Research Article

On Global Existence of Solutions of the Neumann Problem for Spherically Symmetric Nonlinear Viscoelasticity in a Ball

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We examine spherically symmetric solutions to the viscoelasticity system in a ball with the Neumann boundary conditions. Imposing some growth restrictions on the nonlinear part of the stress tensor, we prove the existence of global regular solutions for large data in the weighted Sobolev spaces, where the weight is a power function of the distance to the centre of the ball. First, we prove a global a priori estimate. Then existence is proved by the method of successive approximations and appropriate time extension.

1. Introduction

First, we recall some important facts from the nonlinear theory of viscoelasticity. Among the papers devoted to nonlinear viscoelasticity, we mention below some of them. The global solution (in time) for sufficiently small and smooth data are proved by Ponce (cf. [1]) and Kawashima and Shibata (cf. [2]) for quasilinear hyperbolic system of second-order with viscosity.

In the paper of [3], Kobayashi, Pecher, and Shibata proved global in time solution to a nonlinear wave equation with viscoelasticity under the special assumption about nonlinearity. In the paper of [4], Pawłowski and Zajączkowski showed the existence and uniqueness of global regular solutions to the Cahn-Hilliard system coupled with viscoelasticity.

Finally, in [5], global existence of regular solutions to one-dimensional viscoelasticity is proved. Moreover, many facts on elasticity and viscoelasticity theory can be found in [6–10]. Some existence results in the linear and nonlinear thermoviscoelasticity are shown in [11, 12].

In our paper, we consider a more general nonlinear system of viscoelasticity because the stress tensor is a general nonlinear function depending on a strain. We assume that the stress tensor is a function of a strain at a given instant of time t , but it does not depend on strains at time $t' < t$.

It is worth to emphasize that our constitutive relation for the stress tensor and any another constitutive relation satisfy the rules of continuum mechanics.

In order to prove the global (in time) solution for nonsmall data for nonlinear system of viscoelasticity (cf. formulae (1), (2), and (3)), we consider the spherically symmetric case and use the anisotropic Sobolev spaces with weights.

Speaking more precisely, we consider the motion of viscoelastic medium described by the following system of equations (cf. [7, 11]):

$$\rho u_{,tt} = \operatorname{div} \sigma + \rho f, \quad (1)$$

where $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$ is the displacement vector, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is a given system of the Cartesian coordinates, $t \in \mathbb{R}_+ \cup \{0\}$, ρ is the mass density, $\sigma = \sigma(x, t) \in \mathbb{R}^9$ is the stress tensor, and $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ is the external force field.

We examine system (1) in a bounded domain $\Omega \subset \mathbb{R}^3$ with the Neumann boundary conditions

$$\bar{n} \cdot \sigma|_S = 0, \quad (2)$$

where $S = \partial\Omega$, $\bar{n}$ is the unit outward vector normal to S .

Moreover, we add the initial conditions

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1. \quad (3)$$

We will assume that

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon) + \mu_0 \varepsilon_t, \quad (4)$$

where $\varepsilon = (1/2)(\nabla u + (\nabla u)^T)$ is the linearized strain tensor, $F = F(\varepsilon)$ is some function which will be specified later, and μ_0 is a positive constant.

Our aim is to prove the global existence of solutions to problems (1)–(4) for nonsmall data.

Since we do not know how to show the existence in a general case, we restrict our considerations to the spherically symmetric case. We assume that Ω is a ball B_R with radius R centered at the origin of the introduced Cartesian coordinates. We introduce the spherical coordinates r, φ, ϑ by the relations

$$\begin{aligned} x_1 &= r \cos \varphi \sin \vartheta, \\ x_2 &= r \sin \varphi \sin \vartheta, \\ x_3 &= r \cos \vartheta. \end{aligned} \quad (5)$$

With these coordinates, we connect the orthonormal vectors

$$\begin{aligned} \bar{e}_r &= (\cos \varphi \sin \vartheta, \sin \varphi \sin \vartheta, \cos \vartheta), \\ \bar{e}_\vartheta &= (\cos \varphi \cos \vartheta, \sin \varphi \cos \vartheta, -\sin \vartheta), \\ \bar{e}_\varphi &= (-\sin \varphi, \cos \varphi, 0). \end{aligned} \quad (6)$$

Then, we define $u_r = u \cdot \bar{e}_r$, $u_\vartheta = u \cdot \bar{e}_\vartheta$, $u_\varphi = u \cdot \bar{e}_\varphi$, $\varepsilon_{rr} = \bar{e}_r \cdot \varepsilon \cdot \bar{e}_r = u_{r,r}$, $\varepsilon_{\varphi\varphi} = u_r/r$, $\varepsilon_{\vartheta\vartheta} = u_r/r$. Since the spherically symmetric case is considered, we have $u_\vartheta = u_\varphi = 0$.

To simplify the notation, we introduce

$$w = u_r. \quad (7)$$

Assuming $\varrho = 1$ and transforming (1) to the spherical coordinates, we obtain

$$w_{,tt} = \frac{1}{r^2} (\sigma_{rr} r^2)_{,r} - \frac{1}{r} (\sigma_{\vartheta\vartheta} + \sigma_{\varphi\varphi}) + f_r, \quad (8)$$

where

$$\begin{aligned} \sigma_{rr} &= \frac{\partial F}{\partial \varepsilon_{rr}} + \mu_0 \varepsilon_{rr,t}, & \sigma_{\vartheta\vartheta} &= \frac{\partial F}{\partial \varepsilon_{\vartheta\vartheta}} + \mu_0 \varepsilon_{\vartheta\vartheta,t} \\ \sigma_{\varphi\varphi} &= \frac{\partial F}{\partial \varepsilon_{\varphi\varphi}} + \mu_0 \varepsilon_{\varphi\varphi,t}. \end{aligned} \quad (9)$$

Let us introduce the quantity

$$F(\varepsilon) = \psi(w_r, \eta), \quad (10)$$

where $\eta = w/r$.

Then, (8) takes the form

$$\begin{aligned} w_{,tt} &= \frac{1}{r^2} \left[\left(\frac{\partial \psi}{\partial w_r} + \mu_0 w_{,rt} \right) r^2 \right]_{,r} \\ &\quad - \frac{1}{r} \left(\frac{\partial \psi}{\partial \eta} + \mu_0 \frac{w_{,t}}{r} \right), \end{aligned} \quad (11)$$

and in view of (3), we have the initial conditions

$$w|_{t=0} = w_0, \quad w_{,t}|_{t=0} = w_1, \quad (12)$$

and in view of (2) and (9), the boundary condition

$$\left(\frac{\partial \psi}{\partial w_r} + \mu_0 w_{,rt} \right) \Big|_{S_R} = 0, \quad (13)$$

where $S_R = \partial B_R$.

Global existence of regular solutions to problems (11) and (12) with the Dirichlet boundary condition

$$w|_{S_R} = 0 \quad (14)$$

is proved in [13]. To formulate the main results of this paper we need the following.

Assumptions. Let us introduce the notation $\vartheta = w_r$, $\eta = w/r$. Assume that

- (1) $\psi(\vartheta, \eta) = \psi_1(\vartheta) + \psi_2(\eta)$,
- (2) there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$\begin{aligned} \frac{\partial^2 \psi_1}{\partial \vartheta^2} &\geq \alpha_1, & \frac{\partial^2 \psi_2}{\partial \eta^2} &\geq \alpha_2, \\ \left| \frac{\partial \psi_1}{\partial \vartheta} \right| &\leq \beta_1 |\vartheta|, & \left| \frac{\partial \psi_2}{\partial \eta} \right| &\leq \beta_2 |\eta|, \end{aligned} \quad (15)$$

$$(3) \quad w_0 \in H_{\mu_1}^2(0, R), \quad w_1 \in H_{\mu_1}^2(0, R), \quad \mu_1 \in (2, 1 + \sqrt{5}/2),$$

$$(4) \quad w_1 \in L_{2, \mu_2}(0, R), \quad w_0 \in H_{\mu_2}^1(0, R), \quad \mu_2 \in (0, 3/2), \quad \psi_1 \geq \alpha_1 \vartheta^2, \quad \psi_2 \geq \alpha_2 \eta^2, \quad |\psi_{1,\vartheta}| \leq \beta_1 |\vartheta|, \quad |\psi_{2,\eta}| \leq \beta_2 |\eta|,$$

$$(5) \quad w_1 \in L_{2, \mu_3}(0, R), \quad w_0 \in H_{\mu_3}^1(0, R), \quad \mu_3 \in (0, 1/2).$$

Main Theorem. (1) Let Assumptions (1)–(3) hold. Then there exists a solution to problems (11)–(13) such that

$$\begin{aligned} w_{,tt} &\in B(0, T; L_{2, \mu_1}(0, R)), & w_{,t} &\in B(0, T; H_{\mu_1}^1(0, R)), \\ w &\in B(0, T; H_{\mu_1}^2(0, R)), & w_{,tt} &\in L_2(0, T; H_{\mu_1}^1(0, R)). \end{aligned} \quad (16)$$

(2) Let Assumptions (1) and (4) hold, then

$$\begin{aligned} w_{,t} &\in B(0, T; L_{2, \mu_2}(0, R)) \cap L_2(0, T; H_{\mu_2}^1(0, R)), \\ w &\in B(0, T; H_{\mu_2}^1(0, R)). \end{aligned} \quad (17)$$

(3) Let Assumptions (1), (4), and (5) be satisfied. Then

$$\begin{aligned} w &\in L_\infty((0, R) \times (0, T)), \quad w \in B(0, T; C^{1/2-\mu_3/2}(0, R)), \\ w &\in L_\beta(0, T; C^{(\beta-1)/\beta}(0, T)), \quad \beta \in \left(1, \frac{2}{2\mu_3+1}\right), \\ w_t &\in L_2(0, T; C^{1/2-\mu_3/2}(0, R)). \end{aligned} \quad (18)$$

Our paper is organized as follows. In Section 1, the formulation of the considered problem and the main results are presented. In Section 2, the notation is introduced. Mainly, we define the anisotropic Sobolev spaces with weights. Section 3 is devoted to the proof of energy-type estimates to solutions of problems (11)–(13).

