Research Article

Asymptotic Stability of Solutions to a Nonlinear Urysohn Quadratic Integral Equation

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Here, we prove the existence of L_1 -nondecreasing solution to a nonlinear quadratic integral equation of Urysohn type by applying the technique of weak noncompactness. Also, the asymptotic stability of solutions for that quadratic integral equation is studied.

1. Introduction

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics, and contact mixed problems, and the theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology, and fixed point theory. For details, we refer to [1–23].

Quadratic integral equations often appear in many applications of real world problems, for example, in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, and in the traffic theory (see [12]). The quadratic integral equation can be very often encountered in many applications (see [1, 2, 6–10, 13–26]). However, in most of the previous literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness, the Tychonoff fixed point theorem is used for some quadratic integral equations [20, 26]. Picard and Adomian decomposition methods are used to compare approximate and exact solutions for quadratic integral equations [13, 19, 22]. Also, nondecreasing solution of a quadratic integral of Urysohn-Stieltjes type is studied in [10].

Let $L_1 = L_1[0, T]$ be the class of Lebesgue integrable functions on $I = [0, T]$ with the standard norm.

Here, we are concerned with the nonlinear quadratic functional integral equation

$$x(t) = f(t, x(\phi_1(t))) + g(t, x(\phi_2(t))) \times \int_0^{\alpha(t)} u(t, s, x(\phi_3(s))) ds, \quad t \in I, \quad (1)$$

and we prove the existence of monotonic solutions in L_1 by using the technique of measure of noncompactness. The results of this work generalize those obtained in [18]. Finally, the asymptotic stability of solutions for the quadratic integral equation (1) is studied.

2. Preliminaries

In this section, we collect some definitions and results needed in our further investigations. Assume that the function $f : I \times R \rightarrow R$ satisfies Carathéodory condition that is measurable in t for any x and continuous in x for almost all t . Then, to every function $x(t)$ being measurable on the interval I , we may assign the function

$$(Fx)(t) = f(t, x(t)), \quad t \in I. \quad (2)$$

The operator F defined in such a way is called the superposition operator. This operator is one of the simplest and most important operators investigated in the nonlinear functional

analysis. For this operator, we have the following theorem due to Krasnosel'skii [3].

Theorem 1. *The superposition operator F maps L_1 into itself if and only if*

$$|f(t, x)| \leq c(t) + k|x| \quad \forall t \in I \quad (3)$$

and $x \in R$, where $c(t)$ is a function from L_1 and k is a nonnegative constant.

Now, let E be a Banach space with zero element θ and X a nonempty bounded subset of E . Moreover denote by $B_r = B(\theta, r)$ the closed ball in E centered at θ and with radius r . In the sequel, we will need some criteria for compactness in measure; the complete description of compactness in measure was given by Banaś [3], but the following sufficient condition will be more convenient for our purposes (see [3]).

Theorem 2. *Let X be a bounded subset of L_1 . Assume that there is a family of subsets $(\Omega_c)_{0 \leq c \leq b-a}$ of the interval (a, b) such that $\text{meas } \Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \leq x(t_2)$, $t_1 \in \Omega_c$, $t_2 \notin \Omega_c$; then, the set X is compact in measure.*

The measure of weak noncompactness defined by De Blasi [11, 27] is given by

$$\beta(X) = \inf \left(r > 0; \text{there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + K_r \right). \quad (4)$$

The function $\beta(X)$ possesses several useful properties which may be found in [11].

The convenient formula for the function $\beta(X)$ in L_1 was given by Appell and De Pascale (see [27]) as follows:

$$\beta(X) = \lim_{\epsilon \rightarrow 0} \left(\sup_{x \in X} \left(\sup \left[\int_D |x(t)| dt : D \subset [a, b], \text{meas } D \leq \epsilon \right] \right) \right), \quad (5)$$

where the symbol $\text{meas } D$ stands for Lebesgue measure of the set D .

Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see [3]) defined by

$$\chi(X) = \inf \left(r > 0; \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + K_r \right). \quad (6)$$

In the case when the set X is compact in measure, the Hausdorff and De Blasi measures of noncompactness will be identical. Namely, we have the following (see [11, 27]).

Theorem 3. *Let X be an arbitrary nonempty bounded subset of L_1 . If X is compact in measure, then $\beta(X) = \chi(X)$.*

Finally, we will recall the fixed point theorem due to Banaś [5].

Theorem 4. *Let Q be a nonempty, bounded, closed, and convex subset of E , and let $H : Q \rightarrow Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ ; that is, there exists a constant $\alpha \in [0, 1)$ such that $\chi(HX) \leq \alpha\chi(X)$ for any nonempty subset X of Q . Then, H has at least one fixed point in the set Q .*

3. Existence Theorem

Let the integral operator H be defined as

$$(Hx)(t) = \int_0^{\alpha(t)} u(t, s, x(s)) ds, \quad (7)$$

$$(Fx)(t) = f(t, x(t)).$$

Then, (1) may be written in operator form as

$$(Ax)(t) = (Fx(\phi_1))(t) + (Gx(\phi_2))(t) \cdot (Hx(\phi_3))(t), \quad (8)$$

where $(Gx)(t) = g(t, x(t))$.

Consider the following assumptions.

- (i) $f, g : I \times R \rightarrow R$ are functions such that $f, g : I \times R_+ \rightarrow R_+$. Moreover, the functions f, g satisfy Carathéodory condition (i.e., are measurable in t for all $x \in R$ and continuous in x for all $t \in I$), and there exist two functions $a_1, a_2 \in L_1$ and constants $b_1, b_2 > 0$ such that

$$|f(t, x)| \leq a_1(t) + b_1|x|, \quad |g(t, x)| \leq a_2(t) + b_2|x| \quad \forall (t, x) \in I \times R. \quad (9)$$

Apart from this, the functions f and g are nondecreasing in both variables.

- (ii) $u : I \times I \times R \rightarrow R$ is such that $u(t, s, x) \geq 0$ for $(t, s, x) \in I \times I \times R_+$, and $u(t, s, x)$ satisfies Carathéodory condition (i.e., it is measurable in (t, s) for all $x \in R$ and continuous in x for almost all $(t, s) \in I \times I$).
- (iii) There exist a positive constant b_3 , a function $a_3 \in L_1$, and a measurable (in both variables) function $k(t, s) = k : I \times I \rightarrow R_+$ such that

$$|u(t, s, x)| \leq k(t, s)(a_3(t) + b_3|x|) \quad \forall t, s \in I \text{ and for } x \in R, \quad (10)$$

and the integral operator K , generated by the function k and defined by

$$(Kx)(t) = \int_0^t k(t, s)x(s) ds, \quad t \in I, \quad (11)$$

maps continuously L_1 into L_∞ on I .

- (iv) $t \rightarrow u(t, s, x)$ is a.e. nondecreasing on I for almost all fixed $s \in I$ and for each $x \in R_+$.
- (v) $\alpha : I \rightarrow I$ is continuous.

(vi) $\phi_i : I \rightarrow I, i = 1, 2, 3$, are increasing, absolutely continuous functions on I , and there exist positive constants $B_i, i = 1, 2, 3$, such that $\phi'_i \geq B_i$ a.e. on I .

(vii) Let $d > \sqrt{4Mb_2b_3B_1^2B_2B_3(\|a_1\| + M \cdot \|a_2\| \|a_3\|)}$,
 $M = \|K\|_{L_\infty}$, where $d = B_1B_2B_3 - b_1B_2B_3 - b_2MB_1B_3\|a_3\| - Mb_3B_1B_2\|a_2\|$.

Now, let r be a positive root of the equation

$$b_2b_3B_1Mr^2 - dr + B_1B_2B_3(\|a_1\| + M \cdot \|a_2\| \|a_3\|) = 0, \quad (12)$$

and define the set

$$B_r = \{x \in L_1 : \|x\| \leq r\}. \quad (13)$$

For the existence of at least one L_1 -positive solution of the quadratic functional integral equation (1), we have the following theorem.

