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Research Article

Lexicographic Product and Isoperimetric Number

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The *isoperimetric number* of a graph G, denoted by $i(G)$, was introduced by Mohar (1987). A graph G and a subset X of its vertices are given, and let $\partial(X)$ denote the edge boundary of X, the set of edges which connects vertices in X to vertices not in X. The isoperimetric number of G is defined as $i(G) = \min_{1\le|X|\le|V(G)|/2}(|\partial(X)|/|X|)$. In this paper, some results about the isoperimetric number of graphs obtained by graph operations are given.

1. Introduction

Given a graph G and a subset X of its vertices, let $\partial(X)$ denote the edge boundary of X that is, the set of edges which connects vertices in X with vertices not in X. The isoperimetric number is defined as

$$i(G) = \min_{1\le|X|\le|V(G)|/2} \frac{|\partial(X)|}{|X|}. \tag{1}$$

Clearly, $i(G)$ can be defined in a more symmetric form by

$$i(G) = \min \frac{|E(X,Y)|}{\min\{|X|,|Y|\}}, \tag{2}$$

where the minimum runs over all partitions of $V(G) = X \cup Y$ into nonempty subsets X and Y, and $E(X,Y) = \partial X = \partial Y$ are the edges between X and Y.

As examples of isoperimetric numbers, we consider the following.

(i) The isoperimetric number of the complete graph K_n with n vertices is $i(K_n) = \lceil n/2\rceil$.

(ii) The isoperimetric number of the cycle C_n with n vertices is $i(C_n) = 2/\lfloor n/2\rfloor$.

(iii) The isoperimetric number of the path P_n with n vertices is $i(P_n) = 1/\lfloor n/2\rfloor$.

(iv) The isoperimetric number of the complete bipartite graph with $m+n$ vertices $K_{m,n}$ is

$$i(K_{m,n}) = \begin{cases} \dfrac{mn}{m+n}, & \text{if } m \text{ and } n \text{ are even} \\[2ex] \dfrac{mn+1}{m+n}, & \text{if } m \text{ and } n \text{ are odd} \\[2ex] \dfrac{mn}{m+n-1}, & \text{if } m+n \text{ is odd.} \end{cases} \tag{3}$$

It can be briefly shown as $i(K_{m,n}) = \lceil mn/2\rceil/\lfloor (m+n)/2\rfloor$.

The isoperimetric number is also closely related to the notion of bisection width, $bw(G)$, of a graph G. This is the minimum number of edges that must be removed from G in order to split $V(G)$ into two equal-sized (within one if the number of vertices in G is odd) subsets:

$$bw(G) = \min_{|X|=\lfloor|V(G)|/2\rfloor} |\partial X|, \tag{4}$$

where $X \subset V(G)$. If known, one can use the isoperimetric number of a graph G to establish a lower bound for its bisection width using the fact that

$$\frac{bw(G)}{\lfloor|V(G)|/2\rfloor} \ge i(G). \tag{5}$$

See [1].

The importance of $i(G)$ lies in various interesting interpretations of this number by Mohar as follows [2].

(a) From (2), it is evident that, in trying to determine $i(G)$, we have to find a small edge-cut $E(X, Y)$ separating as large a subset X (assume $|X| \leq |Y|$) as possible from the remaining larger part Y. So, it is evident that $i(G)$ can serve as measure of *connectivity* of graphs. It seems that there might be possible applications in problems concerning connected networks,and the ways to "destroy" them are by removing a large portion of the network by cutting only a few edges.

(b) The problem of the partitioning $V(G)$ into two equally sized subsets (to within one element), in such a way that the number of the edges in the cut is minimal, is known as the *bisectionwidth* problem. It is important in VLSI design and some other practical applications. Clearly, it is related to isoperimetric number.

Theorem 1 (see [2]). *Some of the theorems that Mohar stated are given below.*

(a) $i(G) = 0$ *if and only if G is disconnected.*

(b) *If G is k-edge-connected then* $i(G) \geq 2k/|V(G)|$.

