Research Article

An Upper Bound for the Tensor Rank

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Let $T$ be a tensor of format $(m_1 + 1) \times \cdots \times (m_s + 1)$, $m_1 \geq \cdots \geq m_s > 0$, over $C$. We prove that $T$ has tensor rank at most $\prod_{i=2}^{s+1}(m_i + 1)$.

1. Introduction

Fix integers $s \geq 2$ and $m_i > 0$, $1 \leq i \leq s$, and an algebraically closed base field $K$. Let $T \in \otimes_{i=1}^{s} K^{m_i+1}$ be a tensor of format $(m_1 + 1) \times \cdots \times (m_s + 1)$ over $K$. The tensor rank $r(T)$ of $T$ is the minimal integer $x \geq 0$ such that $T = \sum_{i=1}^{x} v_{i1} \otimes \cdots \otimes v_{is}$ with $v_{ij} \in K^{m_i+1}$ (see [1–6]). Classical papers (e.g., [7]) continue to suggest new results (see [8]). Let $t(m_1, \ldots, m_s)$ be the maximum of all integers $r(T)$, $T \in \otimes_{i=1}^{s} K^{m_i+1}$. In this paper we prove the following result.

**Theorem 1.** For all integers $s \geq 2$ and $m_1 \geq \cdots \geq m_s > 0$ one has $t(m_1, \ldots, m_s) \leq \prod_{i=2}^{s+1}(m_i + 1)$.

This result is not optimal. It is not sharp when $s = 2$, since $t(m_1, m_2) = m_2 + 1$ by elementary linear algebra. For large $s$ the bound should be even worse. In our opinion to get stronger results one should split the set of all $(s; m_1, \ldots, m_s)$ into subregions. For instance, we think that for large $s$ the cases with $m_1 \gg m_2 \gg \cdots \gg m_s > 0$ and the cases with $m_1 = \cdots = m_s$ are quite different.

We make the definitions in the general setting of the Segre-Veronese embeddings of projective spaces (i.e., of partially symmetric tensors), but we only use the case of the usual Segre embedding, that is, the usual tensor rank. The tensor $T = 0$ has zero as its tensor rank. If $\lambda \in K \setminus \{0\}$, then the tensors $T$ and $\lambda T$ have the same rank. Hence it is sufficient to study the function “tensor rank” on the projectivisation of the vector space $\otimes_{i=1}^{s} K^{m_i+1}$. We may translate the tensor rank and the integer $t(m_1, \ldots, m_s)$ in the following language.

For each subset $A$ of a projective space, let $\langle A \rangle$ denote the linear span of $A$. For each integral variety $Y \subset \mathbb{P}^s$ and any $P \in \langle Y \rangle$ the $Y$-rank $r_Y(P)$ of $P$ is the minimal cardinality of a finite set $A \subset Y$ such that $P \in \langle A \rangle$. Now assume $\langle Y \rangle = \mathbb{P}^s$. The maximal $Y$-rank $r_Y$ is the maximum of all integers $r_Y(P)$, $P \in \mathbb{P}^s$. Fix integers $s > 0$, $m_i > 0$, $1 \leq i \leq s$, and $d_i > 0$, $1 \leq i \leq s$. Set $T(m_1, \ldots, m_s) := \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$.

Let $\nu_{d_1, \ldots, d_s} : T(m_1, \ldots, m_s) \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{s} (m_i + d_i)$ be the Segre-Veronese embedding of multidegree $(d_1, \ldots, d_s)$, that is, the embedding of $T(m_1, \ldots, m_s)$ induced by the $K$-vector space of all polynomials $f \in K[x_{ij}]$, $1 \leq i \leq s$, $0 \leq j \leq m_i$, whose nonzero monomials have degree $d_i$ with respect to the variables $x_{ij}$, $0 \leq j \leq m_i$. Set $T(m_1, m_2; d_1, \ldots, d_s) := \nu_{d_1, \ldots, d_s}(T(m_1, \ldots, m_s))$. The variety $T(m_1, m_2; 1, \ldots, 1)$ is the Segre embedding of $T(m_1, \ldots, m_s)$. Fix $P \in \mathbb{P}^r$, $r := -1 + \prod_{i=1}^{s} (m_i + 1)$. Let $\{A_T\}_{T \in K[x_{ij}] \setminus \{0\}}$ be the set of all nonzero tensors of format $(m_1 + 1) \times \cdots \times (m_s + 1)$ associated with $P$. We have $r(T) = r_{T(m_1, \ldots, m_s)}(P)$. Hence $t(m_1, \ldots, m_s) = r_{T(m_1, \ldots, m_s)}$. To prove Theorem 1 we refine the notion of $Y$-rank in the following way.

