}{C} e^{C M_{5} C_{\eta}}  \tag{103}\\
& \quad+\frac{e^{C M_{5} C_{\eta}}}{1-e^{-(C / 2)}}\|g\|^{2}:=\bar{R}_{\sigma} .
\end{align*}
$$

Step 3. Based on Step 1 and Step 2, applying the attraction transitivity lemma given in [28, Theorem 5.1] and noticing the Holder continuity Lemma 9, we can prove our lemma by performing a standard bootstrap argument, whose proof is now simple since Step 1 makes the nonlinear term become subcritical to some extent.
5.4. Proof of Theorem 1. Lemma 14 has shown some asymptotic regularities; however, the radius of $\left\|\bar{B}_{\varepsilon}\right\|_{\mathscr{H}^{1}}$ depends on $\varepsilon$ and the distances only under the $\mathscr{H}_{\varepsilon}^{0}$-norm.

To prove Theorem 1, we first give two lemmas as preliminary.

Lemma 15. There exsits a constant $R_{1}>0$ such that for any bounded (in $\mathscr{H}_{\varepsilon}^{1}$ ) subset $B \subset \mathscr{H}_{\varepsilon}^{1}$, there exsits $T_{1}=T_{1}\left(\|B\|_{\mathscr{H}_{\varepsilon}^{1}}\right)$ such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) B\right\|_{\mathscr{H}_{\varepsilon}^{1}}^{2} \leq R_{1}, \quad \forall t \geq T_{1} . \tag{104}
\end{equation*}
$$

Proof. Multiplying $\left(E_{\varepsilon}\right)$ by $-\Delta u$, we find

$$
\begin{gather*}
\frac{d}{d t}\left(\|\nabla u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+2 \omega\|\nabla u\|^{2} \\
\quad=-2\langle f(u),-\Delta u\rangle+2\langle g,-\Delta u\rangle  \tag{105}\\
2\langle g,-\Delta u\rangle \leq C_{\omega}\|g\|^{2}+\frac{\omega}{2}\|\nabla u\|^{2}
\end{gather*}
$$

Noting $u \in \mathscr{H}_{\varepsilon}^{1} \subset \mathscr{H}_{\varepsilon}^{0}$, from Lemma 6, yields

$$
\begin{align*}
\mid-2 & \langle f(u),-\Delta u\rangle \mid \\
\quad & \leq 2 \int_{\Omega}\left|f^{\prime}(u)\right||\nabla u||\nabla u| d x \\
& \leq C\left(1+\|\nabla u\|^{4 /(N-2)}\right)\|\Delta u\|\|\Delta u\|  \tag{106}\\
& \leq C_{M} \varepsilon\|\nabla u\|^{2}+\frac{\omega}{2}\|\nabla u\|^{2}
\end{align*}
$$

hence, we obtain

$$
\begin{align*}
& \frac{d}{d t}\left(\|\nabla u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+\varepsilon_{2}\left(\|\nabla u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)  \tag{107}\\
& \quad \leq 2 C_{M}\left(\|\nabla u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+C_{\omega}\|g\|^{2}
\end{align*}
$$

where $\varepsilon_{2}$ is a small, positive constant.
Similarly, with using Lemma 4 we finally complete the proof.

Lemma 16. There exists a constant $R_{2}>0$ such that for any bounded (in $\mathscr{H}_{\varepsilon}^{1}$ ) subset $B \subset \mathscr{H}_{\varepsilon}^{1}$, there is a $T_{2}=T_{2}\left(\|B\|_{\mathscr{H}_{\varepsilon}^{1}}\right)$ such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|S_{\varepsilon}(t) B\right\|_{\mathscr{C}^{1}}^{2} \leq R_{2}, \quad \forall t \geq T_{2} \tag{108}
\end{equation*}
$$

Proof. From Lemma 15, we only need to estimate that the bound of $\|\Delta u\|^{2}$ is independent of $\varepsilon \in[0,1]$.