(c) *If* $\delta(G)$ *is the minimal degree of vertices in G then* $i(G) \leq \delta(G)$.

(d) *If* $e = uv$ *is an edge of G and* $|V(G)| \geq 4$ *then* $i(G) \leq [\deg(u) + \deg(v) - 2]/2$.

(e) *If* Δ *is the maximum vertex degree in G then* $i(G) \leq (\Delta - 2) + 2/\lfloor |V(G)|/2 \rfloor$. *If G has a cycle with almost half the vertices of G then* $i(G) \leq \Delta - 2$.

If a set $X \subset V(G)$ with $|X| \leq (1/2)|V(G)|$ reaches the minimum $i(G) = |\partial(X)|/|X|$ we call it an isoperimetric set. For $U \subseteq V(G)$ denoted by $G \mid U$, the subgraph of G is induced on U [2].

Proposition 2 (see [2]). *If G is a connected graph then it has an isoperimetric set X such that* $G \mid X$ *and* $G \mid (V \setminus X)$ *are connected subgraphs of G.*

In the next section, we prove a upper bound for isoperimetric number of lexicographic product of graphs.

2. Lexicographic Product

The lexicographic product $G_1[G_2]$ of two graphs G_1 and G_2 has its vertex set $V(G_1) \times V(G_2)$ with (u_1, u_2) adjacent to (v_1, v_2) if either u_1 adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 are adjacent to v_2 in G_2. Note that unlike the union, join, and Cartesian product, this operation is not commutative.

Theorem 3. *Let G be a graph with m vertices, and let q edges and H be a graph with n vertices. Then,*

$$i(G[H]) \leq \min\left\{i(G)\,n, \frac{2q\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}\right\}. \tag{6}$$

Proof. Let $X_G \subseteq V(G)$ and X_G be the isoperimetric set of G and $\partial_G(X)$ edge boundary of G. We know that $G[H]$ includes n copies of G. If $|X| = |X_G|n$ then $|\partial(X)| \leq |\partial_G(X)|nn$. Hence,

$$\frac{|\partial(X)|}{|X|} \leq \frac{|\partial_G(X)|\,nn}{|X_G|\,n} = \frac{|\partial_G(X)|\,nn}{|X_G|\,n} = \frac{|\partial_G(X)|\,n}{|X_G|}. \tag{7}$$

So,

$$\begin{aligned}\min\left\{\frac{|\partial(X)|}{|X|}\right\} &\leq \min\left\{\frac{|\partial_G(X)|\,n}{|X_G|}\right\} \\ &= \min\left\{\frac{|\partial_G(X)|}{|X_G|}\right\} n = i(G)\,n.\end{aligned} \tag{8}$$

Similarly, let $X_H \subseteq V(H)$ with $|X_H| \leq \lfloor n/2\rfloor$ and $|X_H| = rm$. We know that $G[H]$ includes m copies of H. If $|X| = rm$ then we have $|\partial(X)| \leq r(n-r)2q + |\partial_H(X)|m$. Therefore, $|\partial(X)|/|X| \leq (r(n-r)2q + |\partial_H(X)|m)/rm$. So, $\min\{|\partial(X)|/|X|\} \leq \min\{(r(n-r)2q + |\partial_H(X)|m)/rm\}$. The function $(r(n-r)2q+|\partial_H(X)|m)/rm$ takes its minimum value at $r = \lfloor n/2\rfloor$. Since $|X_H| = \lfloor n/2\rfloor$ then $|\partial(X)| = bw(H)$. We have

$$i(G[H]) \leq \frac{2q\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}. \tag{9}$$

The proof is completed by (8) and (9). □

We have $i(G)n \leq 2q\lceil n/2\rceil)/m+bw(H)/\lfloor n/2\rfloor$ according to the upper bounds of $i(G)$, and hence we get $i(G[H]) \leq i(G)n$.