Theorem 5. *Let the assumptions (i)–(vii) be satisfied.*

If $b_1B_2B_3 + Mb_2B_1B_3\|a_3\| + rb_2b_3M < B_1B_2B_3$, then the quadratic integral equation (1) has at least one solution $x \in L_1$ which is positive and a.e. nondecreasing on I .

Proof. Take an arbitrary $x \in L_1$; then, we get

$$\begin{aligned} |(Ax)(t)| &\leq |a_1(t)| + b_1|x(\phi_1(t))| + (a_2(t) + b_2|x(\phi_2(t))|) \\ &\quad \times \int_0^{\alpha(t)} k(t,s)(a_3(s) + b_3|x(\phi_3(s))|) ds, \end{aligned} \quad (14)$$

which implies that

$$\begin{aligned} \|(Ax)(t)\| &= \int_0^T |(Ax)(t)| dt \\ &\leq \int_0^T |a_1(t)| dt + b_1 \int_0^T |x(\phi_1(t))| dt \\ &\quad + \int_0^T (a_2(t) + b_2|x(\phi_2(t))|) \\ &\quad \times \int_0^{\alpha(t)} k(t,s)(a_3(s) + b_3|x(\phi_3(s))|) ds dt \\ &\leq \|a_1\| + \frac{b_1}{B_1} \int_0^T |x(\phi_1(t))| \cdot \phi'_1(t) dt \\ &\quad + \int_0^T a_2(t) \int_0^{\alpha(t)} k(t,s) a_3(s) ds dt \\ &\quad + b_2 \int_0^T |x(\phi_2(t))| \int_0^{\alpha(t)} k(t,s) a_3(s) ds dt + b_3 \\ &\quad \times \int_0^T a_2(t) \int_0^{\alpha(t)} k(t,s) |x(\phi_3(s))| ds dt \\ &\quad + b_2b_3 \int_0^T |x(\phi_2(t))| \int_0^{\alpha(t)} k(t,s) |x(\phi_3(s))| ds dt \end{aligned}$$

$$\begin{aligned} &\leq \|a_1\| + \frac{b_1}{B_1} \int_0^T |x(\phi_1(t))| \cdot \phi'_1(t) dt \\ &\quad + \int_0^T a_2(t) \int_0^T k(t,s) a_3(s) ds dt \\ &\quad + b_2 \int_0^T |x(\phi_2(t))| \int_0^T k(t,s) a_3(s) ds dt + b_3 \\ &\quad \times \int_0^T a_2(t) \int_0^T k(t,s) |x(\phi_3(s))| ds dt \\ &\quad + b_2b_3 \int_0^T |x(\phi_2(t))| \int_0^T k(t,s) |x(\phi_3(s))| ds dt \\ &\leq \|a_1\| + \frac{b_1}{B_1} \int_{\phi_1(0)}^{\phi_1(T)} |x(\phi_1(t))| \cdot \phi'_1(t) dt \\ &\quad + \int_0^T a_2(t) \int_0^T k(t,s) a_3(s) ds dt \\ &\quad + \frac{b_2}{B_2} \int_{\phi_2(0)}^{\phi_2(T)} |x(\phi_2(t))| \cdot \phi'_2(t) \int_0^T k(t,s) a_3(s) ds dt \\ &\quad + \frac{b_3}{B_3} \int_0^T a_2(t) \int_{\phi_3(0)}^{\phi_3(T)} k(t,s) |x(\phi_3(s))| \cdot \phi'_3(s) ds dt \\ &\quad + \frac{b_2b_3}{B_2B_3} \int_{\phi_2(0)}^{\phi_2(T)} |x(\phi_2(t))| \cdot \phi'_2(t) \\ &\quad \times \int_{\phi_3(0)}^{\phi_3(T)} k(t,s) |x(\phi_3(s))| \cdot \phi'_3(s) ds dt \\ &\leq \|a_1\| + \frac{b_1}{B_1} \int_0^T |x(\theta)| d\theta \\ &\quad + M \|a_2\| \|a_3\| + \frac{Mb_2\|a_3\|}{B_2} \int_0^T |x(\theta)| d\theta \\ &\quad + \frac{Mb_3\|a_2\|}{B_3} \int_0^T |x(\theta)| d\theta \\ &\quad + \frac{Mb_2b_3}{B_2B_3} \int_0^T |x(\theta)| d\theta \cdot \int_0^T |x(\theta)| d\theta \\ &\leq \|a_1\| + \frac{b_1}{B_1} \|x\| + M \|a_2\| \|a_3\| + \frac{Mb_2\|a_3\|}{B_2} \|x\| \\ &\quad + \frac{Mb_3\|a_2\|}{B_3} \|x\| + \frac{Mb_2b_3}{B_2B_3} \|x\|^2. \end{aligned} \quad (15)$$