In Section 4 the existence of the global solution for nonsmall data of problems (11)–(13) is proved.

2. Notation and Auxiliary Results

By c we denote the generic constant which changes from formula to formula. By $c(\sigma)$, $\sigma > 0$, we denote a generic function which is always positive and increasing.

We replace forms of right-hand side (left-hand side) by the abbreviation r.h.s. (l.h.s.). We mark $w_t = \partial_t w$, $w_r = \partial_r w$, and so on.

By $B(I)$ we denote the space of bounded functions on the interval I .

By $H_\mu^k(0, R)$, $\mu \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$, we denote a weighted Sobolev space with the finite norm

$$\|u\|_{H_\mu^k(0, R)} = \left(\sum_{\alpha=0}^k \int_0^R |\partial_r^\alpha u|^2 r^{2(\mu-k+|\alpha|)} dr \right)^{1/2} \quad (19)$$

and $L_{2,\mu}(0, R) = H_\mu^0(0, R)$.

By $C^\alpha(I)$, $\alpha \in (0, 1)$ we denote the Hölder space with the finite norm

$$\|u\|_{C^\alpha(I)} = \sup_{\tau \in I} |u(\tau)| + \sup_{\tau', \tau'' \in I} \frac{|u(\tau') - u(\tau'')|}{|\tau' - \tau''|^\alpha}. \quad (20)$$

Next, we recall the Hardy inequality (see [14, Chapter 1, Section 2.15])

$$\left| \frac{1}{p'} - \mu \right|^p \int_0^\infty r^{p(\mu-1)} |f|^p dr \leq \int_0^\infty r^{p\mu} |f_r|^p dr, \quad (21)$$

where $1/p + 1/p' = 1$, $\mu \in \mathbb{R}$, and $\mu \neq 1/p'$.

The inequality holds also for functions with compact support. Assuming that $\text{supp } f \subset [0, R]$, we introduce $F(x) = \int_x^\infty f(y) dy$ and repeat the proof from [14, Chapter 1, Section 2.15].

From [15, Chapter 2, Section 3], we have the imbedding

$$\begin{aligned} \|u\|_{L_q(0, T; L_p(0, R))} \\ \leq c \left(\|u\|_{L_\infty(0, T; L_2(0, R))} + \|\nabla_r u\|_{L_2(0, T; L_2(0, R))} \right), \end{aligned} \quad (22)$$

where $1/p + 2/q \geq 1/2$.

Finally, we consider the problem

$$u_t - u_{rr} = f, \quad u|_{t=0} = u_0, \quad u_r|_{r=R} = 0. \quad (23)$$

To examine nonstationary problem (23), we need anisotropic weighted Sobolev spaces $V_{p,\nu}^{2,1}((0, R) \times (0, T))$, $p \in (1, \infty)$, $\nu \in \mathbb{R}$, of functions with the finite norm

$$\begin{aligned} \|u\|_{V_{p,\nu}^{2,1}((0, R) \times (0, T))} \\ = \left(\sum_{\alpha+2a \leq 2} \int_0^T \int_0^R |\partial_r^\alpha \partial_t^a u|^p r^{p(\nu-2+\alpha+2a)} dr dt \right)^{1/p}. \end{aligned} \quad (24)$$

Spaces $V_{p,\nu}^2(0, R)$ appropriate for elliptic problems were introduced in [16]. Moreover, we assume that $L_{p,\nu}(0, R) = V_{p,\nu}^0(0, R)$.

The following result is valid.

Lemma 1. *We assume that $f \in L_{p,\nu}((0, R) \times (0, T))$, $u_0 \in V_{p,\nu}^{2-2/p}(0, R)$. Then there exists a solution to problem (23) such that $u \in V_{p,\nu}^{2,1}((0, R) \times (0, T))$ and*

$$\begin{aligned} \|u\|_{V_{p,\nu}^{2,1}((0, R) \times (0, T))} \\ \leq c \left(\|f\|_{L_{p,\nu}((0, R) \times (0, T))} + \|u_0\|_{V_{p,\nu}^{2-2/p}(0, R)} \right). \end{aligned} \quad (25)$$

In the case of elliptic equations, a similar result was proved in [17] for $p = 2$ and in [16] for any $p \in (1, \infty)$.

The weighted Sobolev spaces with fractional derivatives are introduced in [16].

In the nonstationary case, Lemma 1 follows from the methods described in [18] in the case $p = 2$. For the general p Lemma 1 results from considerations in [18–20].

Finally, we introduce spaces used in this paper. We will define them by introducing finite norms.

Besov space $B_{p,\nu}^l(B_R)$, $l \in \mathbb{R}_+$, $p \in (1, \infty)$, $\nu \in \mathbb{R}$,

$$\|u\|_{B_{p,\nu}^l(B_R)} = \|u\|_{B_{p,\nu}^{l-[l]}(B_R)} + \|u\|_{V_{p,\nu}^{[l]}(B_R)}, \quad (26)$$

where $[l]$ is the integer part of l ,

$$\|u\|_{B_{p,\nu}^\alpha(B_R)} = \left(\int_0^R \int_0^R \frac{|u(r_1)r_1^\nu - u(r_2)r_2^\nu|^p}{|r_1 - r_2|^{1+p\alpha}} dr_1 dr_2 \right)^{1/p}, \quad (27)$$

where $\alpha \in (0, 1)$ and

$$\|u\|_{V_{p,\nu}^k(B_R)} = \left(\sum_{\alpha \leq k} \int_0^R \int_0^R |\partial_r^\alpha u|^p r^{p(\nu-k+\alpha)} dr \right)^{1/p}, \quad (28)$$

for k integer.

$B(B_R \times (0, T))$ is the space of bounded functions.

$C^\alpha(\bar{B}_R)$, $C^{\alpha,\beta}(\bar{B}_R \times [0, T])$, $\alpha, \beta \in (0, 1)$, are the Hölder spaces with the finite norms

$$\begin{aligned} \|u\|_{C^\alpha(\bar{B}_R)} &= \|u\|_{B(\bar{B}_R)} + \sup_{r_1, r_2 \in \bar{B}_R} \frac{|u(r_1) - u(r_2)|}{|r_1 - r_2|^\alpha}, \\ \|u\|_{C^{\alpha,\beta}(\bar{B}_R \times [0, T])} &= \|u\|_{B(\bar{B}_R \times [0, T])} \\ &+ \sup_{t \in [0, T]} \sup_{r_1, r_2 \in \bar{B}_R} \frac{|u(r_1, t) - u(r_2, t)|}{|r_1 - r_2|^\alpha} \\ &+ \sup_{r \in \bar{B}_R} \sup_{t', t'' \in [0, T]} \frac{|u(r, t') - u(r, t'')|}{|t' - t''|^\beta}, \end{aligned} \quad (29)$$

where $\bar{B}_R$ is the closure of B_R .

By $L_q^l(0, T; W_p^k(B_R))$, $l, k \in \mathbb{N} \cup \{0\}$, $p, q \in [1, \infty]$, we denote a space of functions with the finite norm

$$\|\partial_t^l u\|_{L_q(0, T; W_p^k(B_R))}. \quad (30)$$

We introduce also the Sobolev spaces

$$W_p^k(B_R) = \left\{ u = u(x) : \|u\|_{W_p^k(B_R)} = \sum_{l \leq k} \|\partial_r^l u\|_{L_p(B_R)} < \infty \right\}, \quad (31)$$

where $k \in \mathbb{Z}_+ \cup \{0\}$, $p \in [1, \infty]$,

$$\begin{aligned} W_p^{k, k/2}(B_R^T) &= \left\{ u = u(r, t) : \|u\|_{W_p^{k, k/2}(B_R^T)} \right. \\ &= \left. \sum_{l+2a \leq k} \|\partial_r^l \partial_t^a u\|_{L_p(B_R^T)} < \infty \right\}, \end{aligned} \quad (32)$$

where $k, k/2 \in \mathbb{Z}_+ \cup \{0\}$, $p \in [1, \infty]$, $B_R^T = B_R \times (0, T)$, $B_R = \{r \in \mathbb{R}_+ : r < R\}$,

$$\begin{aligned} W_p^k(0, T) &= \left\{ u = u(t) : \|u\|_{W_p^k(0, T)} \right. \\ &= \left. \sum_{l \leq k} \|\partial_t^l u\|_{L_p(0, T)} < \infty \right\}, \end{aligned} \quad (33)$$

$k \in \mathbb{Z}_+ \cup \{0\}$, $p \in [0, 1]$. For $k = 0$, we have $W_p^0(B_R) = L_p(B_R)$, $W_p^{0,0}(B_R^T) = L_p(B_R^T)$, $W_p^0(0, T) = L_p(0, T)$, $B_{p,p}^\alpha(B_R) = B_{p,p}^\alpha(B_R)$ is the Besov space introduced in [14, Chapter 4, Section 18] and $W_p^\alpha(B_R)$ is the Sobolev-Slobodetsky space, where $\alpha \notin \mathbb{Z}_+$, $p \in [1, \infty]$.

Moreover,

$$V_{p,0}^k(B_R) \subset W_p^k(B_R), \quad V_{p,0}^{k, k/2}(B_R^T) \subset W_p^{k, k/2}(B_R^T). \quad (34)$$

Finally $H^k(B_R) = W_2^k(B_R)$, $H^{k, k/2}(B_R^T) = W_2^{k, k/2}(B_R^T)$, and so on.