Corollary 4. *Let G be a graph with m vertices, and let q edges and H be a graph with n vertices. If n is even and* $i(G) \leq q/m$ *and* $i(G) \leq 2bw(H)n^2$ *then*

$$i(G)\,n \leq \frac{2q\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}. \tag{10}$$

Proof. If n is even and $i(G) \leq 2bw(H)/n^2$ then $i(G)n^2 \leq 2bw(H)$. Hence, we have $i(G)n^2 + qn^2 \leq 2bw(H) + qn^2$. Since $i(G) \leq q/m$ then $n^2(i(G) + i(G)m) \leq n^2(i(G) + q) \leq i(G)n^2 + qn^2 \leq 2bw(H) + qn^2$. So, we have $i(G)mn^2 \leq n^2(i(G) + i(G)m) \leq 2bw(H) + qn^2$. Therefore, we have

$$\begin{aligned}\frac{i(G)\,mn^2}{mn} \leq \frac{2bw(H) + qn^2}{mn} &= \frac{2q\,(n/2)}{m} + \frac{bw(H)}{n/2} \\ &= \frac{2q\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}.\end{aligned} \tag{11}$$

□

Corollary 5. *Let G be a graph with m vertices, and let q edges and H be a graph with n vertices. If n is odd and* $i(G) \leq 2bw(H)n^2 - n$ *then*

$$i(G)\,n \leq \frac{2q\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}. \tag{12}$$

Corollary 6. *Let T be a tree with m vertices, and let H be a graph with n vertices. If n is even then*

$$i(T)\,n \leq \frac{2(m-1)\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}. \tag{13}$$

Proof. For $n > 1$ and $m > 1$, we know that $0 \le (m-2)n^2 + 2m$. Then, $mn^2 \le (2m-2)n^2 + 2m$. Since $bw(H) \ge 1$ and $i(T) < 1$ then we have $i(T)mn^2 \le mn^2 \le (2m-2)n^2 + 2mbw(H)$. Therefore, we have

$$\begin{aligned} i(T)n &\le \frac{2(m-1)(n/2)}{m} + \frac{bw(H)}{n/2} \\ &= \frac{2(m-1)\lceil n/2\rceil}{m} + \frac{bw(H)}{\lfloor n/2\rfloor}. \end{aligned} \tag{14}$$

□

Corollary 7. *Let T be a tree with m vertices, and let H be a graph with n vertices. If n is odd and $n-1 < m$ then $i(T)n \le 2(m-1)\lceil n/2\rceil/m + bw(H)/\lfloor n/2\rfloor$.*

Corollary 8. *Let G be a graph with m vertices, q edges that is not a tree, and let P_n be a path graph with n vertices. If n is even and $i(G) \le (m-1)/m$ then*

$$i(G)n \le \frac{2q(n/2)}{m} + \frac{1}{(n/2)} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(P_n)}{\lfloor n/2\rfloor}. \tag{15}$$

Proof. If n is even and $i(G) \le (m-1)/m$ then $i(G)m \le m-1$. Thus, $i(G)mn^2 \le (m-1)n^2 \le (m-1)n^2 + 2m$. Therefore, $i(G)mn^2/mn \le ((m-1)n^2 + 2m)/mn = (m-1)n/m + 2/n$. So,

$$\begin{aligned} i(G)n &\le \frac{(m-1)n}{m} + \frac{2}{n} = \frac{2(m-1)n/2}{m} \\ &\quad + \frac{1}{n/2} \le \frac{2q(n/2)}{m} + \frac{1}{n/2} \\ &= \frac{2q\lceil n/2\rceil}{m} + \frac{bw(P_n)}{\lfloor n/2\rfloor}. \end{aligned} \tag{16}$$

□

Corollary 9. *Let G be a graph with m vertices, q edges that is not a tree, and let P_n be a path graph with n vertices. If n is odd and $i(G) \le 2/(n^2-n)$ then*

$$i(G)n \le \frac{2q((n+1)/2)}{m} + \frac{1}{(n-1)/2} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(P_n)}{\lfloor n/2\rfloor}. \tag{17}$$

