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Research Article

Solutions of Nonlocal $(p_1(x), p_2(x))$-Laplacian Equations

Mustafa Avci[1] and Rabil Ayazoglu (Mashiyev)[2]

[1] *Faculty of Economics and Administrative Sciences, Batman University, 72000 Batman, Turkey*
[2] *Faculty of Education, Bayburt University, 69000 Bayburt, Turkey*

Correspondence should be addressed to Mustafa Avci; avcixmustafa@gmail.com

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In view of variational approach we discuss a nonlocal problem, that is, a Kirchhoff-type equation involving $(p_1(x), p_2(x))$-Laplace operator. Establishing some suitable conditions, we prove the existence and multiplicity of solutions.

1. Introduction

We study the existence and multiplicity of solutions of the nonlocal equation

$$\begin{aligned} -M_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)}dx}{p_1(x)}\right)\operatorname{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right) \\ - M_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)}dx}{p_2(x)}\right)\operatorname{div}\left(|\nabla u|^{p_2(x)-2}\nabla u\right) \\ = f(x,u) \quad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega, \end{aligned} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $p_i \in C(\overline{\Omega})$ such that $2 \leq p_i(x) < N$ for any $x \in \overline{\Omega}$, and $i = 1, 2$.

The problem (P) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by [1]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

where ρ, P_0, h, E, and L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. A distinguish feature of the Kirchhoff equation (1) is that the equation contains a nonlocal coefficient $P_0/h + (E/2L)\int_0^L |\partial u/\partial x|^2 dx$ which depends on the average $(E/2L)\int_0^L |\partial u/\partial x|^2 dx$ of the kinetic energy $(1/2)|\partial u/\partial x|^2$ on $[0, L]$, and hence the equation is no longer a pointwise identity. For Kirchhoff-type equations involving the $p(x)$-Laplacian operator, see, for example, [2–4].

The $p(x)$-Laplacian operator $-\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a natural generalization of the p-Laplacian operator $-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ where $p > 1$ is a real constant. The main difference between them is that p-Laplacian operator is $(p-1)$ homogenous, that is, $\Delta_p(\mu u) = \mu^{p-1}\Delta_p u$ for every $\mu > 0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not constant, is not homogeneous. This causes many problems; some classical theories and methods, such as the Lagrange multiplier theorem and the theory of Sobolev spaces, are not applicable. For $p(x)$-Laplacian operator, we refer the readers to [5–9] and references there in. Moreover, the nonlinear problems involving the $p(x)$-Laplacian operator are extremely attractive because they can be used to model dynamical phenomenons which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermorheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes of filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$-Laplacian can be found in [10–14] and the references therein.

In the present paper, by considering the joint effects of different $(p_1(x), p_2(x))$-Laplace operator $-\Delta_{p_1(x)}u - \Delta_{p_2(x)}u$, we study the existence and multiplicity of solutions for

a nonlocal problem, that is, problem (P) via Mountain-Pass theorem and Fountain theorem. As far as we know, there is no paper that deals with a nonlocal problem involving $(p_1(x), p_2(x))$-Laplace operator except [15] in which the authors consider problem (P) for the case $M_1 \equiv 1$ and $M_2 \equiv 1$. Therefore, our paper deals with more general results than those obtained in [15]. Moreover, if we choose the functions $M_1 \equiv M_2 \equiv 1$ in problem (P), we get the equation

$$-\left[\operatorname{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right) + \operatorname{div}\left(|\nabla u|^{p_2(x)-2}\nabla u\right)\right] = f(x,u), \tag{2}$$

which is the well-known anisotropic $\vec{p}(\cdot)$-Laplacian problem (see, e.g., [16] and references therein) in the case $N = 1, 2$, that is,

$$-\sum_{i=1}^{N}\partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\right) = f(x,u). \tag{3}$$

As mentioned above, the $p(x)$-Laplacian can be applied to describe the physical phenomenon with pointwise different properties which earliest arose from the nonlinear elasticity theory. In that context, the systems involving the $(p_1(x), p_2(x))$-Laplacian (or $(p_1(x), \ldots, p_N(x))$-Laplacian) can be good candidates for modeling phenomena which ask for distinct behavior of partial differential derivatives in various directions. For a mathematical model of a real physical phenomenon, one can consider the mean curvature operator

$$-\sum_{i=1}^{2}\partial_{x_i}\left[\left(1+\left|\partial_{x_i}u\right|^2\right)^{(p_i(x)-2)/2}\partial_{x_i}u\right] = f(x,u). \tag{4}$$

It is obvious that problem (P) is a degenerate version of (4) when $M_1 \equiv M_2 \equiv 1$.