3. Estimates

To prove the Main Theorem we have to recall some estimates proved in [13].

Lemma 2 (see [13, Lemma 3.1]). *We consider problems (11)–(13). Assume that w_0, w_1 are initial data defined by (12). Let c_0 be a constant such that*

$$\int_{B_R} \left[\frac{1}{2} w_1^2 + \psi \left(w_{0,r}, \frac{w_0}{r} \right) \right] r^2 dr \equiv c_0^2 < \infty, \quad (35)$$

where ψ is a positive function differentiable with respect to its arguments introduced by (10).

Then, solutions to problems (11)–(13) satisfy the estimate

$$\begin{aligned} \int_{B_R} \left[\frac{1}{2} w_{,t}^2 + \psi \left(w_{,r}, \frac{w}{r} \right) \right] r^2 dr + \int_{B_R^t} \left(w_{,rt}^2 + \frac{w_{,t}^2}{r^2} \right) r^2 dr dt' \\ = \int_{B_R} \left[\frac{1}{2} w_1^2 + \psi \left(w_{0,r}, \frac{w_0}{r} \right) \right] r^2 dr = c_0^2, \quad \text{for any } t > 0. \end{aligned} \quad (36)$$

Next, we need the following.

Lemma 3 (see [13, Lemma 3.2]). *We consider problems (11)–(13). Let $\vartheta = w_{,r}$, $\eta = w/r$. Assume that $\psi(\vartheta, \eta) = \psi_1(\vartheta) + \psi_2(\eta)$. Assume existence of positive constants $\alpha_1, \alpha_2, \beta_1$, and β_2 such that*

$$\psi_1(\vartheta) \geq \alpha_1 \vartheta^2, \quad \psi_2(\eta) \geq \alpha_2 \eta^2, \quad (37)$$

$$\left| \frac{\partial \psi_1}{\partial \vartheta} \right| \leq \beta_1 |\vartheta|, \quad \left| \frac{\partial \psi_2}{\partial \eta} \right| \leq \beta_2 |\eta|. \quad (38)$$

Finally, assume that

$$w_1 \in L_{2,\mu}(0, R), \quad w_0 \in H_\mu^1(0, R), \quad \mu \in \left(0, \frac{3}{2}\right). \quad (39)$$

Then solutions to problems (11)–(13) satisfy the following estimate:

$$\begin{aligned} \|w_{,t}\|_{L_\infty(0,t;L_{2,\mu}(B_R))} + \|w\|_{L_\infty(0,t;H_\mu^1(B_R))} + \|w_{,t}\|_{L_2(0,t;H_\mu^1(B_R))} \\ \leq c(R, \mu, t) \left(\|w_1\|_{L_{2,\mu}(B_R)} + \|w_0\|_{H_\mu^1(B_R)} \right) \\ \leq c_1(R, \mu, T), \end{aligned} \quad (40)$$

where $\mu \in (0, 3/2)$, $t \leq T$.

Continuing, we have the following.

Lemma 4 (see [13, Lemma 3.3]). *We consider problems (11)–(13). Assume that there exist positive constants $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_1, \bar{c}_1$, and $\bar{c}_2$ such that*

$$\begin{aligned} \psi_{1,\vartheta\vartheta} \geq \bar{\alpha}_1, \quad \psi_{2,\eta\eta} \geq \bar{\alpha}_2, \quad |\psi_{1,\vartheta\eta}| \leq \bar{\beta}_1, \\ |\partial_\vartheta^3 \psi_1| \leq \bar{c}_1, \quad |\partial_\eta^3 \psi_2| \leq \bar{c}_2. \end{aligned} \quad (41)$$

Assume also that

$$\eta_{,t}, \vartheta_{,t} \in L_3(0, T; L_{3,2/3\mu}(B_R)), \quad w_1 \in H_\mu^1(B_R), \quad (42)$$

$$w_{,tt}(0) \in L_{2,\mu}(B_R),$$

where

$$w_{,tt}(0) = \frac{1}{r^2} \left[\left(\psi_{,\vartheta} \left(w_{0,r}, \frac{w_0}{r} \right) + \mu_0 w_{1,r} \right) r^2 \right]_{,r} \quad (43)$$

$$- \frac{1}{r} \left[\psi_{,\eta} \left(w_{0,r}, \frac{w_0}{r} \right) + \frac{w_1}{r} \right].$$

Then solutions to (11)–(13) satisfy the following inequality:

$$\begin{aligned} & \|w_{,tt}\|_{L_\infty(0,t;L_{2,\mu}(B_R))} + \|w_{,t}\|_{L_\infty(0,t;H_\mu^1(B_R))} \\ & + \mu_0 \|w_{,tt}\|_{L_2(0,t;H_\mu^1(B_R))} \\ & \leq c(t) \left(\|\vartheta_{,t}\|_{L_3(0,t;L_{3,2/3\mu}(B_R))}^{3/2} + \|\eta_{,t}\|_{L_3(0,t;L_{3,2/3\mu}(B_R))}^{3/2} \right) + c(t) c_2, \end{aligned} \quad (44)$$

where $\mu \in (0, 1 + (1/2)\sqrt{5})$, $t \leq T$ and

$$c_2 = \|w_{,tt}(0)\|_{L_{2,\mu}(B_R)} + \|w_{,t}(0)\|_{H_\mu^1(B_R)}. \quad (45)$$

Remark 5. In general constants α_i and $\bar{\alpha}_i$, $i = 1, 2$, are different. Let us consider the example

$$\psi_1(\vartheta) = \alpha_1 \vartheta^2 + \frac{a}{\sqrt{1 + \vartheta^2}}, \quad (46)$$

where a is a positive constant. Then we have

$$\psi_1(\vartheta) \geq \alpha_1 \vartheta^2. \quad (47)$$

Continuing, we derive

$$\psi_{1,\vartheta\vartheta} = 2\alpha_1 + \frac{a(2\vartheta^2 - 1)}{(1 + \vartheta^2)^{5/2}}. \quad (48)$$

For $\vartheta^2 \geq 1/2$, we obtain $\psi_{1,\vartheta\vartheta} \geq 2\alpha_1$. But for $\vartheta^2 < 1/2$, we have

$$\begin{aligned} \psi_{1,\vartheta\vartheta} & \geq 2\alpha_1 - \frac{a}{(1 + \vartheta^2)^{5/2}} \\ & \geq 2\alpha_1 - a \equiv \bar{\alpha}_1. \end{aligned} \quad (49)$$

It is clear that many examples can be invented.

We need to obtain an estimate from (44). For this purpose, we examine the last two norms on the r.h.s. of (44). We express them in the form

$$\left(\int_0^T \int_{B_R} |w_{,rt}|^3 r^{2\mu} dr dt \right)^{1/2} + \left(\int_0^T \int_{B_R} \left| \frac{w_{,t}}{r} \right|^3 r^{2\mu} dr dt \right)^{1/2} \equiv I. \quad (50)$$

Hence, by the Hardy inequality (see [14, Chapter 1, Section 2.15]), the above expression is bounded by

$$I \leq c \left(\int_0^T \int_{B_R} |w_{,rt}|^3 r^{2\mu} dr dt \right)^{1/2} \equiv I_1. \quad (51)$$

To estimate I_1 , we need the Pego transformation

$$p(r, t) = \int_0^r w_{,t}(r', t) r'^2 dr', \quad (52)$$

$$q(r, t) = \mu_0 w_{,r}(r, t) r^2 - p(r, t).$$

Hence,

$$w_{,rt} r^2 = \frac{1}{\mu_0} (q_{,t} + p_{,t}). \quad (53)$$

To calculate the above expression we need problems for p and q . From Lemma 3.4 from [13], we have the following problems for p and q :

$$p_{,t} - \mu_0 p_{,rr} = \psi_{,\vartheta} r^2 - \int_0^r \left(\psi_{,\eta} + \mu_0 \frac{w_{,t}}{r} \right) r dr$$

$$- 2\mu_0 w_{,t} r,$$

$$p|_{t=0} = \int_0^r w_1(r) r^2 dr \equiv p_0(r), \quad p_{,r}|_{r=R} = w_{,t}(R, t) R^2 \quad (54)$$

$$q_{,t} = -\psi_{,\vartheta} r^2 + \int_0^r \left(\psi_{,\eta} + \mu_0 \frac{w_{,t}}{r} \right) r dr,$$

$$q|_{t=0} = \mu_0 w_{0,r} r^2 \quad (55)$$

$$- \int_0^r w_1(r) r^2 dr \equiv q_0(r).$$

Lemma 6. Let the assumptions of Lemma 3 be satisfied. Let $w_{,r} \in L_{\sigma,2-\gamma}(B_R^T)$, $w_{,t}, \partial_t^{2-1/\sigma} w \in L_\sigma(0, T; L_{2,1}(B_R))$, $\partial_t^{2-1/\sigma}$ means the fractional derivative. Then solutions to problems (54) and (55) satisfy

$$\|p\|_{W_{\sigma,-\gamma}^{2,1}(B_R^T)} \leq c \|w_{,r}\|_{L_{\sigma,2-\gamma}(B_R^T)} + c \|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0}^T)} + cc_1, \quad (56)$$

$$\|q_{,t}\|_{L_{\sigma,-\gamma}(B_R^T)} \leq c \|w_{,r}\|_{L_{\sigma,2-\gamma}(B_R^T)} + cc_1, \quad (57)$$

where c_1 is introduced in (40), $B_{R,R_0} = \{r : 0 < R_0 < r < R\}$, $\gamma < 1/\sigma + 3/2 - \mu$, $\gamma < 1$, $\sigma \in (1, 6]$, and $\mu \in (0, 3/2)$ is introduced in Lemma 3.