Corollary 10. *Let G be a graph with m vertices, q edges that is not a tree, and let C_n be a cycle graph with n vertices. If n is even and $i(G) \le (n^2+4)/n^2$ then*

$$i(G)n \le \frac{2q(n/2)}{m} + \frac{2}{n/2} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(C_n)}{\lfloor n/2\rfloor}. \tag{18}$$

Proof. If n is even and $i(G) \le (n^2+4)/n^2$ then

$$\begin{aligned} i(G)n &\le \frac{n^2+4}{n} = \frac{n^2}{n} + \frac{4}{n} = \frac{mn}{m} + \frac{2}{n/2} \\ &= \frac{2mn/2}{m} + \frac{2}{n/2} \le \frac{2q(n/2)}{m} + \frac{2}{n/2}. \end{aligned} \tag{19}$$

So,

$$i(G)n \le \frac{2q(n/2)}{m} + \frac{2}{n/2} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(C_n)}{\lfloor n/2\rfloor}. \tag{20}$$

□

Corollary 11. *Let G be a graph with m vertices, q edges that is not a tree, and let C_n be a cycle graph with n vertices. If n is odd and $i(G) \le (n^2+3)/(n^2-n)$ then*

$$i(G)n \le \frac{2q((n+1)/2)}{m} + \frac{2}{(n-1)/2} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(C_n)}{\lfloor n/2\rfloor}. \tag{21}$$

Corollary 12. *Let G be a graph with m vertices, q edges that is not a tree and let K_n be a complete graph with n vertices.*

(a) *If n is even and $i(G) \le (q/m) + (1/2)$ then*

$$i(G)n \le \frac{2q(n/2)}{m} + \frac{n^2/4}{n/2} = \frac{2q\lceil n/2\rceil}{m} + \frac{bw(K_n)}{\lfloor n/2\rfloor}. \tag{22}$$

(b) *If n is odd and $i(G) \le (n+1)(2q+m)/mn$ then*

$$i(G)n \le \frac{2q\lceil n/2\rceil}{m} + \frac{bw(K_n)}{\lfloor n/2\rfloor}. \tag{23}$$

Proof. (a) If n is even and $i(G) \le (q/m) + (1/2)$ then we have

$$\begin{aligned} i(G)n &\le \frac{qn}{m} + \frac{n}{2} \\ &= \frac{2q(n/2)}{m} + \frac{n^2/4}{n/2} \\ &= \frac{2q\lceil n/2\rceil}{m} + \frac{bw(K_n)}{\lfloor n/2\rfloor}. \end{aligned} \tag{24}$$

The proof for n even is very similar to the proof for odd. □

Theorem 13. *Let P_m be a path graph with m vertices, and let q edges and H be a graph with n vertices. Then,*

$$i(P_m[H]) = \begin{cases} \dfrac{2n}{m}, & \text{if } m \text{ is even} \\ \dfrac{2n^2+2bw(H)}{mn}, & \text{if } m \text{ is odd and } n \text{ is even and } bw(H)(m-1) < n^2 \\ \dfrac{2n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } bw(H)(m-1) \ge n^2 \\ \dfrac{2n^2+2bw(H)}{mn-1}, & \text{if } m \text{ and } n \text{ are odd and } bw(H)(m-1) < n^2-n \\ \dfrac{2n}{m-1}, & \text{if } m \text{ and } n \text{ are even and } bw(H)(m-1) \ge n^2-n. \end{cases} \tag{25}$$

Proof. Let $V(P_m) = \{1, 2, \dots, m\}$ and $X \subseteq V(P_m[H])$ with $|X| \le \lfloor mn/2\rfloor$. For $i = 1, 2, \dots, m$ let $X_i = X \cap (V(H) \times i)$. Hence, X is the disjoint union of $X_1, X_2, \dots, X_m$. Let $S_0 = \{X_i \mid |X_i| = 0, 1 \le i \le m\}$ and $S_n = \{X_i \mid |X_i| = n, 1 \le i \le m\}$. To prove this theorem, we have three cases.