From this estimate, we show that the operator A maps the ball B_r into itself with

$$r = \frac{d - \sqrt{d^2 - 4Mb_2b_3B_1^2B_2B_3(\|a_1\| + M \cdot \|a_2\| \|a_3\|)}}{2Mb_2b_3B_1}, \quad (16)$$

From assumption (vii) we have

$$0 < d^2 - 4Mb_2b_3B_1^2B_2B_3(\|a_1\| + M \cdot \|a_2\| \|a_3\|) < d^2, \quad (17)$$

which implies that

$$0 < \sqrt{d^2 - 4Mb_2b_3B_1^2B_2B_3(\|a_1\| + M \cdot \|a_2\| \|a_3\|)} < d. \quad (18)$$

Then, d is positive which implies that r is a positive constant.

Now, let Q_r denote the subset of $B_r \in L_1$ consisting of all functions which are positive and a.e. nondecreasing on I .

The set Q_r is nonempty, bounded, convex, and closed (see [3, page 780]). Moreover, this set is compact in measure (see Lemma 2 in [4, page 63]).

From the assumptions, we deduce that the operator A maps Q_r into itself. Since the operator $(Ux)(t) = u(t, s, x)$ is continuous (Theorem 1 in Section 2), then the operator H is continuous, and, hence, the product $G.H$ is continuous. Also, F is continuous. Thus, the operator A is continuous on Q_r .

Let X be a nonempty subset of Q_r . Fix $\epsilon > 0$, and take a measurable subset $D \subset I$ such that $\text{meas } D \leq \epsilon$. Then, for any $x \in X$, using the same reasoning as in [3, 4], we get

$$\begin{aligned} & \|Ax\|_{L_1(D)} \\ &= \int_D |(Ax)(t)| dt \\ &\leq \int_D |a_1(t)| dt + b_1 \int_D |x(\phi_1(t))| dt \\ &\quad + \int_D (a_2(t) dt + b_2 |x(\phi_2(t))|) \\ &\quad \times \int_0^{\alpha(t)} k(t, s) (a_3(s) + b_3 |x(\phi_3(s))|) ds dt \\ &\leq \|a_1\|_{L_1(D)} + \frac{b_1}{B_1} \int_D |x(\phi_1(t))| \cdot \phi_1'(t) dt \\ &\quad + \int_D a_2(t) \int_0^T k(t, s) a_3(s) ds dt \\ &\quad + b_2 \int_D |x(\phi_2(t))| \int_0^T k(t, s) a_3(s) ds dt + Mb_3 \\ &\quad \times \int_D a_2(t) \int_0^T |x(\phi_3(s))| ds dt \\ &\quad + Mb_2b_3 \int_D |x(\phi_2(t))| \int_0^T |x(\phi_3(s))| ds dt \end{aligned}$$

