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Research Article

An Upper Bound for the Tensor Rank

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Let T be a tensor of format $(m_1+1)\times\cdots\times(m_s+1)$, $m_1\ge\cdots\ge m_s>0$, over $\mathbb{C}$. We prove that T has tensor rank at most $\prod_{i\ne 2}(m_i+1)$.

1. Introduction

Fix integers $s\ge 2$ and $m_i>0$, $1\le i\le s$, and an algebraically closed base field $\mathbb{K}$. Let $T\in\otimes_{1\le i\le s}\mathbb{K}^{m_i+1}$ be a tensor of format $(m_1+1)\times\cdots\times(m_s+1)$ over $\mathbb{K}$. The tensor rank $r(T)$ of T is the minimal integer $x\ge 0$ such that $T=\sum_{i=1}^{x}v_{1,i}\otimes\cdots\otimes v_{s,i}$ with $v_{j,i}\in\mathbb{K}^{m_j+1}$ (see [1–6]). Classical papers (e.g., [7]) continue to suggest new results (see [8]). Let $t(m_1,\dots,m_s)$ be the maximum of all integers $r(T)$, $T\in\otimes_{1\le i\le s}\mathbb{K}^{m_i+1}$. In this paper we prove the following result.

Theorem 1. *For all integers $s\ge 2$ and $m_1\ge\cdots\ge m_s>0$ one has $t(m_1,\dots,m_s)\le\prod_{i\ne 2}(m_i+1)$.*

This result is not optimal. It is not sharp when $s=2$, since $t(m_1,m_2)=m_2+1$ by elementary linear algebra. For large s the bound should be even worse. In our opinion to get stronger results one should split the set of all $(s;m_1,\dots,m_s)$ into subregions. For instance, we think that for large s the cases with $m_1\gg m_2\gg\cdots\gg m_s>0$ and the cases with $m_1=\cdots=m_s$ are quite different.

We make the definitions in the general setting of the Segre-Veronese embeddings of projective spaces (i.e., of partially symmetric tensors), but we only use the case of the usual Segre embedding, that is, the usual tensor rank. The tensor $T=0$ has zero as its tensor rank. If $\lambda\in\mathbb{K}\setminus\{0\}$, then the tensors T and λT have the same rank. Hence it is sufficient to study the function "tensor rank" on the projectivisation of the vector space $\otimes_{1\le i\le s}\mathbb{K}^{m_i+1}$. We may translate the tensor rank and the integer $t(m_1,\dots,m_s)$ in the following language. For each subset A of a projective space, let $\langle A\rangle$ denote the linear span of A. For each integral variety $Y\subset\mathbb{P}^n$ and any $P\in\langle Y\rangle$ the Y-rank $r_Y(P)$ of P is the minimal cardinality of a finite set $A\subset Y$ such that $P\in\langle A\rangle$. Now assume $\langle Y\rangle=\mathbb{P}^n$. The maximal Y-rank ρ_Y is the maximum of all integers $r_Y(P)$, $P\in\mathbb{P}^n$. Fix integers $s>0$, $m_i\ge 0$, $1\le i\le s$, and $d_i>0$, $1\le i\le s$. Set $T(m_1,\dots,m_s):=\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_s}$. Let $\nu_{d_1,\dots,d_s}:T(m_1,\dots,m_s)\to\mathbb{P}^r$, $r:=-1+\prod_{1\le i\le s}\binom{m_i+d_i}{m_i}$ be the Segre-Veronese embedding of multidegree $(d_1,\dots,d_s)$, that is, the embedding of $T(m_1,\dots,m_s)$ induced by the $\mathbb{K}$-vector space of all polynomials $f\in\mathbb{K}[x_{i,j}]$, $1\le i\le s$, $0\le j\le m_i$, whose nonzero monomials have degree d_i with respect to the variables $x_{i,j}$, $0\le j\le m_i$. Set $T(m_1,\dots,m_s;d_1,\dots,d_s):=\nu_{d_1,\dots,d_s}(T(m_1,\dots,m_s))$. The variety $T(m_1,\dots,m_s;1,\dots,1)$ is the Segre embedding of $T(m_1,\dots,m_s)$. Fix $P\in\mathbb{P}^r$, $r:=-1+\prod_{1\le i\le s}(m_i+1)$. Let $\{\lambda T\}_{\lambda\in\mathbb{K}\setminus\{0\}}$ be the set of all nonzero tensors of format $(m_1+1)\times\cdots\times(m_s+1)$ associated with P. We have $r(T)=r_{T(m_1,\dots,m_s;1,\dots,1)}(P)$. Hence $t(m_1,\dots,m_s)=\rho_{T(m_1,\dots,m_s;1,\dots,1)}$. To prove Theorem 1 we refine the notion of Y-rank in the following way.