Case 1. Let m be an even integer. To prove this case, we have three subcases.

Subcase 1.1. If $|X| = r$ where $1 \le r < n$ then $|\partial(X)| \ge rn + 1$. Therefore, $|\partial(X)|/|X| \ge (rn+1)/r$. The function $(rn+1)/r$ has its minimum value at $r = n-1$, and we have

$$\min\left\{\frac{|\partial(X)|}{|X|}\right\} \ge \frac{(n-1)n+1}{n-1}. \tag{26}$$

Subcase 1.2. If $S_0 > 0$ and $S_n > 0$ and $|X| = r$ where $n \le r \le mn/2$ then $|\partial(X)| \ge n^2 + (m - S_0 - S_n)|\partial_H(X)|$. Therefore, $|\partial(X)|/|X| \ge (n^2 + (m - S_0 - S_n)|\partial_H(X)|)/r$. The function $(n^2 + (m - S_0 - S_n)|\partial_H(X)|)/r$ has its minimum value at $r = mn/2$. If $r = mn/2$ then $(m - S_0 - S_n)|\partial_H(X)| = 0$. Thus,

$$\min\left\{\frac{|\partial(X)|}{|X|}\right\} \ge \frac{2n}{m}. \tag{27}$$

Subcase 1.3. If $(S_0 = 0, S_n = 0)$ or $(S_0 = 0, S_n > 0)$ or $(S_0 > 0, S_n = 0)$ and $|X| = r$ where $n \le r \le mn/2$ then

$$\begin{aligned}|\partial(X)| &\ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)2q + \left|\partial_H(X)\right|m \\ &\ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)2(m-1) \\ &\quad + \left|\partial_H(X)\right|m \ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)(2m-2) + m.\end{aligned} \tag{28}$$

Thus, $|\partial(X)|/|X| \ge (\lfloor r/m\rfloor(n - \lfloor r/m\rfloor)(2m-2) + m)/r$. The function $(\lfloor r/m\rfloor(n - \lfloor r/m\rfloor)(2m-2) + m)/r$ has its minimum value at $r = mn/2$, and we have

$$\begin{aligned}\min\left\{\frac{|\partial(X)|}{|X|}\right\} &\ge \frac{\lfloor(mn/2)/m\rfloor(n - \lfloor(mn/2)/m\rfloor)(2m-2) + m}{mn/2} \\ &= \frac{n^2m - n^2 + 2m}{mn}.\end{aligned} \tag{29}$$

By (26), (27), and (29), if m is even then $i(P_m[H]) = (2n/m)$.

Case 2. Let m be an odd, and let n be an even integer. To prove this case, we have four subcases.

Subcase 2.1. If $|X| = r$ where $1 \le r < n$ then $|\partial(X)| \ge rn + 1$. Therefore, $|\partial(X)|/|X| \ge (rn+1)/r$. The function $(rn+1)/r$ has its minimum value at $r = n-1$, and we have

$$\min\left\{\frac{|\partial(X)|}{|X|}\right\} \ge \frac{(n-1)n+1}{n-1}. \tag{30}$$

Subcase 2.2. If $S_0 > 0$ and $S_n > 0$ and $|X| = r$ where $n \le r \le (m-1)n/2$ then $|\partial(X)| \ge n^2 + (m - S_0 - S_n)|\partial_H(X)|$. Therefore, $|\partial(X)|/|X| \ge (n^2 + (m - S_0 - S_n)|\partial_H(X)|)/r$. The function $(n^2 + (m - S_0 - S_n)|\partial_H(X)|)/r$ has its minimum value at $r = (m-1)n/2$. If $r = (m-1)n/2$ then $(m - S_0 - S_n)|\partial_H(X)| = 0$. Thus,

$$\min\left\{\frac{|\partial(X)|}{|X|}\right\} \ge \frac{n^2}{(m-1)n/2} = \frac{2n}{m-1}. \tag{31}$$