$$\begin{aligned} &\leq \|a_1\|_{L_1(D)} + \frac{b_1}{B_1} \int_D |x(\theta)| d\theta + M\|a_2\|_{L_1(D)}\|a_3\|_{L_1} \\ &\quad + \frac{Mb_2\|a_3\|_{L_1}}{B_2} \int_D |x(\theta)| d\theta \\ &\quad + \frac{Mb_3\|a_2\|_{L_1(D)}}{B_3} \int_0^T |x(\theta)| d\theta \\ &\quad + \frac{Mb_2b_3}{B_2B_3} \int_D |x(\theta)| d\theta \cdot \int_0^T |x(\theta)| d\theta \\ &\leq \|a_1\|_{L_1(D)} + \frac{b_1}{B_1} \|x\|_{L_1(D)} + M\|a_2\|_{L_1(D)}\|a_3\|_{L_1} \\ &\quad + \frac{Mb_2\|a_3\|_{L_1}}{B_2} \|x\|_{L_1(D)} \\ &\quad + \frac{Mb_3\|a_2\|_{L_1(D)}}{B_3} \|x\| + \frac{Mb_2b_3}{B_2B_3} \|x\|_{L_1(D)} \cdot \|x\| \\ &\leq \|a_1\|_{L_1(D)} + \frac{b_1}{B_1} \|x\|_{L_1(D)} + M\|a_2\|_{L_1(D)}\|a_3\|_{L_1} \\ &\quad + \frac{Mb_2\|a_3\|_{L_1}}{B_2} \|x\|_{L_1(D)} \\ &\quad + \frac{rMb_3\|a_2\|_{L_1(D)}}{B_3} + \frac{rMb_2b_3}{B_2B_3} \|x\|_{L_1(D)}. \end{aligned} \quad (19)$$

Since

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup \left\{ \int_D |a_i(t)| dt : D \subset I, \text{meas } D < \epsilon \right\} \right\} = 0, \quad (20)$$

$i = 1, 2,$

we obtain

$$\beta(Ax(t)) \leq \left[\frac{b_1}{B_1} + \frac{Mb_2\|a_3\|_{L_1}}{B_2} + \frac{Mb_2b_3r}{B_2B_3} \right] \beta(x(t)). \quad (21)$$

This implies that

$$\beta(AX) \leq \left[\frac{b_1}{B_1} + \frac{Mb_2\|a_3\|_{L_1}}{B_2} + \frac{Mb_2b_3r}{B_2B_3} \right] \beta(X), \quad (22)$$

where β is the De Blasi measure of weak noncompactness.

Keeping in mind Theorem 3, we can write (22) in the form

$$\chi(AX) \leq \left[\frac{b_1}{B_1} + \frac{Mb_2\|a_3\|_{L_1}}{B_2} + \frac{Mb_2b_3r}{B_2B_3} \right] \chi(X), \quad (23)$$

where χ is the Hausdorff measure of noncompactness.

Since $(b_1/B_1) + (Mb_2\|a_3\|_{L_1}/B_2) + (Mb_2b_3r/B_2B_3) < 1$, from Theorem 4 follows that A is contraction with respect to the measure of noncompactness χ . Thus, A has at least one fixed point in Q_r which is a solution of the quadratic functional integral equation. $\square$

4. Asymptotic Stability of the Quadratic Integral Equation

We shall show that the solution of the quadratic integral equation (1) is asymptotically stable on $\mathbb{R}_+$.

Definition 6. The function x is said to be asymptotically stable solution of (1) if for any $\epsilon > 0$ there exists $T' = T'(\epsilon) > 0$ such that for every $t \geq T'$ and for every other solution y of (1),

$$|x(t) - y(t)| \leq \epsilon. \quad (24)$$

Proof. Let r be defined by (16), and consider the following assumptions.