Proof. For solutions to problem (54), we have (see Lemma 1),

$$\begin{aligned} & \|p\|_{W_{\sigma,-\gamma}^{2,1}(B_R^T)} \\ & \leq c \left(\|w_{,r} r^2\|_{L_{\sigma,-\gamma}(B_R^T)} + \left\| \int_0^r (|w| + |w_{,t}|) dr \right\|_{L_{\sigma,-\gamma}(B_R^T)} \right. \\ & \quad + \|w_{,t} r\|_{L_{\sigma,-\gamma}(B_R^T)} + \|p_0\|_{W_{\sigma,-\gamma}^{2-2/\sigma}(B_R)} \\ & \quad \left. + \|w_{,t}(R, t) R^2\|_{W_{\sigma,-\gamma}^{1-1/\sigma, 1/2-1/2\sigma}(S_R^T)} \right). \end{aligned} \quad (58)$$

Now, we examine the terms from the r.h.s. of (58).

The first norm on the r.h.s. of (58) equals

$$\left(\int_{B_R^T} |w_{,r}|^\sigma r^{(2-\nu)\sigma} dr dt \right)^{1/\sigma}. \quad (59)$$

Applying the Hölder inequality, the second integral on the r.h.s. of (58) is bounded by

$$\begin{aligned} & \left(\int_{B_R^T} \left| \left(\int_0^r r^{-2\nu_1} dr \right)^{1/2} \right. \right. \\ & \quad \times \left. \left. \left(\int_0^r (w^2 + w_{,t}^2) r^{2\nu_1} dr \right)^{1/2} \right| r^{-\nu\sigma} dr dt \right)^{1/\sigma} \leq cc_1, \end{aligned} \quad (60)$$

where the last inequality holds in virtue of Lemma 3 and under the assumption

$$\nu_1 + \nu < \frac{1}{\sigma} + \frac{1}{2}, \quad \nu_1 = \mu - 1, \quad (61)$$

where $\mu \in (0, 3/2)$ is introduced in Lemma 3. Hence,

$$\mu + \nu < \frac{1}{\sigma} + \frac{3}{2}. \quad (62)$$

The third integral on the r.h.s. of (58) we express in the form

$$\left(\int_{B_R^T} |w_{,t}|^\sigma r^{(1-\nu)\sigma} dr dt \right)^{1/\sigma} \equiv J_1. \quad (63)$$

Assuming $\nu < 1$, setting $\mu = 1 - \nu > 0$, and recalling imbedding (22) and Lemma 3, we obtain for $\sigma \leq 6$ the estimate

$$\begin{aligned} J_1 & \leq c \sup_t \left(\int_{B_R} |(wr^\mu)_{,t}|^2 dr \right)^{1/2} \\ & \quad + c \left(\int_{B_R^T} |\partial_r (wr^\mu)_{,t}|^2 dr dt \right)^{1/2} \leq cc_1. \end{aligned} \quad (64)$$

Finally, the last term on the r.h.s. of (58) is bounded by

$$\|w_{,t} r^2\|_{W_\sigma^{1-1/\sigma, 1/2-1/2\sigma}(\mathbb{S}_R^T)} \leq c \|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0}^T)}. \quad (65)$$

Using the above estimates in the r.h.s. of (58) yields (56).

Repeating the considerations leading to (56) to problem (55) gives (57). This concludes the proof. $\square$

Using (56) and (57) in (53) gives

$$\begin{aligned} & \|w_{,rt}\|_{L_{\sigma,2-\nu}(B_R^T)} \\ & \leq c \|w_{,r}\|_{L_{\sigma,2-\nu}(B_R^T)} + c(T) \|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0}^T)} + cc_1. \end{aligned} \quad (66)$$

Applying the equality

$$w_{,r}(t) = \int_0^t w_{,rt'} dt' + w_{,r}(0) \quad (67)$$

in the first term on the r.h.s. of (66) and using the Gronwall lemma, we get

$$\|w_{,rt}\|_{L_{\sigma,2-\nu}(B_R^T)} \leq c(T) \|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0}^T)} + c(T) c_1. \quad (68)$$

Now, we have to estimate the norm on the r.h.s. of (68). For the purpose we need the following.

Lemma 7. *Let w be a solution to (11)–(13). Assume that $\psi = \psi_1(\vartheta) + \psi_2(\eta)$ and (37), (38) are satisfied. Assume that $w_1 \in W_\sigma^{1-1/\sigma}(B_R)$, $w_0 \in W_\sigma^1(B_{R,R_0/2})$, where $\sigma \in (1, \infty)$, $0 < R_0 < R$. Then*

$$\begin{aligned} & \|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0}^T)} \\ & \leq c(T) \left(c_0 + \|w_0\|_{W_\sigma^1(B_{R,R_0/2})} + \|w_1\|_{W_\sigma^{1-1/\sigma}(B_R)} \right), \end{aligned} \quad (69)$$

where c_0 is introduced in (36).

Proof. Let us introduce a smooth function $\zeta = \zeta(r)$ such that $\zeta(r) = 0$ for $r < R_0/2$ and $\zeta(r) = 1$ for $r > R_0$, where $R_0 < R$.

First, we obtain a local estimate in B_{R,R_0} for solutions to problems (11)–(13). Multiplying (11) by $\zeta^2 w_{,t} r^2$ and integrating over B_R yields

$$\begin{aligned} & \int_{B_R} w_{,tt} \zeta^2 w_{,t} r^2 dr + \int_{B_R} (\psi_\vartheta + \mu_0 w_{,rt}) (\zeta^2 w_{,t})_{,r} r^2 dr \\ & \quad + \int_{B_R} (\psi_{,\eta} + \mu_0 \eta_{,t}) \zeta^2 w_{,t} r dr = 0. \end{aligned} \quad (70)$$

Introducing the notation

$$\bar{w} = w\zeta, \quad (71)$$

we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} (\psi_{,\vartheta\vartheta,t} + \psi_{,\eta\eta,t}) \zeta^2 r^2 dr \\ & \quad + 2 \int_{B_R} \psi_{,\vartheta} w_{,t} \zeta \zeta_{,r} r^2 dr \\ & \quad + \mu_0 \int_{B_R} \left[w_{,rt} (\zeta^2 w_{,t})_{,r} + \frac{\bar{w}_{,t}}{r^2} \right] r^2 dr = 0. \end{aligned} \quad (72)$$

Continuing, we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} \psi \zeta^2 r^2 dr \right) \\ & \quad + \mu_0 \int_{B_R} \left(w_{,rt}^2 \zeta^2 + \frac{\bar{w}_{,t}^2}{r^2} \right) r^2 dr \\ & = -2\mu_0 \int_{B_R} w_{,rt} w_{,t} \zeta \zeta_{,r} r^2 dr - 2 \int_{B_R} \psi_{,\vartheta} w_{,t} \zeta \zeta_{,r} r^2 dr. \end{aligned} \quad (73)$$

Applying the Hölder and the Young inequalities to the r.h.s. of the above equality and using (38) yields

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} \psi \zeta^2 r^2 dr \right) \\ & + \frac{\mu_0}{2} \int_{B_R} \left(w_{,rt}^2 \zeta^2 + \frac{\bar{w}_{,t}^2}{r^2} \right) r^2 dr \\ & \leq c\mu_0 \int_{B_R} w_{,t}^2 \zeta_{,r}^2 r^2 dr + c \int_{B_R} w_{,r}^2 \zeta_{,r}^2 r^2 dr. \end{aligned} \quad (74)$$

Using that $w_{,rt}^2 \zeta^2 = (\bar{w}_{,rt} - w_{,t} \zeta_{,r})^2$ in the above inequality implies

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} \psi \zeta^2 r^2 dr \right) \\ & + \mu_0 \int_{B_R} \left(\bar{w}_{,rt}^2 + \frac{\bar{w}_{,t}^2}{r^2} \right) r^2 dr \\ & \leq c\mu_0 \int_{B_R} w_{,t}^2 \zeta_{,r}^2 r^2 dr + c \int_{B_R} w_{,r}^2 \zeta_{,r}^2 r^2 dr. \end{aligned} \quad (75)$$

Integrating the above inequality with respect to time and using (37), we have

$$\begin{aligned} & \frac{1}{2} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} \left(\alpha_1 w_{,r}^2 \zeta^2 + \alpha_2 \frac{\bar{w}_{,t}^2}{r^2} \right) r^2 dr \\ & + \mu_0 \int_{B_R^t} \left(w_{,rt}^2 + \frac{\bar{w}_{,t}^2}{r^2} \right) r^2 dr dt' \\ & \leq c\mu_0 \int_{B_R^t} w_{,t}^2 \zeta_{,r}^2 r^2 dr dt' + c \int_{B_R^t} w_{,r}^2 \zeta_{,r}^2 r^2 dr dt' \\ & + \frac{1}{2} \int_{B_R} \bar{w}_1 r^2 dr + \int_{B_R} \psi \left(w_{0,r}, \frac{w_0}{r} \right) r^2 dr \\ & \equiv c \int_{B_R^t} (w_{,t}^2 + w_{,r}^2) \zeta_{,r}^2 r^2 dr dt' + c c_0^2. \end{aligned} \quad (76)$$

Since $\bar{w}_{,r}^2 \leq 2((w_{,r} \zeta)^2 + w^2 \zeta_{,r}^2)$, the above inequality yields

$$\begin{aligned} & \frac{1}{2} \int_{B_R} \bar{w}_{,t}^2 r^2 dr + \int_{B_R} \left(\frac{1}{2} \alpha_1 \bar{w}_{,r}^2 r^2 + \alpha_2 \bar{w}^2 \right) dr \\ & + \mu_0 \int_{B_R^t} (\bar{w}_{,rt}^2 r^2 + \bar{w}_{,t}^2) dr dt' \\ & \leq c \int_{B_R^t} (w_{,t}^2 + w_{,r}^2 + w^2) \zeta_{,r}^2 r^2 dr dt' + c c_0^2 \leq c c_0^2, \end{aligned} \quad (77)$$

where (36) was used again.