Applying Lemma 15 again, we have

$$
\begin{equation*}
\frac{d}{d t}\left(\|\nabla u\|^{2}+\varepsilon\|\nabla u\|^{2}\right)+\left(2 \omega-2 C C_{M}-1\right)\|\nabla u\|^{2} \leq C_{\omega}\|g\|^{2} \tag{109}
\end{equation*}
$$

Taking $\omega=\max \left\{1,2 C C_{M}\right\}$ which may provide that $1 / \omega<$ $\varepsilon$ and $2 \omega-2 C C_{M}-1>0$, integrating (109) on $[t, t+1]$, and from Lemma 15 , when $t \geq T_{1}$ we yield

$$
\begin{equation*}
\int_{t}^{t+1}\|\Delta u(\tau)\|^{2} d \tau \leq \frac{C_{\omega, R_{1}}}{2 \omega-2 C C_{M}-1} \tag{110}
\end{equation*}
$$

Hence, multiplying $\left(E_{\varepsilon}\right)$ by $-\Delta u_{t}$, we can complete our proof by applying the uniform Gronwall lemma.

Now, we are ready to prove Theorem 1.
Proof of Theorem 1. Set

$$
\begin{equation*}
\mathbb{B}=\left\{u \in \mathscr{H}^{1}:\|u\|_{\mathscr{H}^{1}}^{2} \leq R_{2}\right\}, \tag{111}
\end{equation*}
$$

where the constant $R_{2}$ comes from Lemma 16.
From Lemmas 16 and 14, we know that there is a $t_{0}$ such that $S_{\varepsilon}(t) \bar{B}_{\varepsilon} \subset \mathbb{B}$ (recall that $\bar{B}_{\varepsilon}$ is given in (78)) for all $t \geq t_{0}$ and any $\varepsilon \in[0,1]$.

On the other hand, note that $\exists c_{1}, c_{2}>0$ such that

$$
\begin{equation*}
c_{1}\|\cdot\|_{\mathscr{H}_{\varepsilon}^{0}} \leq\|\cdot\|_{\mathscr{H}} \leq c_{2}\|\cdot\|_{\mathscr{H}_{\varepsilon}^{1}} \tag{112}
\end{equation*}
$$

Then, from Lemma 9, there exists $t_{1}$ which depends only on $\left\|B_{0}\right\|_{\mathscr{H}}$ and $\left\|\bar{B}_{\varepsilon}\right\|_{\mathscr{\ell _ { \varepsilon } ^ { 1 }}}$ (so only on $M, \bar{R}$ ) such that

$$
\begin{array}{r}
\forall \varepsilon \in[0,1], \\
\left\|S_{\varepsilon}(t) u_{1}-S_{\varepsilon}(t) u_{2}\right\|_{\mathscr{H}} \leq e^{C_{\bar{R}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{1 / 2}, \\
\forall t \geq t_{1}, \\
u_{1} \in B_{0} \subset \mathscr{H} \subset \mathscr{H}_{\varepsilon}^{0}, \quad u_{2} \in \bar{B}_{\varepsilon} \subset \mathscr{H}_{\varepsilon}^{1} \subset \mathscr{H}_{\varepsilon}^{0}, \\
\forall \varepsilon \in[0,1], \quad S_{\varepsilon}(t) B_{0} \subset B_{0}, \quad \forall t \geq t_{1}, \tag{114}
\end{array}
$$

Therefore, from Lemma 14, we have

$$
\begin{align*}
& \operatorname{dist}_{\mathscr{H}}\left(S_{\varepsilon}\left(t+t_{0}+t_{1}\right) B_{0}, \mathbb{B}\right) \\
& \quad \leq \operatorname{dist}_{\mathscr{H}}\left(S_{\varepsilon}\left(t+t_{0}+t_{1}\right) B_{0}, S_{\varepsilon}\left(t_{0}+t_{1}\right) \bar{B}_{\varepsilon}\right) \\
& \quad \leq C_{M, \bar{R}, t_{0}+t_{1}} \operatorname{dist}_{\mathscr{H}}^{1 / 2}\left(S_{\varepsilon}(t) B_{0}, \bar{B}_{\varepsilon}\right)  \tag{115}\\
& \quad \leq C_{M, \bar{R}, t_{0}+t_{1}} \sqrt{Q_{1}\left(\left\|B_{0}\right\|_{\mathscr{H}}\right)} e^{-(\bar{v} / 2) t} .
\end{align*}
$$

Hence, noting that $t_{0}, t_{1}$, and $\bar{R}$ are all fixed, we can complete the proof by taking $\nu=\bar{\nu} / 2$ and applying Lemma 11.