(*) There exist constants l_1 and l_2 satisfying that

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq l_1 |x - y|, \\ |g(t, x) - g(t, y)| &\leq l_2 |x - y|, \\ \forall t \in I, x, y \in \mathbb{R}_+. \end{aligned} \quad (25)$$

$$\begin{aligned} (**) \quad &2MB_1B_2B_3\|a_2\|\|a_3\| + 2rb_2MB_1B_3\|a_2\| + \\ &2rMb_3B_1B_2\|a_2\| + 2Mr^2b_2b_3B_1 < \epsilon(B_1B_2B_3 - \\ &l_1B_2B_3 - Ml_2B_1B_3\|a_3\| - Mrb_2b_3B_1). \end{aligned}$$

For solutions $x = x(t)$ and $y = y(t)$ of (1) in B_r , by the assumptions (*) and (**), we deduce that

$$\begin{aligned} &|x(t) - y(t)| \\ &= |(Ax)(t) - (Ay)(t)| \\ &\leq |f(t, x(\phi_1(t))) - f(t, y(\phi_1(t)))| \\ &\quad + |g(t, x(\phi_2(t))) - g(t, y(\phi_2(t)))| \\ &\quad \times \int_0^{\alpha(t)} |u(t, s, y(\phi_3(s)))| ds \\ &\quad + |g(t, y(\phi_2(t)))| \\ &\quad \times \int_0^{\alpha(t)} |u(t, s, y(\phi_3(s))) - u(t, s, x(\phi_3(s)))| ds \\ &\leq l_1 |x(\phi_1(t)) - y(\phi_1(t))| + l_2 |x(\phi_2(t)) - y(\phi_2(t))| \\ &\quad \times \int_0^T k(t, s) (a_3(s) + b_3 |y(\phi_3(s))|) ds \\ &\quad + 2(a_2(t) + b_2 |y(\phi_2(t))|) \\ &\quad \times \int_0^T k(t, s) (a_3(s) + b_3 |x(\phi_3(s))|) ds; \end{aligned} \quad (26)$$

for any $t \in I$, using (**), we have

$$\begin{aligned} \|x - y\| &= \int_0^T |x(t) - y(t)| dt \\ &\leq \frac{l_1}{B_1} \|x - y\| + \frac{Ml_2}{B_2} \|x - y\| \\ &\quad \times \left[\|a_3\| + \frac{b_3 r}{B_3} \right] + 2M \|a_2\| \|a_3\| \\ &\quad + \frac{2rb_2 M \|a_3\|}{B_2} + \frac{2Mr^2 b_2 b_3}{B_2 B_3} + \frac{2Mrb_3 \|a_2\|}{B_3}. \end{aligned} \quad (27)$$

Then,

$$\begin{aligned} &\|x - y\| \\ &\leq \left[2M \|a_2\| \|a_3\| + \frac{2rb_2 M \|a_3\|}{B_2} + \frac{2Mr^2 b_2 b_3}{B_2 B_3} + \frac{2Mrb_3 \|a_2\|}{B_3} \right] \\ &\quad \cdot \left[1 - \frac{l_1}{B_1} - \frac{Mrb_3 l_2}{B_2 B_3} - \frac{l_2 M \|a_3\|}{B_2} \right]^{-1} \\ &\leq \epsilon, \quad t \in I. \end{aligned} \quad (28)$$

That is, the solution $x = x(t)$ of (1) is asymptotically stable on $\mathbb{R}_+$. This completes the proof. $\square$

5. Applications

As particular cases of Theorem 5, we can obtain theorems on the existence of positive and a.e. nondecreasing solutions belonging to the space $L_1(I)$ of the following quadratic integral equations.

(1) If $\alpha(t) = 1$, then we obtain the quadratic integral equation

$$\begin{aligned} x(t) &= f(t, x(\phi_1(t))) + g(t, x(\phi_2(t))) \\ &\quad \times \int_0^1 u(t, s, x(\phi_3(s))) ds, \quad t \in I. \end{aligned} \quad (29)$$

(2) If $\alpha(t) = 1$ and $f(t, x) = a(t)$, then we obtain the quadratic integral equation

$$\begin{aligned} x(t) &= a(t) + g(t, x(\phi_2(t))) \int_0^1 u(t, s, x(\phi_3(s))) ds, \\ &\quad t \in I. \end{aligned} \quad (30)$$

(3) If $f(t, x) = a(t)$, $u(t, s, x) = h(t, s, x)$, $\alpha(t) = 1$, and $g(t, s) = 1$, then we obtain the quadratic integral equation

$$x(t) = a(t) + \int_0^1 h(t, s, x(\phi_3(s))) ds, \quad t \in I, \quad (31)$$

which was proved by Banaś in [4].