Subcase 2.3. If $S_0 > 0$ and $S_n > 0$ and $|X| = r$ where $(m-1)n/2 \le r \le mn/2$ then $|\partial(X)| = n^2 + bw(H)$. Therefore, $|\partial(X)|/|X| = (n^2 + bw(H))/r$. The function $(n^2 + bw(H))/r$ has its minimum value at $r = mn/2$. Thus,

$$\min\left\{\frac{|\partial(X)|}{|X|}\right\} = \frac{n^2 + bw(H)}{mn/2} = \frac{2n^2 + 2bw(H)}{mn}. \tag{32}$$

Subcase 2.4. If $(S_0 = 0, S_n = 0)$ or $(S_0 = 0, S_n > 0)$ or $(S_0 > 0, S_n = 0)$ and $|X| = r$ where $n \le r \le mn/2$ then

$$\begin{aligned}|\partial(X)| &\ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)2q + \left|\partial_H(X)\right|m \\ &\ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)2(m-1) + \left|\partial_H(X)\right|m \\ &\ge \left\lfloor\frac{r}{m}\right\rfloor\left(n - \left\lfloor\frac{r}{m}\right\rfloor\right)(2m-2) + m.\end{aligned} \tag{33}$$

Thus, $|\partial(X)|/|X| \ge (\lfloor r/m\rfloor(n - \lfloor r/m\rfloor)(2m-2) + m)/r$. The function $(\lfloor r/m\rfloor(n - \lfloor r/m\rfloor)(2m-2) + m)/r$ takes minimum value at $r = mn/2$, and we have

$$\begin{aligned}\min\left\{\frac{|\partial(X)|}{|X|}\right\} &\ge \frac{\lfloor(mn/2)/m\rfloor(n - \lfloor(mn/2)/m\rfloor)(2m-2) + m}{mn/2} \\ &= \frac{n^2m - n^2 + 2m}{mn}.\end{aligned} \tag{34}$$

By (30), (31), (32), and (34), we have that if m is odd and n is even and $bw(H)(m-1) < n^2$ then $i(P_m[H]) = (2n^2 + 2bw(H))/mn$, and if m is odd and n is even and $bw(H)(m-1) \ge n^2$ then $i(P_m[H]) = 2n/(m-1)$.

Case 3. The proofs of the case in which m and n are odd are similar to that of Case 2. □

The isoperimetric number of $i(P_m[P_n])$ is given in the following corollary.

Corollary 14. *Let m and n be positive integers. Then,*

$$i(P_m[P_n]) = \begin{cases} \dfrac{2n}{m}, & \text{if } m \text{ is even} \\ \dfrac{2n^2+2}{mn}, & \text{if } m \text{ is odd and } n \text{ is even and } (m-1) < n^2 \\ \dfrac{2n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } (m-1) \ge n^2 \\ \dfrac{2n^2+2}{mn-1}, & \text{if } m \text{ is odd and } n \text{ is odd and } (m-1) < n^2 - n \\ \dfrac{2n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } (m-1) \ge n^2 - n. \end{cases} \tag{35}$$

The isoperimetric number of $i(P_m[C_n])$ is given in the following corollary.

Corollary 15. *Let $m > 5$ and n be positive integers. Then,*

$$i(P_m[C_n]) = \begin{cases} \dfrac{2n}{m}, & \text{if } m \text{ is even} \\ \dfrac{2n^2+4}{mn}, & \text{if } m \text{ is odd and } n \text{ is even and } 2(m-1) < n^2 \\ \dfrac{2n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } 2(m-1) \geq n^2 \\ \dfrac{2n^2+4}{mn-1}, & \text{if } m \text{ is odd and } n \text{ is odd and } 2(m-1) < n^2 - n \\ \dfrac{2n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } 2(m-1) \geq n^2 - n. \end{cases} \quad (36)$$

The isoperimetric number of $i(C_m[P_n])$ is given in the following corollary.