## 6. Applications of Theorem 1

As for the applications of Theorem 1, in this subsection, we consider the existence of finite dimensional exponential attractors and the upper semicontinuity of global attractors for problem $\left(E_{\varepsilon}\right)$ under assumptions (1), (2), and $\omega>1$.
6.1. A Priori Estimates. For the subset $\mathbb{B}$ defined in (113), and from Lemmas 6 and 8 we know that there is a $t_{\mathbb{B}}$ such that

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|u_{t}\right\|^{2}+\varepsilon\left\|\nabla u_{t}\right\|^{2} \leq M_{3}, \quad \forall t \geq t_{\mathbb{B}}, u_{0} \in \mathbb{B}, \tag{116}
\end{equation*}
$$

where $u(t)=S_{\varepsilon}(t) u_{0}$.
Now, for each $\varepsilon \in[0,1]$, define $\widehat{B}_{\varepsilon}$ as follows:

$$
\begin{equation*}
\widehat{B}_{\varepsilon}=\bigcup_{t \geq t_{\mathbb{B}}+T_{2}} S_{\varepsilon}(t) \mathbb{B}, \tag{117}
\end{equation*}
$$

where $T_{2}$ is the time given in Lemma 16 corresponding to $\mathbb{B}$. Then, for each $\varepsilon \in[0,1]$ we have $\widehat{B}_{\varepsilon}$ as a positive invariant under $S_{\varepsilon}(t)$ (i.e., $S_{\varepsilon}(t) \widehat{B}_{\varepsilon}=\widehat{B}_{\varepsilon}$, for all $t \geq 0$ ) (from Lemma 16)

$$
\begin{equation*}
\forall \varepsilon \in[0,1], \quad\left\|\widehat{B}_{\varepsilon}\right\|_{\mathscr{H}^{1}}^{2} \leq R_{2} \tag{118}
\end{equation*}
$$

Moreover, we have the following results.
Lemma 17. Under assumptions (1), (2), and $\omega>1$, there exists a constant $\mathscr{T}>0$ such that for every $\varepsilon \in[0,1]$, the semigroup $S_{\varepsilon}(t)$ satisfies the following properties: $S_{\varepsilon}(\mathscr{T})$ admits a decomposition of the form

$$
\begin{equation*}
S_{\varepsilon}(\mathscr{T})=L_{\varepsilon}+N_{\varepsilon}, \quad L_{\varepsilon}: \widehat{B}_{\varepsilon} \longrightarrow \mathscr{H}_{\varepsilon}^{0}, \quad \widehat{B}_{\varepsilon} \longrightarrow \mathscr{H}_{\varepsilon}^{\theta} \tag{119}
\end{equation*}
$$

where $L_{\varepsilon}$ and $N_{\varepsilon}$ satisfy the estimates

$$
\begin{gather*}
\left\|L_{\varepsilon} u_{1}-L_{\varepsilon} u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \leq \frac{1}{4}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}, \quad \forall u_{1}, u_{2} \in \widehat{B}_{\varepsilon} \\
\left\|N_{\varepsilon} u_{1}-N_{\varepsilon} u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{\theta}} \leq C_{R_{2} \mathscr{T}}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}, \quad \forall u_{1}, u_{2} \in \widehat{B}_{\varepsilon} \tag{120}
\end{gather*}
$$

where the constants $C_{R_{2} \mathscr{T}}$ are independent of $\varepsilon$ and

$$
\theta= \begin{cases}1, & N=3,4,5,6  \tag{121}\\ \frac{4}{N-2}, & N>6\end{cases}
$$