2. Preliminaries

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain (for details, see, e.g., [17–19]).

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p : p \in C\left(\overline{\Omega}\right),\ p(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}. \tag{5}$$

For any $p \in C_+(\overline{\Omega})$, denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x), \qquad p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty, \tag{6}$$

and define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u \text{ is a measurable real function on } \Omega : \int_\Omega |u(x)|^{p(x)}dx < \infty\right\}. \tag{7}$$

We define a norm, the so-called Luxemburg norm, on $L^{p(x)}(\Omega)$ by the formula

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}dx \le 1\right\}, \tag{8}$$

and then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space.

Define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{9}$$

then it can be equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{10}$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)}, \tag{11}$$

since Poincaré inequality holds; that is, there exists a positive constant $C > 0$ such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \tag{12}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proposition 1 (see [18, 19]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $(1/p'(x)) + (1/p(x)) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right)|u|_{p(x)}|v|_{p'(x)}. \tag{13}$$

Proposition 2 (see [18, 19]). *Denote $\rho(u) = \int_\Omega |u(x)|^{p(x)}dx$, for all $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$; one has*

(i) $|u|_{p(x)} > 1 (\le 1) \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+} (|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-})$;

(ii) $\lim_{n\to\infty}|u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty}\rho(u_n) = 0$; $\lim_{n\to\infty}|u_n|_{p(x)} \to \infty \Leftrightarrow \lim_{n\to\infty}\rho(u_n) \to \infty$.

Proposition 3 (see [18, 19]). *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$, then the following statements are equivalent:*

(i) $\lim_{n\to\infty}|u_n - u|_{p(x)} = 0$;

(ii) $\lim_{n\to\infty}\rho(u_n - u) = 0$;

(iii) $u_n \to u$ *in measure* Ω *and* $\lim_{n\to\infty}\rho(u_n) = \rho(u)$.

Proposition 4 (see [18, 19]). (i) *If $1 < p^- \le p^+ < \infty$, then spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$ ($p^*(x) = Np(x)/N - p(x)$ if $p(x) < N$ and $p^*(x) = +\infty$ if $p(x) \ge N$), then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*

Definition 5. Let X be a Banach space and $J : X \to \mathbb{R}$ a C^1 functional. We say that a functional J satisfies the Palais-Smale condition ((**PS**) for short), if any sequence $\{u_n\}$ in X such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence.

3. Main Results and Proofs

Let us consider the functional

$$J(u) = \int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx + \int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx, \quad \forall u \in X, \tag{14}$$

where $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega)$ with its norm given by $\|u\| := \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, for all $u \in X$. It is obvious that $(X, \|\cdot\|)$ is also a separable and reflexive Banach space.

By using standard arguments, it can be proved that $J \in C^1(X, \mathbb{R})$ (see [20]), and the $(p_1(x), p_2(x))$-Laplace operator is the derivative operator of J in the weak sense. Denote $L := J' : X \to X^*$; then

$$\begin{aligned} \langle L(u), \varphi \rangle = & \int_\Omega |\nabla u|^{p_1(x)-2} \nabla u \nabla \varphi \, dx \\ & + \int_\Omega |\nabla u|^{p_2(x)-2} \nabla u \nabla \varphi \, dx, \quad \forall u, \varphi \in X, \end{aligned} \tag{15}$$

where $\langle \cdot, \cdot \rangle$ is the dual pair between X and its dual X^*.

Let us denote

$$\begin{aligned} p_M(x) &= \max\{p_1(x), p_2(x)\}, \\ p_m(x) &= \min\{p_1(x), p_2(x)\}, \quad \forall x \in \overline{\Omega}. \end{aligned} \tag{16}$$

By the definition, it is not difficult to see that $p_M(x), p_m(x) \in C_+(\overline{\Omega})$. For $q(x) \in C_+(\overline{\Omega})$ such that $q(x) < p_M^*(x)$ for any $x \in \overline{\Omega}$, we have $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega) = W_0^{1,p_M(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous and compact.