(4) If $g(t, x) = 0$, then we obtain the functional equation

$$x(t) = f(t, x(\phi_1(t))), \quad t \in I, \quad (32)$$

which is the same results proved by Banaś in [3].

(5) If $f(t, x) = a(t)$, and $u(t, s, x) = k(t, s)h(t, x)$, then we obtain the quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^1 k(t, s) h(s, x(\phi_3(s))) ds, \quad (33)$$

$$t \in I,$$

which is the same result proved in [16].

(6) If $f(t, x) = a(t)$, $u(t, s, x) = h(t, x)$, and $\alpha(t) = t$, then we obtain the quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t h(s, x(\phi_3(s))) ds, \quad t \in I, \quad (34)$$

which is the same result proved in [18].

Example 7. Let us consider the quadratic integral equation of Urysohn type having the form

$$x(t) = a(t) + x(t) \int_0^1 \frac{t}{t+s} u(t, s, x(s)) ds, \quad t \in [0, 1]. \quad (35)$$

This equation represents the Hammerstein counterpart of the famous Chandrasekhar quadratic integral equation which has numerous application (cf. [1, 2, 6, 24]). It arose originally in connection with scattering through a homogeneous semi-infinite plane atmosphere [24].

In case $a(t) = 1$ and $u(t, s, x(s)) = \lambda \phi(s)x(s)$, λ is a positive constant. Then, (35) has the form

$$x(t) = 1 + \lambda x(t) \int_0^1 \frac{t \phi(s)}{t+s} x(s) ds. \quad (36)$$

In order to apply our results, we have to impose an additional condition that the so-called “characteristic” function ϕ is continuous on I .

In this case, $r = (1 - \sqrt{1 - 4\lambda k_1})/2\lambda k$, and the assumption (vii) may be reduced to $4\lambda k_1 \leq 1$ where $\sup_{s \in I} \phi(s) = k_1$.

Example 8. Consider the following quadratic functional integral equation:

$$x(t) = \frac{1}{6}x(t) + \left[t + \frac{1}{3+t}x(\sin(t^2 + 3t)) \right] \times \int_0^t \left[1 + \frac{1}{3+s}x(\sin(s^2 + 4s)) \right] ds \quad t \in [0, 1]. \quad (37)$$

Taking

$$f(t, x) = \frac{1}{6}x(t), \quad g(t, x) = 1 + \frac{1}{3+t}x, \quad (38)$$

$$u(t, s, x) = t + \frac{1}{3+t}x,$$

then we can easily deduce that

(i) $|u(t, s, x)| \leq 1 + (1/4)|x|$ and $|g(t, x)| \leq t + (1/4)|x|$ (i.e., $\alpha(t) = t, a_1(t) = 0, a_2(t) = t$, and $a_3(t) = 1$ which implies that $\|a_1\| = 0, \|a_2\| = \|a_3\| = 1$, and $b_1 = 1/6, b_2 = b_3 = 1/4$);

(ii) $\phi_1(t) = t, \phi_2(t) = \sin(t^2 + 3t)$, and $\phi_3(t) = \sin(t^2 + 4t)$, and then $\phi'_1(t) = 1, \phi'_2(t) = (2t + 3)\cos(t^2 + 3t) > 2$, and $\phi'_3(t) = (2t + 4)\cos(t^2 + 4t) > 3$ (i.e., $B_1 = 1/2, B_2 = 2, B_3 = 3$, and $M = 1$).

Now, we will calculate r .

Then, $r = 2.30228 > 0$ and $(b_1/B_1) + (Mb_2\|a_3\|_{L_1}/B_2) + (Mb_2b_3r/B_2B_3) = 0.48231546246258117 < 1$.

Thus, all the assumptions of Theorem 5 are satisfied; so, the quadratic functional integral equation (37) possesses at least one solution being positive, a.e. nondecreasing, and integrable in $[0, 1]$.

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