Proof. For any two initial data $u_{i} \in \widehat{B}_{\varepsilon}$ with solution $S_{\varepsilon}(t) u_{i}=$ $u^{i}(i=1,2)$, we decompose the difference $S_{\varepsilon}(t) u_{1}-S_{\varepsilon}(t) u_{2}$ as follows:

$$
\begin{equation*}
S_{\varepsilon}(t) u_{1}-S_{\varepsilon}(t) u_{2}=L_{\varepsilon}(t)\left(u_{1}-u_{2}\right)+N_{\varepsilon}(t)\left(u_{1}-u_{2}\right) \tag{122}
\end{equation*}
$$

where $L_{\varepsilon}(t)\left(u_{1}-u_{2}\right)=\widetilde{v}$ solves

$$
\begin{gather*}
\widetilde{v}_{t}-\varepsilon \Delta \widetilde{v}_{t}-\omega \Delta \widetilde{v}=0 \quad \text { in } \Omega \times \mathbb{R}^{+}, \\
\widetilde{v}(x, 0)=u_{1}-u_{2},  \tag{123}\\
\left.\widetilde{v}\right|_{\partial \Omega}=0 \\
\widetilde{w}_{t}-\varepsilon \Delta \widetilde{w}_{t}-\omega \Delta \widetilde{w}+f\left(u^{1}\right)-f\left(u^{2}\right)=0 \quad \text { in } \Omega \times \mathbb{R}^{+}, \\
\widetilde{w}(x, 0)=0, \\
\left.\widetilde{w}\right|_{\partial \Omega}=0 . \tag{124}
\end{gather*}
$$

Next, for clarity, we decompose the remainder proof into two steps.
Step 1 . For $\widetilde{v}(t)$, multiplying (123) by $\widetilde{v}(t)$, we have

$$
\begin{align*}
& \frac{d}{d t}\left(\|\widetilde{v}\|^{2}+\varepsilon\|\nabla \widetilde{v}\|^{2}\right)+2 \mu^{1}\left(\|\widetilde{v}\|^{2}+\varepsilon\|\nabla \widetilde{v}\|^{2}\right)  \tag{125}\\
& \quad \leq 2 \omega \lambda_{1}\left(\|\widetilde{v}\|^{2}+\varepsilon\|\nabla \widetilde{v}\|^{2}\right)
\end{align*}
$$

where $\mu_{1}$ is a small positive constant such that $\mu_{1}<\omega \lambda_{1}$.
Using Lemma 4 we can deduce that

$$
\begin{equation*}
\left\|L_{\varepsilon} u_{1}-L_{\varepsilon} u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2}=\|\widetilde{v}\|_{\mathscr{H}_{\varepsilon}^{0}}^{2} \leq Q\left(\left\|\widehat{B}_{\varepsilon}\right\|_{\mathscr{H}}\right)\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2} e^{-\mu_{1} t} \tag{126}
\end{equation*}
$$

Hence, by taking $T^{\prime}>0$ large enough, we get

$$
\begin{align*}
& \left\|L_{\varepsilon}\left(t+T^{\prime}\right) u_{1}-L_{\varepsilon}\left(t+T^{\prime}\right) u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \\
& \quad \leq \frac{1}{4}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}, \quad \forall t \geq 0 . \tag{127}
\end{align*}
$$

Step 2. For $\widetilde{w}(t)$, multiplying (124) by $A^{\theta} \widetilde{w}(t)$ (where $\theta$ is given in (121)), we obtain

$$
\begin{gather*}
\frac{1}{2} \frac{d}{d t}\left(\left\|A^{\theta / 2} \widetilde{w}\right\|^{2}+\varepsilon\left\|A^{(1+\theta) / 2} \widetilde{w}\right\|^{2}\right)+\omega\left\|A^{(1+\theta) / 2} \widetilde{w}\right\|^{2}  \tag{128}\\
=-\left\langle f\left(u^{1}\right)-f\left(u^{2}\right), A^{\theta} \widetilde{w}\right\rangle
\end{gather*}
$$