Definition 2. Fix positive integers s, m_i, $1\le i\le s$, and d_i, $1\le i\le s$. A small box of $T(m_1,\dots,m_s)$ is a closed set $L_1\times\cdots\times L_s\subset T(m_1,\dots,m_s)$ with L_i being a hyperplane of $\mathbb{P}^{m_i}$ for all i. A large box of $T(m_1,\dots,m_s)$ is a product $L_1\times\cdots\times L_s\subset T(m_1,\dots,m_s)$ such that there is $j\in\{1,\dots,s\}$ with $L_j\subset\mathbb{P}^{m_j}$ being a hyperplane, while $L_i=\mathbb{P}^{m_i}$ for all $i\ne j$. A small polybox (resp., large polybox) of $T(m_1,\dots,m_s)$ is a finite union of small (resp., large) boxes of $T(m_1,\dots,m_s)$. A small box (resp., small polybox, resp., large box, resp., large polybox) $B \subset T(m_1, \dots, m_s; d_1, \dots, d_s)$ is the image by $\nu_{d_1,\dots,d_s}$ of a small box (resp., small polybox, resp., large box, resp., large polybox) of $T(m_1, \dots, m_s)$.

Definition 3. Fix positive integers s, m_i, $1 \le i \le s$, and d_i, $1 \le i \le s$, and set $r := -1 + \prod_{1\le i\le s} \binom{m_i+d_i}{m_i}$. Fix $P \in \mathbb{P}^r$. The rank $r_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$ of P is the minimal cardinality of a finite set $A \subset T(m_1, \dots, m_s; d_1, \dots, d_s)$ such that $P \in \langle A\rangle$. The unboxed rank (resp., small unboxed rank) $r'_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$ (resp., $r''_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$) of P is the minimal integer $t > 0$ such that for each large polybox (resp., small polybox) $B \subset T(m_1, \dots, m_s; d_1, \dots, d_s)$ there is a finite set $A \subset T(m_1, \dots, m_s; d_1, \dots, d_s) \setminus B$ with $P \in \langle A\rangle$ and $\sharp(A) = t$. Let $t(m_1, \dots, m_s; d_1, \dots, d_s)$ (resp., $t'(m_1, \dots, m_s; d_1, \dots, d_s)$, resp., $t''(m_1, \dots, m_s; d_1, \dots, d_s)$) be the maximum of all integers $r_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$ (resp., $r'_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$, resp., $r''_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$), $P \in \mathbb{P}^r$.

Notice that $t(m_1, \dots, m_s) = t(m_1, \dots, m_s; 1, \dots, 1)$.

Since $r_{m_1,\dots,m_s;d_1,\dots,d_s}(P) \le r''_{m_1,\dots,m_s;d_1,\dots,d_s}(P) \le r'_{m_1,\dots,m_s;d_1,\dots,d_s}(P)$ for all P, we have $t(m_1, \dots, m_s; d_1, \dots, d_s) \le t''(m_1, \dots, m_s; d_1, \dots, d_s)$. Hence Theorem 1 is an immediate corollary of the following result.

Theorem 4. *For all integers $s \ge 2$ and $m_1 \ge \cdots \ge m_s > 0$ one has $t''(m_1, \dots, m_s) \le \prod_{i\ne 2}(m_i + 1)$.*

We hope that the definitions of unboxed rank and small unboxed rank are interesting in themselves, not just as a tool. As far as we know the best upper bound for the symmetric tensor rank is due to Białynicki-Birula and Schinzel ([9, 10]). In [9] Białynicki-Birula and Schinzel used the corresponding notion in the case $s = 1$.

2. Proof of Theorem 4

Remark 5. Fix integers $s \ge 2$ and $m_i > 0$, $1 \le i \le s$. Fix $j \in \{1, \dots, s\}$ and let $\pi_j : T(m_1, \dots, m_s) \to T(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_s)$ be the projection. For any small polybox $B \subset T(m_1, \dots, m_s)$ the set $\pi_j(B)$ is a small polybox of the Segre variety $T(m_1, m_{j-1}, m_{j+1}, \dots, m_s)$.