Now, we introduce the following Pego transformation

$$\begin{aligned} \tilde{p}(r, t) &= \int_0^r \bar{w}_{,t}(r', t) r'^2 dr', \\ \tilde{q}(r, t) &= \mu_0 \bar{w}_{,r} r^2 - \tilde{p}(r, t). \end{aligned} \quad (78)$$

Moreover, we have the initial-boundary value problem for $\tilde{p}$,

$$\begin{aligned} \tilde{p}_{,t} - \mu_0 \tilde{p}_{,rr} &= \int_0^r \bar{w}_{,tt} r^2 dr - \mu_0 (\bar{w}_{,t} r^2)_{,r} \equiv I, \\ \tilde{p}|_{t=0} &= \int_0^r \bar{w}_1 r^2 dr, \quad \tilde{p}(R, t) = \int_0^R \bar{w}_{,t} r^2 dr. \end{aligned} \quad (79)$$

The nonhomogeneous Dirichlet boundary condition is not convenient so we introduce the new function

$$\tilde{p}' = \tilde{p} - \tilde{p}(R, t), \quad (80)$$

which is a solution to the problem

$$\begin{aligned} \tilde{p}'_{,t} - \mu_0 \tilde{p}'_{,rr} &= I - \int_0^R \bar{w}_{,tt} r^2 dr \equiv I_1, \\ \tilde{p}'|_{t=0} &= \int_0^r \bar{w}_1 r^2 dr - \int_0^R \bar{w}_1 r^2 dr \equiv \tilde{p}'_0, \\ \tilde{p}'|_{r=R} &= 0. \end{aligned} \quad (81)$$

We have to calculate I_1 . For this purpose, we need the expressions which follow from (11) multiplied by ζ

$$\begin{aligned} \bar{w}_{,tt} &= \frac{1}{r^2} [(\psi_{,9} \zeta + \mu_0 w_{,rt} \zeta) r^2]_{,r} - (\psi_{,9} + \mu_0 w_{,rt}) \zeta_{,r} \\ &\quad - \frac{1}{r} \left(\psi_{,\eta} \zeta + \mu_0 \frac{\bar{w}_{,t}}{r} \right), \end{aligned} \quad (82)$$

and from (13) multiplied by ζ

$$(\psi_{,9} + \mu_0 w_{,rt}) \zeta|_{r=R} = 0. \quad (83)$$

Using (82) and (83) in I_1 yields

$$\begin{aligned} I_1 &= \int_0^r [(\psi_{,9} \zeta + \mu_0 w_{,rt} \zeta) r^2]_{,r} dr \\ &\quad - \int_0^r (\psi_{,9} + \mu_0 w_{,rt}) \zeta_{,r} r^2 dr \\ &\quad - \int_0^r (\psi_{,\eta} \zeta r + \mu_0 \bar{w}_{,t}) dr - \mu_0 (\bar{w}_{,t} r^2)_{,r} \\ &\quad - \int_0^R [(\psi_{,9} \zeta + \mu_0 w_{,rt} \zeta) r^2]_{,r} dr \\ &\quad + \int_0^R (\psi_{,9} + \mu_0 w_{,rt}) \zeta_{,r} r^2 dr \\ &\quad + \int_0^R (\psi_{,\eta} \zeta r + \mu_0 \bar{w}_{,t}) dr \end{aligned}$$

$$\begin{aligned}
&= \psi_{,9} \zeta r^2 + \mu_0 \bar{w}_{,rt} r^2 - \mu_0 w_{,t} \zeta_{,r} r^2 \\
&\quad + \int_r^R (\psi_{,9} + \mu_0 w_{,rt}) \zeta_{,r} r^2 dr \\
&\quad + \int_r^R (\psi_{,9} \zeta + \mu_0 \bar{w}_{,t}) dr - \mu_0 (\bar{w}_{,rt} r^2 + 2r \bar{w}_{,t}) \\
&= \psi_{,9} \zeta r^2 - \mu_0 w_{,t} \zeta_{,r} r^2 - 2\mu_0 r \bar{w}_{,t} \\
&\quad + \int_r^R (\psi_{,9} \zeta_{,r} r^2 + \psi_{,9} \zeta r + \mu_0 \bar{w}_{,t}) dr \\
&\quad - \mu_0 \int_r^R w_{,t} (\zeta_{,r} r^2)_{,r} dr.
\end{aligned} \tag{84}$$

For solutions to problem (81), we have

$$\|\tilde{p}'\|_{W_\sigma^{2,1}(B_R^T)} \leq c \left(\|I_1\|_{L_\sigma(B_R^T)} + \|\tilde{p}'_0\|_{W_\sigma^{2-2/\sigma}(B_R)} \right). \tag{85}$$

Hence,

$$\begin{aligned}
&\|\tilde{p}\|_{W_\sigma^{2,1}(B_R^T)} \\
&\leq c \left(\|I_1\|_{L_\sigma(B_R^T)} + \|\tilde{p}_{,t}(R, t)\|_{L_\sigma(B_R^T)} + \|\tilde{p}'_0\|_{W_\sigma^{2-2/\sigma}(B_R)} \right).
\end{aligned} \tag{86}$$

From the expression of I_1 , we have

$$\begin{aligned}
&\|I_1\|_{L_\sigma(B_R^T)} \\
&\leq c \left(\|w_{,r}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_{,t}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w\|_{L_\sigma(B_{R,R_0/2}^T)} \right).
\end{aligned} \tag{87}$$

In view of (82) and (83), we obtain

$$\begin{aligned}
\tilde{p}_{,t}(R, t) &= \int_0^R \bar{w}_{,tt} r^2 dr \\
&= \int_0^R (\psi_{,9} + \mu_0 w_{,rt}) \zeta_{,r} r^2 dr \\
&\quad - \int_0^R (\psi_{,9} \zeta r + \mu_0 \bar{w}_{,t}) dr.
\end{aligned} \tag{88}$$

Hence,

$$\begin{aligned}
&|\tilde{p}_{,t}(R, t)| \\
&\leq \left| \int_0^R (\psi_{,9} \zeta_{,r} r^2 + \psi_{,9} \zeta r + \mu_0 \bar{w}_{,t}) dr \right| \\
&\quad + \left| \int_0^R \mu_0 w_{,t} (\zeta_{,r} r^2)_{,r} dr \right|.
\end{aligned} \tag{89}$$

Employing (38) yields

$$\begin{aligned}
&\|\tilde{p}_{,t}(R, t)\|_{L_\sigma(B_R^T)} \\
&\leq c \left(\|w_{,r}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_{,t}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w\|_{L_\sigma(B_{R,R_0/2}^T)} \right).
\end{aligned} \tag{90}$$

From the form of $\tilde{p}'_0$ (see (81)₂), we derive

$$\begin{aligned}
\|\tilde{p}'_0\|_{W_\sigma^{2-2/\sigma}(B_R)} &= \left\| \int_0^r \bar{w}_1 r^2 dr \right\|_{W_\sigma^{2-2/\sigma}(B_R)} \\
&\leq \|\bar{w}_1 r^2\|_{W_\sigma^{2-2/\sigma}(B_R)} \leq c \|\bar{w}_1\|_{W_\sigma^{1-2/\sigma}(B_R)}.
\end{aligned} \tag{91}$$

Using (90) in (86) yields

$$\begin{aligned}
\|\tilde{p}\|_{W_\sigma^{2,1}(B_R^T)} &\leq c \left(\|w_{,r}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_{,t}\|_{L_\sigma(B_{R,R_0/2}^T)} \right. \\
&\quad \left. + \|w\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_1\|_{W_\sigma^{1-2/\sigma}(B_{R,R_0/2}^T)} \right).
\end{aligned} \tag{92}$$

Since $\tilde{p}_{,r} = \bar{w}_{,t} r^2$ and that

$$\|\tilde{p}_{,r}\|_{W_\sigma^{1,1/2}(B_R^T)} \leq c \|\tilde{p}\|_{W_\sigma^{2,1}(B_R^T)}, \tag{93}$$

we obtain from (92) the inequality

$$\begin{aligned}
&\|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0/2}^T)} \\
&\leq c(T) \left(\|w_{,r}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_{,t}\|_{L_\sigma(B_{R,R_0/2}^T)} \right. \\
&\quad \left. + \|w_0\|_{L_\sigma(B_{R,R_0/2})} + \|w_1\|_{W_\sigma^{1-2/\sigma}(B_{R,R_0/2})} \right),
\end{aligned} \tag{94}$$

where the relation $w(t) = \int_0^t w_{,t'} dt' + w_0$ was exploited.

Using the interpolation inequality

$$\begin{aligned}
&\|w_{,t}\|_{L_\sigma(B_{R,R_0/2}^T)} \\
&\leq \varepsilon \|w_{,rt}\|_{L_\sigma(B_{R,R_0/2}^T)} + c \left(\frac{1}{\varepsilon} \right) \|w_{,t}\|_{L_2(B_{R,R_0/2}^T)}
\end{aligned} \tag{95}$$

in (94), assuming that ε is sufficiently small, and using (77), we obtain

$$\begin{aligned}
&\|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0/2}^T)} \\
&\leq c(T) \left(\|w_{,r}\|_{L_\sigma(B_{R,R_0/2}^T)} + \|w_0\|_{L_\sigma(B_{R,R_0/2})} \right. \\
&\quad \left. + \|w_1\|_{W_\sigma^{1-2/\sigma}(B_{R,R_0/2})} + c_0 \right).
\end{aligned} \tag{96}$$

Employing

$$w_{,r}(t) = \int_0^t w_{,rt'} dt' + w_{,r}(0) \tag{97}$$

in (96) and applying the Gronwall lemma, we get

$$\begin{aligned}
&\|w_{,t} r^2\|_{W_\sigma^{1,1/2}(B_{R,R_0/2}^T)} \\
&\leq c(T) \left(\|w_0\|_{W_\sigma^1(B_{R,R_0/2})} + \|w_1\|_{W_\sigma^{1-2/\sigma}(B_{R,R_0/2})} + c_0 \right).
\end{aligned} \tag{98}$$

From (98), we obtain (69). This concludes the proof. $\square$

To derive an estimate for (44), we have to take into account Lemmas 6 and 7. The lemmas imply an estimate for the first two terms on the r.h.s. of (44). For this purpose, we need the following.