Case $1(N=3,4)$. Then by using the embedding $D(A) \hookrightarrow$ $L^{p}(\Omega)$ for any $p \geq 1$, we have

$$
\begin{align*}
\mid- & \left\langle f\left(u^{1}\right)-f\left(u^{2}\right), A \widetilde{w}\right\rangle \mid \\
\leq & C\left(1+\left\|u^{1}\right\|_{L^{4 N /(N-2)}}^{4 /(N-2)}+\left\|u^{2}\right\|_{L^{L^{4 N /(N-2)}}}^{4 /(N-2)}\right) \\
& \times\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C\left(1+\left\|u^{1}\right\|_{\mathscr{H}^{1}}^{4 /(N-2)}+\left\|u^{2}\right\|_{\mathscr{\epsilon ^ { 1 }}}^{4 /(N-2)}\right)  \tag{129}\\
& \times\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C_{R_{2}}\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C_{R_{2}} e^{C_{R_{2}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}\|A \widetilde{w}\| \\
\leq & C_{R_{2}, \omega} e^{C_{R_{2}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2}+\frac{\omega}{2}\|A \widetilde{w}\|^{2},
\end{align*}
$$

where we have used (118) and (46).
Case $2(N=5,6)$. Since $4 N /(N-2) \leq 2 N /(N-4)$ and embedding $D(A) \hookrightarrow L^{2 N /(N-4)}(\Omega)$, we also have

$$
\begin{align*}
\mid- & \left\langle f\left(u^{1}\right)-f\left(u^{2}\right), A \widetilde{w}\right\rangle \mid \\
\leq & C\left(1+\left\|u^{1}\right\|_{L^{L^{N /(N-2)}}}^{4 /(N-2)}+\left\|u^{2}\right\|_{L^{4 N /(N-2)}}^{4 /(N-2)}\right) \\
& \times\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C\left(1+\left\|u^{1}\right\|_{L^{2 N /(N-4)}}^{4 /(N-2)}+\left\|u^{2}\right\|_{L^{2 N /(N-4)}}^{4 /(N-2)}\right)  \tag{130}\\
& \times\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C\left(1+\left\|u^{1}\right\|_{\mathscr{H}^{1}}^{4 /(N-2)}+\left\|u^{2}\right\|_{\mathscr{C}^{1}}^{4 /(N-2)}\right) \\
& \times\left\|\nabla\left(u_{1}-u_{2}\right)\right\|\|A \widetilde{w}\| \\
\leq & C_{R_{2}, \omega} e^{C_{R_{2}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2}+\frac{\omega}{2}\|A \widetilde{w}\|^{2} .
\end{align*}
$$

Case $3(N>6)$. Noticing that embedding $D(A) \hookrightarrow$ $L^{2 N /(N-4)}(\Omega)$ and $(4-2 \theta) / 2 N+(N-2) / 2 N+(N-(1-$ 2日)) $/ 2 N=1$, we have

$$
\begin{align*}
\mid- & \left\langle f\left(u^{1}\right)-f\left(u^{2}\right), A^{\theta} \widetilde{w}\right\rangle \mid \\
& \leq C_{R_{2}, \omega} e^{C_{R_{2}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2}+\frac{\omega}{2}\left\|A^{(1+\theta) / 2} \widetilde{w}\right\|^{2} \tag{131}
\end{align*}
$$

therefore, for any $N \geq 3$, we have

$$
\begin{align*}
\frac{d}{d t}( & \left.\left\|A^{\theta / 2} \widetilde{w}\right\|^{2}+\varepsilon\left\|A^{(1+\theta) / 2} \widetilde{w}\right\|^{2}\right) \\
& +\omega\left\|A^{(1+\theta) / 2} \widetilde{w}\right\|^{2}  \tag{132}\\
\leq & 2 C_{R_{2}, \omega} e^{C_{R_{2}} t}\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}^{2}, \quad \forall t \geq 0
\end{align*}
$$

Hence, taking

$$
\begin{equation*}
\mathscr{T}=T^{\prime}, \quad L_{\varepsilon}=L_{\varepsilon}(\mathscr{T}), \quad N_{\varepsilon}=N_{\varepsilon}(\mathscr{T}) \tag{133}
\end{equation*}
$$

then, from (127) and (132), we can see that $\mathscr{T}, L_{\varepsilon}$, and $N_{\varepsilon}$ satisfy Lemma 17.