In the case $s = 2$ we also need the following notation. Fix integers $m_1 \ge m_2 > 0$. For each $P \in \mathbb{P}^r$, $r = (m_1 + 1)(m_2 + 1) - 1$, let $\tilde{t}_{m_1,m_2}(P)$ be the minimal integer $t > 0$ with the following property: for each finite union $E \subset \mathbb{P}^{m_1}$ of hyperplanes there is a set $A \subset T(m_1, m_2) \setminus E \times \mathbb{P}^{m_2}$ such that $\sharp(A) = t$ and $P \in \langle \nu_{1,1}(A)\rangle$. Let $\tilde{t}(m_1, m_2)$ be the maximum of all integers $\tilde{t}_{m_1,m_2}(P)$, $P \in \mathbb{P}^r$. Obviously $t(m_1, m_2) \le t''(m_1, m_2) \le \tilde{t}(m_1, m_2) \le t'(m_1, m_2)$. Linear algebra gives $t(m_1, m_2) = 1 + \min\{m_1, m_2\} = m_2 + 1$.

Lemma 6. *For all integers $m_1 \ge m_2 > 0$ one has $t''(m_1, m_2) \le \tilde{t}(m_1, m_2) \le m_1 + 1$.*

Proof. It is sufficient to prove the inequality $\tilde{t}(m_1, m_2) \le m_1 + 1$. Without losing generality we may assume $m_1 = m_2$. Set $m := m_1$ and $V := \mathbb{K}^{m+1}$. Fix $P \in \mathbb{P}^r$, $r = m^2 + 2m$, and a union $E \subset \mathbb{P}^m$ of finitely many hyperplanes. Fix $v \in V \otimes V$ inducing P and $E' \subsetneq V$ inducing E. Fix a basis $e_0, \dots, e_m$ of V such that $e_i \notin E'$ for all i. We may write $v = \sum_{i=0}^m e_i \otimes w_i$ for some $w_i \in V$. Hence $\tilde{t}_{m,m}(P) \le m + 1$. □

Proof of Theorem 4. Lemma 6 gives the case $s = 2$. Hence we may assume $s \ge 3$ and use induction on s. Fix $P \in \mathbb{P}^r$ and a small polybox $B \subset T(m_1, \dots, m_s; 1, \dots, 1)$. For a fixed integer s we also use induction on m_s, starting from the case $m_s = 0$ (in which we use $s - 1$ instead of s).

Take a general hyperplane $L \subset \mathbb{P}^{m_s}$. Set $T(m_1, \dots, m_s; s, L) := T(m_1, \dots, m_{s-1}) \times L \subset T(m_1, \dots, m_s)$, $E := \nu_{1,\dots,1}(T(m_1, \dots, m_s; s, L))$, $F := \langle E\rangle$, and $R := -1 + \prod_{1\le i\le s-1}(m_i + 1)$. We have $\dim(F) = -1 + m_s\prod_{1\le i\le s-1}(m_i + 1)$. Let $\ell : \mathbb{P}^r \setminus F \to \mathbb{P}^R$ denote the linear projection from F. Notice that $F \cap T(m_1, \dots, m_s; 1, \dots, 1) = E$. If $m_s = 1$, then we have $E = T(m_1, \dots, m_{s-1}; 1, \dots, 1)$ and hence we use induction on s to apply Theorem 4 to E. If $m_s \ge 2$, then we use induction on m_s to apply Theorem 4 to E. Set $\ell' := \ell \mid T(m_1, \dots, m_s; 1, \dots, 1) \setminus E$. Notice that ℓ' induces a surjection $\ell' : T(m_1, \dots, m_s; 1, \dots, 1) \setminus E \to T(m_1, \dots, m_{s-1}; 1, \dots, 1)$ (projection onto the first $s - 1$ factors). Let B_1 denote the closure of $\ell'(B\setminus B\cap E)$ in $T(m_1, \dots, m_{s-1}; 1, \dots, 1)$. Since B is a small polybox, B_1 is a small polybox (Remark 5). For general L we may also assume that $B \cap E$ is a small polybox of E. First assume $P \in E$. Since $B \cap E$ is a small polybox of E, the inductive assumption gives the existence of a set $A \subset E\setminus A\cap B$ such that $P \in \langle A\rangle$ and $\sharp(A) \le m_s \times \prod_{1\le i\le s-1, i\ne 2}(m_i + 1)$. Hence $r''_{T(m_1,\dots,m_s;1,\dots,1)}(P) < \prod_{i\ne 2}(m_i+1)$. Now assume $P \notin F$. Hence $\ell(P)$ is defined. Since B_1 is a small polybox, there is $B \subset T(m_1, \dots, m_{s-1}; 1, \dots, 1) \setminus B_1$ such that $\ell(P) \in \langle B\rangle$ and $\sharp(B) \le t''(m_1, \dots, m_{s-1}) \le (m_1+1)\times\prod_{3\le i\le s-1}(m_i+1)$. Since ℓ' is surjective, there is $B_2 \subset E$ such that $\ell'(B_2) = B$. Since $B_2 \cap E = \emptyset$ and $F\cap T(m_1, \dots, m_s; 1, \dots, 1) = E$, we have $B_2\cap F = \emptyset$. Hence ℓ is defined at each point of B_2. Since $P \in \langle B\rangle$ and $\ell(B_2) = B$, there is $O \in F$ such that $P \in \langle\{O\}\cup B_2\rangle$. Since $B\cap E$ is a small polybox, there is $B_3 \subset E \setminus B \cap E$ such that $O \in \langle B_3\rangle$ and $\sharp(B_3) \le t''(m_1, \dots, m_s - 1) \le m_s \times \prod_{1\le i\le s, i\ne 2}(m_i + 1)$. We have $P \in \langle B_2 \cup B_3\rangle$ and $\sharp(B_2 \cup B_3) \le \prod_{i\ne 2}(m_i + 1)$. □