Lemma 8. *Let w be a solution to problems (11)–(13). Assume that there exist constants $\bar{\alpha}_i, \bar{c}_i, i = 1, 2, \bar{\beta}_1$ such that (41) holds, where $\psi = \psi_1(\vartheta) + \psi_2(\eta)$, $\vartheta = w_r, \eta = w/r$. Assume that $w_{,tt}(0) \in L_{2,\mu_1}(B_R)$, $w_0 \in W_3^1(B_{R,R_0/2}) \cap H_{\mu_2}^1(B_R)$, $w_1 \in H_{\mu_1}^1(B_R) \cap L_{2,\mu_2}(B_R) \cap W_3^{2/3}(B_{R,R_0/2})$, c_0 is defined in (36),*

$$\begin{aligned} c_1 &= \|w_1\|_{L_{2,\mu_2}(B_R)} + \|w_0\|_{H_{\mu_2}^1(B_R)}, \\ c_2 &= \|w_{,tt}(0)\|_{L_{2,\mu_1}(B_R)} + \|w_1\|_{H_{\mu_1}^1(B_R)}, \\ \mu_1 &\in \left(\frac{3}{2}, 1 + \frac{1}{2}\sqrt{5}\right), \quad \mu_2 \in \left(\frac{5}{6}, \frac{3}{2}\right). \end{aligned} \quad (99)$$

Then the following estimate holds:

$$\begin{aligned} &\|w_{,tt}\|_{L_{\infty}(0,t;L_{2,\mu_1}(B_R))} + \|w_{,t}\|_{L_{\infty}(0,t;H_{\mu_1}^1(B_R))} + \|w_{,tt}\|_{L_2(0,t;H_{\mu_1}^1(B_R))} \\ &\leq c(t) \left[\left(c_1 + c_0 + \|w_0\|_{W_3^1(B_{R,R_0/2})} \right. \right. \\ &\quad \left. \left. + \|w_1\|_{W_3^{2/3}(B_{R,R_0/2})} \right)^{3/2} + c_2 \right], \quad t \leq T. \end{aligned} \quad (100)$$

Proof. From (44) and (51), we have

$$\begin{aligned} &\|w_{,tt}\|_{L_{\infty}(0,t;L_{2,\mu_1}(B_r))} + \|w_{,t}\|_{L_{\infty}(0,t;H_{\mu_1}^1(B_r))} + \|w_{,tt}\|_{L_2(0,t;H_{\mu_1}^1(B_r))} \\ &\leq c(t) \left[\left(\int_{B_R^T} |w_{,rt}|^3 r^{2\mu_1} dr dt \right)^{1/2} + c_2 \right], \end{aligned} \quad (101)$$

where $\mu_1 \in (0, 1 + (1/2)\sqrt{5})$, $t \leq T$, $c_2 = \|w_{,tt}(0)\|_{L_{2,\mu_1}(B_r)} + \|w_{,t}(0)\|_{H_{\mu_1}^1(B_r)}$.

Inequality (68) takes the form

$$\|w_{,rt}\|_{L_{\sigma,2-\nu}(B_{R_0}^t)} \leq c(t) \left(\|w_{,t}r^2\|_{W_{\sigma}^{1,1/2}(B_{R_0}^t)} + c_1 \right), \quad (102)$$

where $c_1 = \|w_1\|_{L_{2,\mu_2}(B_R)} + \|w_0\|_{H_{\mu_2}^1(B_R)}$, $\mu_2 \in (0, 3/2)$.

Finally, (69) yields

$$\begin{aligned} &\|w_{,t}r^2\|_{W_{\sigma}^{1,1/2}(B_{R_0}^t)} \\ &\leq c(t) \left(c_0 + \|w_0\|_{W_{\sigma}^1(B_{R,R_0/2})} + \|w_1\|_{W_{\sigma}^{1-1/\sigma}(B_{R,R_0/2})} \right). \end{aligned} \quad (103)$$

To apply (102) and (103) in the r.h.s. of (101), we assume $\nu = 2 - (2/3)\mu_1$, $\sigma = 3$. In view of the assumptions of Lemma 6, we have that $\nu < 1$ so $2 - (2/3)\mu_1 < 1$. This implies that $\mu_1 > 3/2$. Hence, $\mu_1 \in (3/2, 1 + (1/2)\sqrt{5})$. Since $\nu = 2 - (2/3)\mu_1$, assumptions of Lemma 6 imply that $\mu_2 < 1/3 + 3/2 - \nu = 11/6 - 2 + (2/3)\mu_1$, so $\mu_2 \in (5/6, 1/2 + (1/3)\sqrt{5}) \cap (0, 3/2) = (5/6, 3/2)$. Employing (102) and (103) in (101) yields (100). This concludes the proof. $\square$

Remark 9. In view of the expression of $w_{,tt}(0)$ appeared in the assumptions of Lemma 4 we derive

$$\|w_{,tt}(0)\|_{L_{2,\mu_1}(B_R)} \leq c \left(\|w_0\|_{H_{\mu_1}^2(B_R)} + \|w_1\|_{H_{\mu_1}^2(B_R)} \right). \quad (104)$$

Since $\mu_1 - 2 \in (-1/2, (1/2)\sqrt{5} - 1)$ can be chosen positive, we see that all weights in the norms $\|w_0\|_{H_{\mu_1}^2(B_R)}$, $\|w_1\|_{H_{\mu_1}^2(B_R)}$ can be chosen as power functions with positive exponents.

Assuming that the r.h.s. of (104) is finite assumptions (99) can be replaced by

$$\begin{aligned} w_1, w_0 &\in H_{\mu_1}^2(B_R), \quad \mu_1 \in \left(\frac{3}{2}, 1 + \frac{\sqrt{5}}{2}\right), \\ \mu_2 &\geq \mu_1 - 1, \quad \mu_2 \in \left(\frac{5}{6}, \frac{3}{2}\right). \end{aligned} \quad (105)$$

Up to now there is no estimate for $w_{,rr}$. Hence, to get the space regularity, we need the following.

Lemma 10. *Let (105) hold. Let*

$$\psi_{,99} \geq \alpha_1, \quad |\psi_{,9}| \leq c|\vartheta|, \quad |\psi_{,\eta}| \leq c|\eta|. \quad (106)$$

Then solutions to problems (11)–(13) satisfy

$$\int_0^T \int_0^R w_{,rr}^2 r^{2\mu} dr dt + \int_0^R w_{,rr}^2 r^{2\mu} dr \leq \varphi(c_0, c_1, c_3, T), \quad (107)$$

where $2 < \mu < (1/3)(5 + \sqrt{5})$, c_0 is from (36), c_1 from (40) and

$$c_3 = \|w_0\|_{H_{\mu}^2(B_R)} + \|w_1\|_{H_{\mu}^2(B_R)}. \quad (108)$$

Proof. Multiplying (11) by $w_{,rr}r^{2\mu}$ and integrating over B_R , we obtain

$$\begin{aligned} \int_0^R w_{,tt} w_{,rr} r^{2\mu} dr &= \int_0^R \frac{1}{r^2} [(\psi_{,9} + \mu_0 w_{,rt}) r^2]_{,r} w_{,rr} r^{2\mu} dr \\ &\quad - \int_0^R \left(\frac{1}{r} \psi_{,\eta} + \mu_0 \frac{w_{,t}}{r} \right) w_{,rr} r^{2\mu} dr. \end{aligned} \quad (109)$$

Continuing, we have

$$\begin{aligned} &\int_0^R (\psi_{,99} w_{,rr}^2 + \mu_0 w_{,rrt} w_{,rr}) r^{2\mu} dr \\ &= \int_0^R w_{,tt} w_{,rr} r^{2\mu} dr \\ &\quad - 2 \int_0^R (\psi_{,9} + \mu_0 w_{,rt}) w_{,rr} r^{2\mu-1} dr \\ &\quad + \int_0^R (\psi_{,\eta} + \mu_0 w_{,t}) w_{,rr} r^{2\mu-1} dr. \end{aligned} \quad (110)$$

Using (106), we obtain

$$\begin{aligned} r\alpha_1 \int_0^R w_{,rr}^2 r^{2\mu} dr + \frac{\mu_0}{2} \frac{d}{dt} \int_0^R w_{,rr}^2 r^{2\mu} dr \\ \leq \varepsilon \int_0^R w_{,rr}^2 r^{2\mu} dr + c \left(\frac{1}{\varepsilon} \right) \int_0^R w_{,tt}^2 r^{2\mu} dr \\ + c \int_0^R \left(w_{,r}^2 + w_{,rt}^2 + w_{,t}^2 + w^2 r^{-2} \right) r^{2\mu-2} dr. \end{aligned} \quad (111)$$

Integrating the inequality with respect to time, assuming that ε is sufficiently small, using that $\mu = 1 + \nu$, $\nu \in (0, 3/2)$, and employing (40) with $\mu = \nu$ and (102), (103) with $\sigma = 2$, $\nu = 2 - (2/3)\mu_1$, $\mu = (2/3)\mu_1 + 1$, we obtain (101). Hence, $\mu \in (1, 5/2) \cap (2, 5/3 + \sqrt{5}/3) = (2, (1/3)(5 + \sqrt{5}))$. This concludes the proof. $\square$

Finally, we recall some local properties of solutions to problems (11)–(13) which are proved in Lemma 3.11 in [13].