We say that $u \in X$ is a weak solution of (P) if

$$\begin{aligned} & M_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx\right) \int_\Omega |\nabla u|^{p_1(x)-2} \nabla u \nabla \varphi \, dx \\ & \quad + M_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx\right) \int_\Omega |\nabla u|^{p_2(x)-2} \nabla u \nabla \varphi \, dx \\ & = \int_\Omega f(x, u) \varphi \, dx, \end{aligned} \tag{17}$$

for any $\varphi \in X$.

We associate to the problem (P) the energy functional, defined as $I : X \to \mathbb{R}$:

$$\begin{aligned} I(u) = & \widehat{M}_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx\right) \\ & + \widehat{M}_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx\right) - \int_\Omega F(x, u) \, dx, \end{aligned} \tag{18}$$

where $\widehat{M}_i(t) = \int_0^t M_i(\xi) d\xi$ $(i = 1, 2.)$ and $F(x, u) = \int_0^u f(x, s) ds$. We know that from $(\mathbf{M}_0)$ and $(\mathbf{f}_0)$ (see below) I is well defined and in a standard way we can prove that $I \in C^1(X, \mathbb{R})$ and that the critical points of I are solutions of (P).

Moreover, the derivative of I is given by

$$\begin{aligned} & \langle I'(u), \varphi \rangle \\ & \quad = M_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx\right) \int_\Omega |\nabla u|^{p_1(x)-2} \nabla u \nabla \varphi \, dx \\ & \quad + M_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx\right) \int_\Omega |\nabla u|^{p_2(x)-2} \nabla u \nabla \varphi \, dx \\ & \quad - \int_\Omega f(x, u) \varphi \, dx, \end{aligned} \tag{19}$$

for any $u, \varphi \in X$.

Now, we are ready to set and prove the first main result of the present paper.

Theorem 6. *Assume that the following assumptions hold:*

$(\mathbf{M}_0)$ $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous functions and satisfy the conditions*

$$\begin{aligned} C_1 t^{\alpha-1} &\le M_1(t), \\ C_2 t^{\alpha-1} &\le M_2(t), \end{aligned} \tag{20}$$

for all $t > 0$, where C_1 and C_2 are positive constants and $\alpha > 1$;

$(\mathbf{f}_0)$ $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies the growth condition*

$$|f(x, t)| \le C_3 + C_4 |t|^{q(x)-1}, \quad \forall (x, t) \in \overline{\Omega} \times \mathbb{R}, \tag{21}$$

where C_3 and C_4 are positive constants and $p_1, p_2, q \in C_+(\overline{\Omega})$ such that $q^+ < \alpha p_m^- < p_M^*(x)$, for all $x \in \overline{\Omega}$.

Then problem (P) *has a weak solution.*

Proof. Let $\|u\| > 1$. By the assumptions $(\mathbf{M}_0)$ and $(\mathbf{f}_0)$, we have

$$\begin{aligned} I(u) = & \widehat{M}_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx\right) \\ & + \widehat{M}_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx\right) - \int_\Omega F(x, u) \, dx \\ \ge & C_1 \int_0^{(1/p_1^+)\rho_1(\nabla u)} t^{\alpha-1} dt + C_2 \int_0^{(1/p_2^+)\rho_2(\nabla u)} t^{\alpha-1} dt \end{aligned}$$

$$
\begin{aligned}
&- C_3 \int_\Omega |u|^{q(x)} dx - C_4 \\
&= \frac{C_1}{\alpha(p_1^+)^\alpha}\left(\int_\Omega |\nabla u|^{p_1(x)} dx\right)^\alpha \\
&\quad + \frac{C_2}{\alpha(p_2^+)^\alpha}\left(\int_\Omega |\nabla u|^{p_2(x)} dx\right)^\alpha \\
&\quad - C_3 \int_\Omega |u|^{q(x)} dx - C_4 \\
&\geq \frac{C_1}{\alpha(p_1^+)^\alpha}\|u\|^{\alpha p_1^-} + \frac{C_2}{\alpha(p_2^+)^\alpha}\|u\|^{\alpha p_2^-} \\
&\quad - C_3\|u\|^{q^+} - C_4 \\
&\geq \frac{c}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_3\|u\|^{q^+} \\
&\quad - C_4 \longrightarrow +\infty \quad (\|u\| \longrightarrow \infty),
\end{aligned} \tag{22}
$$