Lemma 18. Under assumptions (1), (2), and $\omega>1$, for an arbitrary fixed time $T>0$ and any $\varepsilon \in[0,1]$, the semigroup $S_{\varepsilon}(t)$ is Lipchitz continuous on $[0, T] \times \widehat{B}_{\varepsilon}$ in the following sense: there exists a positive constant $\bar{C}_{T, R_{2}}$ such that for any $u_{i} \in \widehat{B}_{\varepsilon}$, $t_{i} \in[0, T], i=1,2$,

$$
\begin{equation*}
\left\|S_{\varepsilon}\left(t_{1}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \leq \bar{C}_{T, R_{2}}\left(\left|t_{1}-t_{2}\right|+\left\|u_{1}-u_{2}\right\|_{\mathscr{H}_{\varepsilon}^{0}}\right) . \tag{134}
\end{equation*}
$$

Proof. For $u_{1}, u_{2} \in \widehat{B}_{\varepsilon}$ and $t_{1}, t_{2} \in[0, T]$ we have

$$
\begin{align*}
& \left\|S_{\varepsilon}\left(t_{1}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{2}\right\|_{\mathscr{C}_{\varepsilon}^{0}} \\
& \quad \leq\left\|S_{\varepsilon}\left(t_{1}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{1}\right\|_{\mathscr{H}_{\varepsilon}^{0}}  \tag{135}\\
& \quad+\left\|S_{\varepsilon}\left(t_{2}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{2}\right\|_{\mathscr{C}_{\varepsilon}^{0}} .
\end{align*}
$$

The second term of above inequality is handled by estimate (46). Concerning the first one

$$
\begin{align*}
& \left\|S_{\varepsilon}\left(t_{1}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{1}\right\|_{\mathscr{H}_{\varepsilon}^{0}} \\
& \quad=\left\|\int_{t_{1}}^{t_{2}} \frac{d}{d t} S_{\varepsilon}(t) u_{1} d s\right\|_{\mathscr{H}_{\varepsilon}^{0}}  \tag{136}\\
& \quad \leq\left|\int_{t_{1}}^{t_{2}}\left\|\frac{d}{d t} S_{\varepsilon}(t) u_{1}\right\|_{\mathscr{H}_{\varepsilon}^{0}} d s\right| .
\end{align*}
$$

Then from (116) and (117) we can deduce

$$
\begin{equation*}
\left\|S_{\varepsilon}\left(t_{1}\right) u_{1}-S_{\varepsilon}\left(t_{2}\right) u_{1}\right\|_{\mathscr{R _ { \varepsilon } ^ { 0 }}} \leq \sqrt{M_{3}}\left|t_{1}-t_{2}\right| \tag{137}
\end{equation*}
$$

So, the proof is completed immediately.
6.2. Exponential Attractors. We are now ready to prove the following result about the existence of exponential attractors.

Lemma 19. Under assumptions (1), (2), and $\omega>1$, for every $\varepsilon \in[0,1]$, there exists a compact subset $\mathscr{E} \subset \mathscr{H}^{1}$, uniformly bounded in $\mathscr{H}^{1}$, which satisfies the following conditions:
(i) $\mathscr{E}_{\varepsilon}$ is semi-invariant with respect to the semigroup $\left\{S_{\varepsilon}(t)\right\}_{t \geq 0}$, that is,

$$
\begin{equation*}
S_{\varepsilon}(t) \mathscr{E}_{\varepsilon} \subset \mathscr{E}_{\varepsilon}, \quad \forall t \geq 0 \tag{138}
\end{equation*}
$$

(ii) the fractal dimension of $\mathscr{E}_{\varepsilon}$ is finite, that is,

$$
\begin{equation*}
\operatorname{dim}_{F}\left(\mathscr{E}_{\varepsilon}, \mathscr{H}\right) \leq \Lambda_{\varepsilon}<\infty, \quad \forall \varepsilon \in[0,1] ; \tag{139}
\end{equation*}
$$

(iii) for each $\varepsilon \in[0,1], \mathscr{E}_{\varepsilon}$ enjoys a uniform exponential attraction property of the following form: for any bounded (in $\mathscr{H}$ ) subset $B \subset \mathscr{H}$,

$$
\begin{equation*}
\operatorname{dist}_{\mathscr{H}}\left(S_{\varepsilon}(t) B, \mathscr{E}_{\varepsilon}\right) \leq Q_{\varepsilon}\left(\|B\|_{\mathscr{E}}\right) e^{-\nu^{\prime} t}, \quad \forall t \geq 0 \tag{140}
\end{equation*}
$$

Here, $\Lambda_{\varepsilon}$ and $Q_{\varepsilon}(\cdot)$ may depend on $\varepsilon$, but $\nu^{\prime}$ is independent of $\varepsilon$.