Lemma 11. *Let the assumptions of Lemma 3 be satisfied. Then*

$$\begin{aligned} |w(r, t)| &\leq \frac{R^{1/2-\mu}}{\sqrt{1-2\mu}} c_1, \quad \mu \in \left(0, \frac{1}{2}\right), \\ \|w_{,t}\|_{L_2(0,T;C^{1/2-\mu/2}(B_R))} &\leq cc_1, \quad \mu \in \left(0, \frac{1}{2}\right), \\ \|w\|_{B(0,T;C^{1/2-\mu/2}(B_R))} &\leq cc_1, \quad \mu \in \left(0, \frac{1}{2}\right), \\ \|w\|_{L_{\beta}(0,R;C^{(\beta-1)/\beta}(0,T))} &\leq cc_1, \quad 1 < \beta < \frac{2}{2\mu+1}, \mu \in \left(0, \frac{1}{2}\right). \end{aligned} \quad (112)$$

4. Existence

We prove the existence of solutions to problems (11)–(13) by the following method of successive approximations:

$$\begin{aligned} w_{,tt}^{(n+1)} - \mu_0 \frac{1}{r^2} \left(w_{,rt}^{(n+1)} r^2 \right)_{,r} + \mu_0 \frac{1}{r^2} w_{,t}^{(n+1)} \\ = \frac{1}{r^2} \left(\psi_{,g^{(n)}} r^2 \right)_{,r} - \frac{1}{r} \psi_{,n^{(n)}}, \\ w_{,t}^{(n+1)}|_{t=0} = w_0, \quad w_{,t}^{(n+1)}|_{t=0} = w_1, \\ w_{,r}^{(n+1)}|_{r=R} = 0, \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \quad (113)$$

where $\vartheta^{(n)} = w_r^{(n)}$, $n^{(n)} = w^{(n)}/r$, and we assume that the zero approximation is defined by

$$w^{(0)} = w_0 + tw_1. \quad (114)$$

The boundary condition (113)₃ implies (13) because

$$\left| \frac{\partial \psi}{\partial \vartheta} \right| \leq c |\vartheta|. \quad (115)$$

First, we examine the linear problem

$$\begin{aligned} u_{,t} - \mu_0 u_{,rr} - \frac{2\mu_0 u_{,r}}{r} + \mu_0 \frac{u}{r^2} &= f \quad \text{in } B_R^T, \\ u|_{t=0} &= u_1 \quad \text{in } B_R, \\ u_{,r}|_{r=R} &= 0 \quad \text{on } S_R. \end{aligned} \quad (116)$$

Lemma 12. *Assume that $f \in L_{2,\nu}(B_R^T)$, $u_1 \in H_\nu^1(B_R)$, $\nu \in (2 - (1/2)\sqrt{5}, 2 + (1/2)\sqrt{5})$. Then there exists a solution to problem (116) such that $u \in H_\nu^{2,1}(B_R^T)$ and the estimate holds*

$$\|u\|_{H_\nu^{2,1}(B_R^T)} \leq c \left(\|f\|_{L_{2,\nu}(B_R^T)} + \|u_1\|_{H_\nu^1(B_R)} \right). \quad (117)$$

Proof. The existence can be proved by the Galerkin method. Therefore, we shall restrict the proof to show only estimate (117).

Multiplying (116)₁ by $ur^{2\mu}$ and integrating the result over B_R yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{B_R} u^2 r^{2\mu} dr + \mu_0 \int_{B_R} u_{,r}^2 r^{2\mu} dr \\ + 2(\mu - 1) \mu_0 \int_{B_R} u_{,r} u r^{2\mu-1} dr \\ + \mu_0 \lim_{r \rightarrow 0} (u_{,r} u r^{2\mu}) + \mu_0 \int_{B_R} u^2 r^{2\mu-2} dr \\ = \int_{B_R} f u r^{2\mu} dr. \end{aligned} \quad (118)$$

Looking for solutions that $u_{,r}u$ is finite at $r = 0$, the last but one term on the l.h.s. of (118) vanishes for $\mu > 0$. Vanishing of the term can be guaranteed by choosing an appropriate base functions.

Integrating by parts in the third term on the l.h.s. of (118) implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{B_R} u^2 r^{2\mu} dr + \mu_0 \int_{B_R} u_{,r}^2 r^{2\mu} dr \\ + \mu_0 \mu (3 - 2\mu) \int_{B_R} u^2 r^{2\mu-2} dr + (1 - \mu) \mu_0 u^2 r^{2\mu-1} \Big|_{r=0} \\ = (1 - \mu) \mu_0 u^2(R, t) R^{2\mu-1} + \int_{B_R} f u r^{2\mu} dr. \end{aligned} \quad (119)$$

For $\mu \in (0, 1)$, the last term on the l.h.s. is positive so it can be omitted in the estimate implied by (119). Then we show the existence of solutions such that

$$\int_0^T \int_0^R u^2 r^{2\mu-2} dr < \infty. \quad (120)$$

Hence, $u \sim r^\varepsilon$ near $r = 0$, where $\varepsilon > -\mu + 1/2$. Then the last term on the l.h.s. of (119) vanishes. For $\mu > 1$, the last term on the l.h.s. of (119) vanishes under the assumption that we are looking for solutions such that $u(r, t) \sim r^\varepsilon$ near $r = 0$, $\varepsilon > -\mu + 1/2$.

Finally, we estimate the first term on the r.h.s. of (119). Let us recall the set

$$B_{R,R_0} = \{r \in (0, R) : 0 < R_0 < r < R\}. \quad (121)$$

Then by imbedding, we get

$$\begin{aligned} |u(R, t)|^2 &\leq \varepsilon \int_{B_{R,R_0}} u_{,r}^2 dr + c \left(\frac{1}{\varepsilon} \right) \int_{B_{R,R_0}} u^2 dr \\ &\leq \varepsilon' \int_{B_R} u_{,r}^2 r^{2\mu} dr + c \left(\frac{1}{\varepsilon'} \right) \int_{B_R} u^2 r^{2\mu} dr, \end{aligned} \quad (122)$$

where $\varepsilon' = \varepsilon/R_0^{2\mu}$.

Using that ε' is sufficiently small, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{B_R} u^2 r^{2\mu} dr + \mu'_0 \int_{B_R} u_{,r}^2 r^{2\mu} dr \\ &\quad + \mu'_0 \mu (3 - 2\mu) \int_{B_R} u^2 r^{2\mu-2} dr \\ &\leq c \int_{B_R} u^2 r^{2\mu} dr + \left(\int_{B_R} f^2 r^{2\mu} dr \right)^{1/2} \left(\int_{B_R} u^2 r^{2\mu} dr \right)^{1/2} \\ &\leq c \int_{B_R} u^2 r^{2\mu} dr + c \int_{B_R} f^2 r^{2\mu} dr. \end{aligned} \quad (123)$$

Integrating (123) with respect to time, we get the estimate

$$\begin{aligned} &\int_{B_R} u^2 r^{2\mu} dr + \mu'_0 \int_{B_R^t} u_{,r}^2 r^{2\mu} dr dt \\ &\quad + \mu'_0 \mu (3 - 2\mu) \int_{B_R^t} u^2 r^{2\mu-2} dr dt \\ &\leq e^{ct} \left(\int_{B_R^t} f^2 r^{2\mu} dr + \int_{B_R} u_1^2 r^{2\mu} dr \right) \equiv \alpha_1(t). \end{aligned} \quad (124)$$

The above inequality implies the estimate for $\mu \in [0, 3/2]$. To increase the interval, we use the Hardy inequality

$$\left| \frac{1}{2} - \mu \right|^{1/2} \int_0^R u^2 r^{2\mu-2} dr \leq \int_0^R u_{,r}^2 r^{2\mu} dr. \quad (125)$$

Combining (124) and (125), we see that (124) gives an estimate for μ satisfying the inequality

$$\alpha \left| \frac{1}{2} - \mu \right|^2 + \mu (3 - 2\mu) > 0, \quad \alpha = \frac{\mu'_0}{\mu_0}. \quad (126)$$

Solving (126), we see that $\mu \in ((3 - \alpha - \sqrt{9 - 4\alpha})/2(2 - \alpha), (3 - \alpha + \sqrt{9 - 4\alpha})/2(2 - \alpha))$. Since α is very close to 1, we can assume that

$$\mu \in \left(1 - \frac{1}{2} \sqrt{5}, 1 + \frac{1}{2} \sqrt{5} \right). \quad (127)$$

Hence, $[0, 3/2] \subset (1 - (1/2)\sqrt{5}, 1 + (1/2)\sqrt{5})$.