**Definition 2.** Fix positive integers $s$, $m_i$, $1 \leq i \leq s$, and $d_i$, $1 \leq i \leq s$. A small box of $T(m_1, \ldots, m_s)$ is a closed set $L_1 \times \cdots \times L_s \subset T(m_1, \ldots, m_s)$ with $L_i$ being a hyperplane of $\mathbb{P}^{m_i}$ for all $i$. A large box $T(m_1, \ldots, m_s)$ is a product $L_1 \times \cdots \times L_s \subset T(m_1, \ldots, m_s)$ such that there is $j \in \{1, \ldots, s\}$ with $L_j \subset \mathbb{P}^{m_j}$ being a hyperplane, while $L_i = \mathbb{P}^{m_i}$ for all $i \neq j$. A small polybox (resp., large polybox) of $T(m_1, \ldots, m_s)$ is a finite union of small (resp., large) boxes of $T(m_1, \ldots, m_s)$. A small box (resp., small polybox, resp., large box, resp., large
polybox) \( B \subset T(m_1, \ldots, m_s; d_1, \ldots, d_s) \) is the image by \( \nu_{d_1, \ldots, d_s} \) of a small box (resp., small polybox, resp., large box, resp., large polybox) of \( T(m_1, \ldots, m_s) \).

**Definition 3.** Fix positive integers \( s, m_i, 1 \leq i \leq s \), and \( d_i, 1 \leq i \leq s \), and set \( r := -1 + \prod_{1 \leq i \leq s} (m_i + d_i) \). Fix \( P \in \mathbb{P}^r \).

The rank \( r_{m_1, m_2, \ldots, m_s}((P) \bigcap T(m_1, m_2, \ldots, m_s; d_1, \ldots, d_s)) \) of \( P \) is the minimal cardinality of a finite set \( A \subset T(m_1, m_2, \ldots, m_s; d_1, \ldots, d_s) \) such that for each finite union \( \#(A) \leq t(\#(C)) \) there is a finite set \( A \subset T(m_1, m_2, \ldots, m_s; d_1, \ldots, d_s) \) with \( \#(A) = t \).

**Lemma 6.** For all integers \( m_i \geq m_2 > 0 \) one has \( t''(m_1, \ldots, m_s) \leq t(m_1, \ldots, m_s) \). Proof. It is sufficient to prove the inequality \( t(m_1, m_2) \leq t(m_1 + 1, m_2) \). Without losing generality we may assume \( m_1 = m_2 = m \). Set \( m := m_1 \) and \( V := \mathbb{P}^{m-1} \). Fix \( P \in \mathbb{P}^r \), \( r = m^2 + 2m \), and a union \( E \subset \mathbb{P}^m \) of finitely many hyperplanes. Fix \( v \in V \otimes V \) inducing \( P \) and \( E' \subset V \) inducing \( E \). Fix a basis \( e_0, \ldots, e_m \) of \( V \) such that \( e_i \notin E' \) for all \( i \). We may write \( v = \sum_{i=0}^m e_i \otimes w_i \) for some \( w_i \in V \). Hence \( t_{m_1, m_2}((P) \bigcap T(m_1, m_2; d_1, d_2)) \leq \prod_{1 \leq i \leq t} (m_i + d_i) \) for all \( i \). Without losing generality we may assume \( P \) and \( E' \subset \mathbb{P}^m \) are a small box (Remark 5). For a fixed integer \( s \) we also use induction on \( m_s \), starting from the case \( m_1 = 0 \) (in which we use \( s \) instead of \( s \)).

**Proof of Theorem 4.** Lemma 6 gives the case \( s = 2 \). Hence we may assume \( s \geq 3 \) and use induction on \( s \). Fix \( P \in \mathbb{P}^r \) and a small polybox \( B \subset T(m_1, \ldots, m_s; 1, \ldots, 1) \). For a fixed integer \( s \) we also use induction on \( m_s \), starting from the case \( m_1 = 0 \) (in which we use \( s \) instead of \( s \)).

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**References**


