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Research Article

Dynamic Analysis of Rotating Pendulum by Hamiltonian Approach

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A conservative system always admits Hamiltonian invariant, which is kept unchanged during oscillation. This property is used to obtain the approximate frequency-amplitude relationship of the governing equation with sinusoidal nonlinearity. Here, we applied Hamiltonian approach to obtain natural frequency of the nonlinear rotating pendulum. The problem has been solved without series approximation and other restrictive assumptions. Numerical simulations are then conducted to prove the efficiency of the suggested technique.

1. Introduction

The rotational pendulum equation [1, 2] arises in a number of models describing the phenomenon in engineering. This equation has been described in the wind-excited vibration absorber [3] and mechanical and civil structure [4, 5] and has received much attention recently. To improve the understanding of dynamical systems, it is important to seek their exact solution. Most dynamical systems can not be solved exactly; numerical or approximate methods must be used. Numerical methods are often costly and time consuming to get a complete dynamics of the problem.

Little progress was made on the integrability of the rotational pendulum by Lai et al. [6]. Lai and his colleagues used Taylor's series and Chebyshev's polynomials to convert the trigonometric nonlinearity to algebraic nonlinearity. They used Mickens iteration method [7] and found the approximate explicit formulas. Various alternative approaches have been proposed for solving nonlinear dynamical system, parameter-expanding method [8], frequency-amplitude formulation [9], max-min approach [10], harmonic balance method [11], variational approach [12], homotopy perturbation method [13–15], Lindstedt-Poincare method [16], and Hamiltonian approach [17–19].

2. Governing Equation of a Rotational Pendulum

Let us consider a pendulum revolving about a vertical axis and swinging horizontally as shown in Figure 1. The rotational pendulum is assumed to have a length l and a lumped mass m and turn at constant speed ω_0. The kinetic energy and potential energy are

$$T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\omega_0^2\sin^2\theta\right),$$
$$V = mgl\left(1-\cos\theta\right), \tag{1}$$

where θ is the angular displacement of the pendulum in the vertical direction. The equation of rotational pendulum can be derived using the Lagrange equation. From the Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = 0, \tag{2}$$

where $L = T - V$. We have

$$ml^2\ddot{\theta} + mgl\sin\theta - \frac{1}{2}ml^2\omega_0^2\sin 2\theta = 0. \tag{3}$$

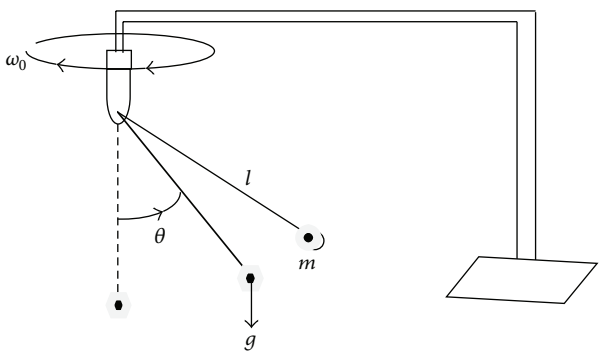

Figure 1: Rotational pendulum at a constant speed.

The second-order differential equation of the rotational pendulum system with initial conditions is

$$\ddot{\theta} + \tilde{\omega}^2 \sin\theta - \tilde{\omega}^2 \Lambda \sin\theta \cos\theta = 0,$$
$$\theta(0) = A \in (0, \pi), \qquad \dot{\theta}(0) = 0, \qquad \dot{u}(0) = 0, \tag{4}$$

where $\tilde{\omega}^2 = g/l$, $\Lambda = \omega_0^2 l/g$.

According to (1), we have

$$T(-\theta) = -T(\theta), \qquad V(-\theta) = V(\theta). \tag{5}$$

Consequently the rotational pendulum equation has a conservative behavior and a periodic solution. The variational principle for (4) can be written as

$$J(\theta) = \int_0^{\tilde{T}/4} \left(-\frac{\dot{\theta}^2}{2} - F(\theta) \right) dt, \tag{6}$$

where $\tilde{T}$ is period of the nonlinear oscillator and $(\partial F/\partial\theta) = \tilde{\omega}^2 \sin\theta - \tilde{\omega}^2 \Lambda \sin\theta \cos\theta$. The least Lagrangian action, from which we can write the Hamiltonian, is

$$H = \frac{\dot{\theta}^2}{2} + F(\theta) = \text{constant}. \tag{7}$$

From (7), we have

$$\frac{\partial H}{\partial A} = 0. \tag{8}$$

Introducing a new function, $\widehat{H}(u)$, is defined as

$$\widehat{H}(\theta) = \int_0^{T/4} \left(\frac{\dot{\theta}^2}{2} + F(\theta) \right) dt. \tag{9}$$

Equation (8) is, then, equivalent to the following one:

$$\frac{\partial}{\partial A}\left(\frac{\partial \widehat{H}}{\partial T} \right) = 0 \quad \text{or} \quad \frac{\partial}{\partial A}\left(\frac{\partial \widehat{H}}{\partial (1/w)} \right) = 0. \tag{10}$$

From (10), we can obtain approximate frequency-amplitude relationship of rotating pendulum.