Finally, for μ satisfying (127), we have estimate (124). In the next step, we consider the parabolic problem

$$\begin{aligned} u_{,t} - \mu_0 u_{,rr} &= 2\mu_0 \frac{u_{,r}}{r} - \mu_0 \frac{u}{r^2} + f, \\ u|_{t=0} &= u_1, \quad u_{,r}|_{r=R} = 0. \end{aligned} \quad (128)$$

For solutions to problem (128), we have the inequality (see Lemma 1)

$$\begin{aligned} \|u\|_{H_{\mu'}^{2,1}(B_R^T)} &\leq c \left(\left\| \frac{u_{,r}}{r} \right\|_{L_{2,\mu'}(B_R^T)} + \left\| \frac{u}{r^2} \right\|_{L_{2,\mu'}(B_R^T)} \right. \\ &\quad \left. + \|f\|_{L_{2,\mu'}(B_R^T)} + \|u_1\|_{H_{\mu'}^1(B_R)} \right). \end{aligned} \quad (129)$$

The first two norms on the r.h.s. of (129) can be estimated in view of (124). Then we obtain some restriction on μ' . We have

$$\left\| \frac{u_{,r}}{r} \right\|_{L_{2,\mu'}(B_R^T)} \leq \|u_{,r}\|_{L_{2,\mu'-1}(B_R^T)} \leq \alpha_1(T), \quad (130)$$

where the last inequality holds for $\mu' - 1 = \mu$, so $\mu' \in (2 - (1/2)\sqrt{5}, 2 + (1/2)\sqrt{5})$. Hence, μ' can be chosen less than 1. Similarly, the inequalities

$$\left\| \frac{u}{r^2} \right\|_{L_{2,\mu'}(B_R^T)} \leq \|u\|_{L_{2,\mu'-2}(B_R^T)} \leq \alpha_1(T) \quad (131)$$

hold for $\mu' - 2 = \mu - 1$ so $\mu' - 1 = \mu$. Hence, the same restriction on μ' as before is satisfied. Since the existence follows from the Galerkin method, Lemma 12 is proved. $\square$

In view of Lemma 12, we have the following.

Lemma 13. Assume that $w_0 \in H_v^2(B_R)$, $w_1 \in H_v^2(B_R)$, $\nu \in (2, 1 + (1/2)\sqrt{5})$. Assume the Lipschitz continuity

$$\begin{aligned} |\psi_{,\vartheta}(u) - \psi_{,\vartheta}(v)| &\leq c|u - v|, \\ |\psi_{,\eta}(u) - \psi_{,\eta}(v)| &\leq c|u - v|. \end{aligned} \quad (132)$$

Then there exists a solution to problems (11)–(13) such that $w_{,t} \in H_{\nu}^{2,1}(B_R^T)$ and

$$\|w_{,t}\|_{H_{\nu}^{2,1}(B_R^T)} \leq c \left(\|w_0\|_{H_{\nu}^2(B_R)} + \|w_1\|_{H_{\nu}^1(B_R)} \right). \quad (133)$$

Proof. Applying Lemma 12 to problem (113) yields

$$\begin{aligned} &\|w_{,t}^{(n+1)}\|_{H_{\nu}^{2,1}(B_R^T)} \\ &\leq c \left(\|w_{,rr}^{(n)}\|_{L_{2,\nu}(B_R^T)} + \|w_{,r}^{(n)}\|_{L_{2,\nu-1}(B_R^T)} \right. \\ &\quad \left. + \|w^{(n)}\|_{L_{2,\nu-2}(B_R^T)} + \|w_1\|_{H_{\nu}^1(B_R)} \right), \end{aligned} \quad (134)$$

where $\nu \in (2 - (1/2)\sqrt{5}, 2 + (1/2)\sqrt{5})$ and we used that

$$|\psi_{,\vartheta\vartheta}| \leq c, \quad |\psi_{,\vartheta}| \leq c|\vartheta|, \quad |\psi_{,\eta}| \leq c|\eta|. \quad (135)$$

In view of the relation

$$w = \int_0^t w_{,t'} dt' + w_0, \quad (136)$$

we obtain from (134) the inequality

$$\begin{aligned} & \|w_{,t}^{(n+1)}\|_{H_v^{2,1}(B_R^T)} \\ & \leq cT^{1/2} \|w_{,t}^{(n)}\|_{H_v^{2,1}(B_R^T)} + c\|w_0\|_{H_v^2(B_R)} + c\|w_1\|_{H_v^1(B_R)}. \end{aligned} \quad (137)$$

Hence, for sufficiently small T , we have

$$\|w_{,t}^{(n)}\|_{H_v^{2,1}(B_R^T)} \leq c \left(\|w_0\|_{H_v^2(B_R)} + \|w_1\|_{H_v^1(B_R)} \right), \quad (138)$$

for any $n \in \mathbb{N}$.

To show convergence of the sequence $\{w^{(n)}\}_{n \in \mathbb{N}}$ we introduce the differences

$$W^{(n)} = w^{(n)} - w^{(n-1)}, \quad (139)$$

which are solutions to the problems

$$\begin{aligned} & W_{,tt}^{(n+1)} - \frac{\mu_0}{r^2} (W_{,rt}^{(n+1)} r^2)_{,r} + \frac{\mu_0}{r^2} W_{,t}^{(n+1)} \\ & = \frac{1}{r^2} [(\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}) r^2]_{,r} - \frac{1}{r} (\psi_{,\eta^{(n)}} - \psi_{,\eta^{(n-1)}}), \end{aligned} \quad (140)$$

$$W^{(n+1)}|_{t=0} = 0, \quad W_{,t}^{(n+1)}|_{t=0} = 0, \quad W_{,r}^{n+1}|_{r=R} = 0.$$

To show convergence we multiply (140) by $W_{,t}^{(n+1)} r^{2\nu}$ and integrate the result over B_R . Then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu} dr + \mu_0 \int_{B_R} |W_{,tr}^{(n+1)}|^2 r^{2\nu} dr \\ & + \mu_0 \nu (3 - 2\nu) \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu-2} dr \\ & = \int_{B_R} \frac{1}{r^2} [(\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}) r^2]_{,r} W_{,t}^{(n+1)} r^{2\nu} dr \\ & - \int_{B_R} \frac{1}{r} (\psi_{,\eta^{(n)}} - \psi_{,\eta^{(n-1)}}) W_{,t}^{(n+1)} r^{2\nu} dr \equiv I_1 + I_2. \end{aligned} \quad (141)$$

Integrating by parts in I_1 yields

$$\begin{aligned} I_1 &= - \int_{B_R} (\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}) W_{,tr}^{(n+1)} r^{2\nu} dr \\ & - 2(\nu - 1) \int_{B_R} (\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}) W_{,t}^{(n+1)} r^{2\nu-1} dr \equiv I_1^1 + I_1^2, \end{aligned} \quad (142)$$

where

$$\begin{aligned} |I_1^1| &\leq \varepsilon_1 \int_{B_R} |W_{,rt}^{(n+1)}|^2 r^{2\nu} dr \\ & + c \left(\frac{1}{\varepsilon_1} \right) \int_{B_R} |\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}| r^2 dr, \end{aligned} \quad (143)$$

and the second integral in view of the assumption of the Lipschitz continuity is bounded by

$$c \int_{B_R} |W_{,r}^{(n)}|^2 r^{2\nu} dr. \quad (144)$$

Next, we examine

$$\begin{aligned} |I_1^2| &\leq c \int_{B_R} |\psi_{,g^{(n)}} - \psi_{,g^{(n-1)}}| |W_{,t}^{(n+1)}| r^{2\nu-1} dr \\ &\leq \varepsilon_2 \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu-2} dr + c \left(\frac{1}{\varepsilon_2} \right) \int_{B_R} |W_{,r}^{(n)}|^2 r^{2\nu} dr. \end{aligned} \quad (145)$$

Finally, we estimate I_2 ,

$$\begin{aligned} |I_2| &\leq c \int_{B_R} \frac{1}{r^2} |W^{(n)}| |W_{,t}^{(n+1)}| r^{2\nu} dr \\ &\leq \varepsilon_3 \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu-2} dr + c \left(\frac{1}{\varepsilon_3} \right) \int_{B_R} |W^{(n)}|^2 r^{2\nu-2} dr. \end{aligned} \quad (146)$$

Using the above estimates in (141) and assuming that $\varepsilon_1 - \varepsilon_3$ are sufficiently small, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu} dr + \mu_0' \int_{B_R} |W_{,rt}^{(n+1)}|^2 r^{2\nu} dr \\ & + \mu_0 \nu (3 - 2\nu) \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu-2} dr \\ & \leq c \left(\int_{B_R} |W_{,r}^{(n)}|^2 r^{2\nu} dr + \int_{B_R} |W^{(n)}|^2 r^{2\nu-2} dr \right), \end{aligned} \quad (147)$$

where $\mu_0' < \mu_0$ and the same considerations as in (125)–(127) are repeated. Integrating (147) with respect to time and employing the formula

$$W^{(n)}(t) = \int_0^t W_{,t}^{(n)} dt, \quad (148)$$

we derive from (147) the inequality

$$\begin{aligned} & \int_{B_R} |W_{,t}^{(n+1)}|^2 r^{2\nu} dr + \mu_0' \int_{B_R^T} |W_{,rt}^{(n+1)}|^2 r^{2\nu} dr dt \\ & + \mu_0 \nu (3 - 2\nu) \int_{B_R^T} |W_{,t}^{(n+1)}|^2 r^{2\nu-2} dr dt \\ & \leq cT \left(\int_{B_R^T} |W_{,rt}^{(n)}|^2 r^{2\nu} dr dt + \int_{B_R^T} |W_{,t}^{(n)}|^2 r^{2\nu-2} dr dt \right). \end{aligned} \quad (149)$$

Hence, we have that the sequence $\{w^{(n)}\}_{n \in \mathbb{N}}$ converges and then we show the existence of solutions to problems (11)–(13) on a small interval $(0, T)$. Since we have the estimate on the interval $(0, T)$ with arbitrary T , the above considerations can be repeated step by step.

In view of Lemmas 8 and 10 and Remark 9, the extension can be made under the restriction that $\nu \in (2, 1 + (1/2)\sqrt{5})$. This concludes the proof. $\square$

Proof. Proof of the Main Theorem. In view of assumptions of Lemmas 8 and 10, Remark 9, and also by Lemmas 12 and 13, we have the first assertion of the theorem. In view of Lemma 3 and the Galerkin method, we prove the second assertion. In view of Lemmas 3 and 11, the third assertion follows. This concludes the proof. $\square$

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