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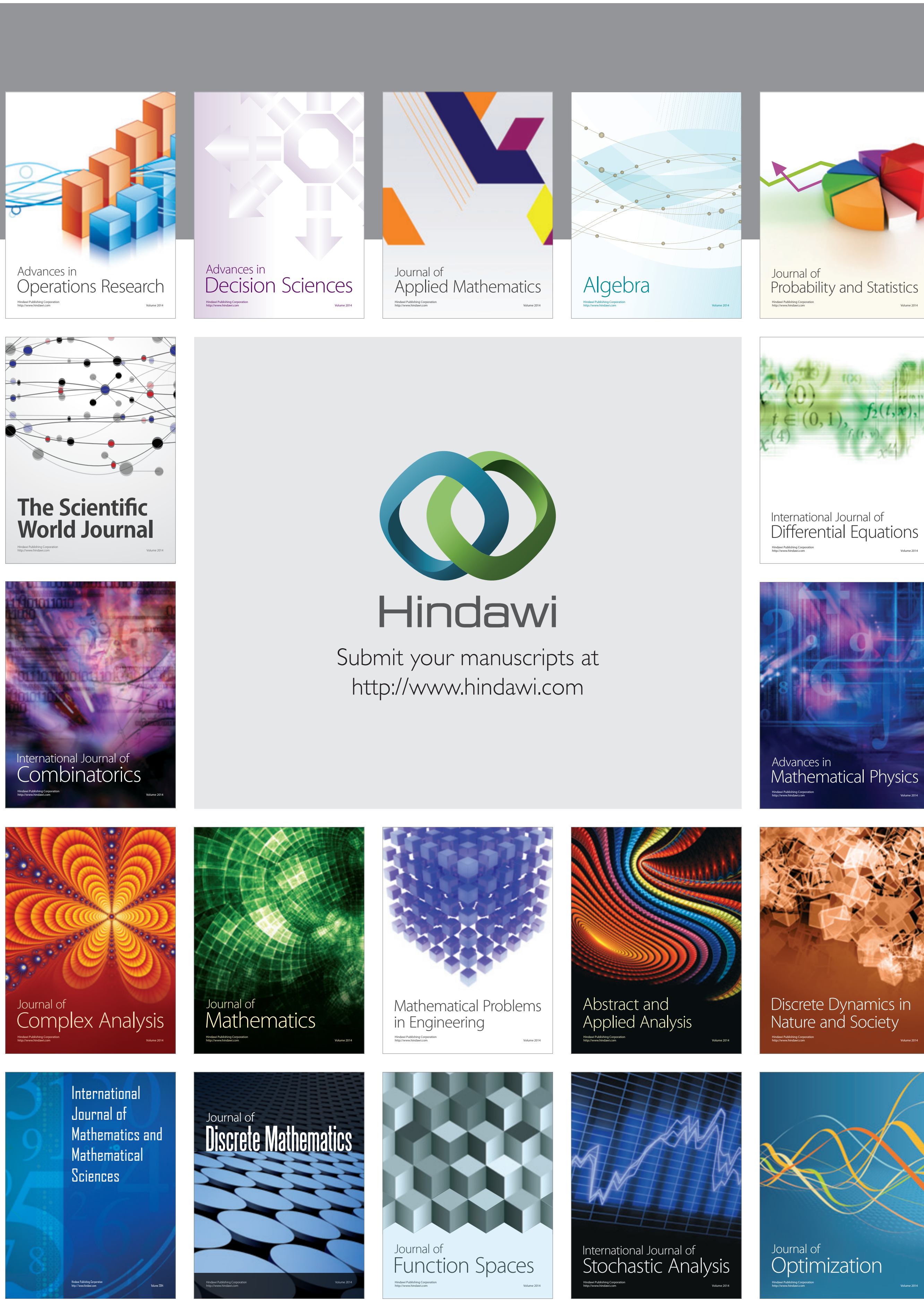
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