3. Solution

The variational formulation of rotating pendulum can be written as

$$J(\theta) = \int_0^{\tilde{T}/4} \left(-\frac{\dot{\theta}^2}{2} - \tilde{\omega}^2 \cos\theta - \frac{1}{2}\tilde{\omega}^2 \Lambda \sin^2\theta \right) dt, \tag{11}$$

and $\widehat{H}(\theta)$ can be written in the form of

$$\widehat{H}(\theta) = \int_0^{\tilde{T}/4} \left(\frac{\dot{\theta}^2}{2} - \tilde{\omega}^2 \cos\theta - \frac{1}{2}\tilde{\omega}^2 \Lambda \sin^2\theta \right) dt. \tag{12}$$

Assume that the solution can be expressed as $\theta = A\cos\omega t$, substituting

$$\begin{aligned}
\widehat{H}(\theta) &= \int_0^{\tilde{T}/4} \left(\frac{1}{2}(A\omega \sin\omega t)^2 - \tilde{\omega}^2 \cos(A\cos\omega t) \right. \\
&\qquad \left. - \frac{1}{2}\tilde{\omega}^2 \sin^2(A\cos\omega t) \right) dt, \\
\widehat{H}(\theta) &= \int_0^{\pi/2} \left(\frac{1}{2}\omega(A\sin t)^2 - \frac{\tilde{\omega}^2}{\omega} \cos(A\cos t) \right. \\
&\qquad \left. - \frac{1}{2\omega}\tilde{\omega}^2 \sin^2(A\cos t) \right) dt, \\
\widehat{H}(\theta) &= \frac{A^2\pi\omega}{8} - \frac{\pi\Lambda\tilde{\omega}^2}{8\omega} - \frac{\pi\omega_0^2 \text{Bessels}J(0, A)}{2\omega} \\
&\qquad + \frac{\pi\lambda\omega_0^2 \text{Bessels}J(0, 2A)}{8\omega}.
\end{aligned} \tag{13}$$

Setting

$$\begin{aligned}
&\frac{\partial}{\partial A}\left(\frac{\partial \widehat{H}}{\partial (1/\omega)} \right) \\
&\quad = \frac{-A\pi\omega^2}{4} + \frac{\pi\tilde{\omega}^2 \text{Bessels}J(1, A)}{2} - \frac{\pi\lambda\tilde{\omega}^2 \text{Bessels}J(1, 2A)}{4} \\
&\quad = 0.
\end{aligned} \tag{14}$$

We obtained the following frequency-amplitude relationship for nonlinear rotating pendulum:

$$\omega = \tilde{\omega}\sqrt{\frac{2\text{Bessels}J(1, A) - \Lambda \text{Bessels}J(1, 2A)}{A}}. \tag{15}$$

4. Closing Comments

The present method is an extremely simple method, leading to accuracy of the obtained results. The main advantage of the method is that the obtained results are valid for the whole solution domain. For graphical comparison, variations of the frequency against the amplitude of motion with four different

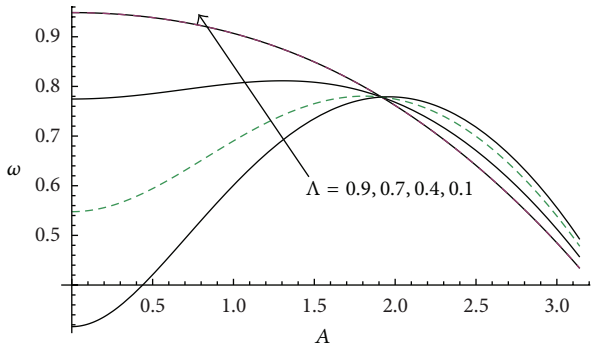

Figure 2: Amplitude versus frequency of rotational pendulum for different values of A.

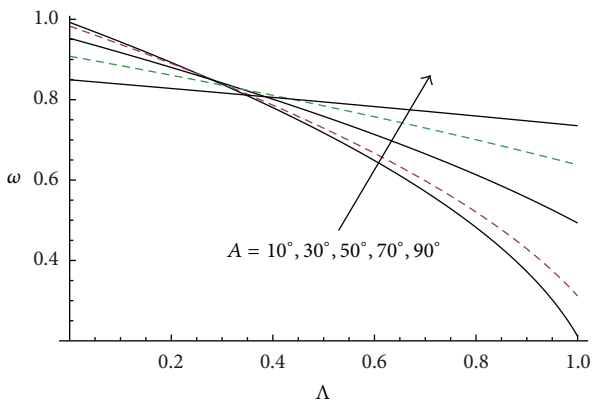

Figure 3: Λ versus frequency of rotational pendulum for different values of amplitude.

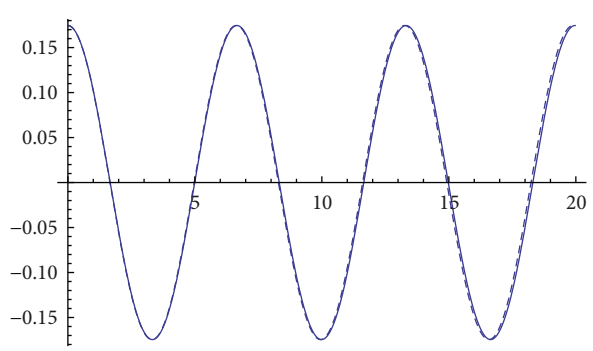

Figure 4: Comparison of approximate solutions with exact solution for $\tilde{\omega} = 1$, $\Lambda = 0.1$, and $A = 10^\circ$ (time versus angular displacement).

constants $\Lambda = 0.1, 0.4, 0.7,$ and 0.9 are almost the same at $A = 1.916$ in Figure 2. The influence of A on the frequency is examined in Figure 3. It is quite obvious that a large Λ has a lower frequency; therefore, an increase in Λ tends to decrease the frequency. An excellent agreement is found in Figure 4 between the two solutions.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

The work including proofreading was done by all the authors.

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