Proof. For each $\varepsilon \in[0,1]$, we know that $\widehat{B}_{\varepsilon}$ is invariant and compact in $\mathscr{H}_{\varepsilon}^{0}$. Hence, applying the abstract results established in [23, 24], from Lemmas 17 and 18 we can first construct an exponential attractor on $\widehat{B}_{\varepsilon}$ with respect to the $\mathscr{H}_{\varepsilon}^{0}$-norm. Then, we can complete the proof by using the attraction transitivity lemma given in [28, Theorem 5.1] from Lemma 14 and the Hölder continuity (47).
6.3. Upper Semicontinuity of Global Attractors. Since $\mathscr{A}_{\varepsilon} \subset$ $\mathscr{E}_{\varepsilon}$, (ii) of Lemma 19 implies that the fractal dimension of the global attractor $\mathscr{A}_{\varepsilon}$ is finite too. Moreover, we have the following upper semicontinuity result of $\mathscr{A}_{\varepsilon}$ at $\varepsilon=0$.

Lemma 20. Under assumptions (1), (2), and $\omega>1$, the global attractors $\left\{\mathscr{A}_{\varepsilon}\right\}_{\varepsilon \in[0,1]}$ are upper semicontinuous at $\varepsilon=0$.

$$
\begin{equation*}
\operatorname{dist}_{\mathscr{H}}\left(\mathscr{A}_{\varepsilon}, \mathscr{A}_{0}\right) \longrightarrow 0 \quad \text { as } \varepsilon \longrightarrow 0^{+} \tag{141}
\end{equation*}
$$

Proof. Since the global attractor $\mathscr{A}_{\varepsilon}$ is strictly invariant, that is, $S_{\varepsilon}(t) \mathscr{A}_{\varepsilon}=\mathscr{A}_{\varepsilon}$ for all $t \geq 0$, it is obvious to see that

$$
\begin{equation*}
\bigcup_{\varepsilon \in[0,1]} \mathscr{A}_{\varepsilon} \subset \mathbb{B} \text { and compact in } \mathscr{H} \text {. } \tag{142}
\end{equation*}
$$

Therefore, to apply Lemma 3, we can take $K=\mathrm{cl}_{\mathscr{H}_{1}}(\mathbb{B})$ and we only need to verify condition (14). Let $\varepsilon \in[0,1]$ and $\widehat{u}=S_{\varepsilon}(t) u_{\varepsilon}$ with $u_{\varepsilon} \in \mathscr{A}_{\varepsilon}$; also let $\widehat{v}=S_{0}(t) u_{0}$ with $u_{0} \in$ $\mathbb{B}$. Denote $\widehat{w}(t)=\widehat{u}(t)-\widehat{v}(t)$. Then $\widehat{w}$ solves the following equation:

$$
\begin{gather*}
\widehat{w}_{t}-\omega \Delta \widehat{w}+f(\widehat{u})-f(\widehat{v})=\varepsilon \Delta \widehat{u}_{t}, \\
\widehat{w}(x, 0)=u_{\varepsilon}-u_{0}  \tag{143}\\
\left.\widehat{w}\right|_{\partial \Omega}=0
\end{gather*}
$$