Corollary 16. *Let $m > 5$ and n be positive integers. Then,*

$$i(C_m[P_n]) = \begin{cases} \dfrac{4n}{m}, & \text{if } m \text{ and } n \text{ are even and } m > \dfrac{4n^2}{n^2+2} \\ \dfrac{4n^2+2}{mn}, & \text{if } m \text{ is odd and } n \text{ is even and } m \leq 2n^2+1 \\ \dfrac{4n}{m-1}, & \text{if } m \text{ is odd and } n \text{ is even and } m > 2n^2+1 \\ \dfrac{n^2-1}{n}, & \text{if } m \text{ is even and } n \text{ is odd and } m \leq \dfrac{4n^2}{n^2-1} \\ \dfrac{4n}{m}, & \text{if } m \text{ is even and } n \text{ is odd and } m > \dfrac{4n^2}{n^2-1} \\ \dfrac{4n^2}{(m-1)n}, & \text{if } m \text{ is odd and } n \text{ is odd and } m \leq 2n^2-2n+1 \\ \dfrac{4n^2+2}{mn-1}, & \text{if } m \text{ is odd and } n \text{ is odd and } m > 2n^2-2n+1. \end{cases} \quad (37)$$

The isoperimetric number of $i(K_m[P_n])$ is given in the following corollary.

Corollary 17. *Let m and n be positive integers. Then,*

$$i(K_m[P_n]) = \begin{cases} \dfrac{n^2m-n^2+4}{2n}, & \text{if } n \text{ is even} \\ \dfrac{n^2m^2-n^2m+3m}{2mn}, & \text{if } m \text{ is even and } n \text{ is odd} \\ \dfrac{n^2m^2-n^2m+5m-1}{2(mn-1)}, & \text{if } m \text{ and } n \text{ are odd.} \end{cases} \quad (38)$$

The isoperimetric number of $i(K_{1,m}[P_n])$ is given in the following corollary.

Corollary 18. *Let m and n be positive integers. Then,*

$$i(K_{1,m}[P_n]) = \begin{cases} \dfrac{n^2m+2m+2}{(m+1)n}, & \text{if } n \text{ is even and } 2m+2 < n^2 \\ n, & \text{if } n \text{ is even and } 2m+2 \geq n^2 \\ \dfrac{n^2m+2m+3}{(m+1)n}, & \text{if } n \text{ and } m \text{ are odd and } 2m+3 < n^2 \\ n, & \text{if } n \text{ and } m \text{ are odd and } 2m+3 \geq n^2 \\ \dfrac{n^2m+2m+2}{(m+1)n-1}, & \text{if } n \text{ is odd and } m \text{ is even and } 2m+2 < n^2-n \\ n, & \text{if } n \text{ is odd and } m \text{ is even and } 2m+2 \geq n^2-n. \end{cases} \quad (39)$$

The isoperimetric number of $i(K_{1,m}[C_n])$ is given in the following corollary.

Corollary 19. *Let m and n be positive integers. Then,*

$$i(K_{1,m}[C_n]) = \begin{cases} \dfrac{n^2m+4m+4}{(m+1)n}, & \text{if } n \text{ is even and } 4m+4 < n^2 \\ n, & \text{if } n \text{ is even and } 4m+4 \geq n^2 \\ \dfrac{n^2m+4m+5}{(m+1)n}, & \text{if } n \text{ and } m \text{ are odd and } 4m+5 < n^2 \\ n, & \text{if } n \text{ and } m \text{ are odd and } 4m+5 \geq n^2 \\ \dfrac{n^2m+4m+4}{(m+1)n-1}, & \text{if } n \text{ is odd and } m \text{ is even and } 4m+4 < n^2-n \\ n, & \text{if } n \text{ is odd and } m \text{ is even and } 4m+4 \geq n^2-n. \end{cases} \quad (40)$$

References

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