Multiplying (143) by $\widehat{w}_{t}$, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left(\omega\|\nabla \widehat{w}\|^{2}+2\langle\varepsilon \Delta \widehat{u}, \widehat{w}\rangle\right)+\left\|\widehat{w}_{t}\right\|^{2}  \tag{144}\\
& \quad=-\left\langle f(\widehat{u})-f(\widehat{v}), \widehat{w}_{t}\right\rangle \\
& \left|-\left\langle f(\widehat{u})-f(\widehat{v}), \widehat{w}_{t}\right\rangle\right| \\
& \quad \leq C\left(1+\|\widehat{u}\|_{\mathscr{H}^{1}}^{4 /(N-2)}+\|\widehat{v}\|_{\mathscr{C}^{1}}^{4 /(N-2)}\right)\|\nabla \widehat{w}\|\left\|\widehat{w}_{t}\right\|  \tag{145}\\
& \quad \leq C_{\|B\|_{\mathscr{C}}}\|\nabla \widehat{w}\|^{2}+\frac{1}{2}\left\|\widehat{w}_{t}\right\|^{2},
\end{align*}
$$

where we used Lemma 19 and (118) and noticed the process of Step 2 of Lemma 17 for $N$.

On the other hand, for (145), using Poincaré inequality, we have

$$
\begin{align*}
(\omega-1) & \|\nabla \widehat{w}\|^{2}-\varepsilon^{2}\|\nabla \widehat{u}\|^{2} \\
& \leq \omega\|\nabla \widehat{w}\|^{2}+2\langle\varepsilon \Delta \widehat{u}, \widehat{w}\rangle  \tag{146}\\
& \leq\left(\omega+\frac{1}{\lambda_{1}}\right)\|\nabla \widehat{w}\|^{2}+\varepsilon^{2}\|\nabla \widehat{u}\|^{2}
\end{align*}
$$

So, we obtain

$$
\begin{align*}
\frac{d}{d t} & \left(\omega\|\nabla \widehat{w}\|^{2}+2\langle\varepsilon \Delta \widehat{u}, \widehat{w}\rangle\right)+\left\|\widehat{w}_{t}\right\|^{2} \\
& \leq 2 C_{\|B\|_{g_{1}}}\|\nabla \widehat{w}\|^{2}  \tag{147}\\
& \leq 2 C_{\|B\|_{\xi^{1}}}\left(\omega\|\nabla \widehat{w}\|^{2}+2\langle\varepsilon \Delta \widehat{u}, \widehat{w}\rangle\right)
\end{align*}
$$

Noticing $\mathbb{B} \subset \mathscr{H}^{1} \subset \mathscr{H}_{\varepsilon}^{1} \subset \mathscr{H}_{\varepsilon}^{0}$, using (24), Lemma 15, and Gronwall inequality, yields

$$
\begin{align*}
& \|\nabla \widehat{w}\|^{2} \leq \frac{1}{(\omega-1)} 2 C_{\|B\|_{t, \mathscr{C ^ { 1 }}}}\left(\left(\omega+\frac{1}{\lambda_{1}}\right)\left\|u_{\varepsilon}-u_{0}\right\|_{\mathscr{H}}^{2}+\varepsilon R_{1}\right) \\
& \quad+\varepsilon M_{1}, \quad \forall t \geq T_{1 B}+T_{1} \tag{148}
\end{align*}
$$

Hence, we know that there exists $t_{1}=t_{1}\left(\|\mathbb{B}\|_{\mathscr{H}^{1}}\right) \geq T_{1 B}+$ $T_{1}$ such that

$$
\begin{align*}
&\left\|\nabla \widehat{w}\left(t_{1}+1\right)\right\|^{2} \\
& \leq \frac{1}{(\omega-1)} 2 C_{t_{1},\|B\|_{\mathscr{E}^{1}}}\left(\left(\omega+\frac{1}{\lambda_{1}}\right)\left\|u_{\varepsilon}-u_{0}\right\|_{\mathscr{H}}^{2}+\varepsilon R_{1}\right) \\
&+\frac{M_{1}}{\omega-1} \varepsilon \tag{149}
\end{align*}
$$

which implies

$$
\begin{align*}
& \text { if } \varepsilon_{n} \longrightarrow 0^{+}, \text {and } \mathscr{A}_{\varepsilon_{n}} \ni u_{n} \longrightarrow u_{0} \\
& \text { then } S_{\varepsilon_{n}}\left(t_{1}+1\right) u_{n} \longrightarrow S_{0}\left(t_{1}+1\right) u_{0} \tag{150}
\end{align*}
$$

Therefore, from (142) and (150), we can directly apply Lemma 3 to complete the proof.

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