where $\rho_1(\nabla u) = \int_\Omega |\nabla u|^{p_1(x)} dx$, $\rho_2(\nabla u) = \int_\Omega |\nabla u|^{p_2(x)} dx$ and $c = \min\{C_1, C_2\}$. So, I is coercive. Since I is sequentially weakly lower semicontinuous, I has a minimum point u in X and u is a weak solution of (P). □

Theorem 7. *Assume that the following assumptions hold:*

($\mathbf{M}_1$) $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous functions and satisfy the conditions*

$$
\begin{aligned}
C_5 t^{\alpha-1} &\leq M_1(t) \leq C_6 t^{\alpha-1}, \\
C_7 t^{\alpha-1} &\leq M_2(t) \leq C_8 t^{\alpha-1},
\end{aligned} \tag{23}
$$

for all $t > 0$, where C_5, C_6, C_7, C_8, and α are positive constants such that $C_5 \leq C_6 \leq C_7 \leq C_8$ and $\alpha > 1$;

($\mathbf{f}_1$) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies the growth condition*

$$
|f(x,t)| \leq C_9 + C_{10}|t|^{\beta(x)-1}, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}; \tag{24}
$$

($\mathbf{f}_2$) $f(x,t) = o(|t|^{\alpha p_M^+ - 1})$, $t \to 0$ *uniformly for* $x \in \overline{\Omega}$, *where C_9 and C_{10} are positive constants and $\beta \in C_+(\overline{\Omega})$ such that $\alpha p_M^+ < \beta^- \leq \beta^+ < p_M^*(x)$, for all $x \in \overline{\Omega}$;*

($\mathbf{f}_3$) $f(x,-t) = -f(x,t), \forall (x,t) \in \overline{\Omega} \times \mathbb{R}$;

(**AR**) $\exists K > 0$, $\theta > (p_M^+)^\alpha$ *such that*

$$
0 < \theta F(x,t) \leq f(x,t)\, t, \qquad |t| \geq K \ \text{a.e.}\ x \in \overline{\Omega}. \tag{25}
$$

Then problem (P) *has at least one nontrivial weak solution.*

To obtain the result of Theorem 7, we need to show Lemmas 8 and 9 hold.

Lemma 8. *Suppose* ($\mathbf{M}_1$), ($\mathbf{f}_1$), *and* (**AR**) *hold. Then I satisfies* (**PS**) *condition.*

Proof. Let us assume that there exists a sequence $\{u_n\}$ in X such that

$$
|I(u_n)| \leq c, \qquad I'(u_n) \longrightarrow 0. \tag{26}
$$

Then by the assumptions (26), ($\mathbf{M}_1$), and (**AR**), we get

$$
\begin{aligned}
c + \|u_n\| &\geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{C_5}{\alpha(p_1^+)^\alpha} - \frac{C_6}{\theta\alpha(p_1^-)^{\alpha-1}}\right)\left(\int_\Omega |\nabla u_n|^{p_1(x)} dx\right)^\alpha \\
&\quad + \left(\frac{C_7}{\alpha(p_2^+)^\alpha} - \frac{C_8}{\theta\alpha(p_2^-)^{\alpha-1}}\right)\left(\int_\Omega |\nabla u_n|^{p_2(x)} dx\right)^\alpha \\
&\quad + \int_\Omega \left(\frac{1}{\theta} f(x,u_n)\, u_n - F(x,u_n)\right) dx \\
&\geq \lambda \|u_n\|^{\alpha p_m^-} + c,
\end{aligned} \tag{27}
$$

where $\lambda = ((C_5/\alpha(p_1^+)^\alpha) - (C_6/\theta\alpha(p_1^-)^{\alpha-1})) + ((C_7/\alpha(p_2^+)^\alpha) - (C_8/\theta\alpha(p_2^-)^{\alpha-1}))$. Since $\theta > (p_M^+)^\alpha$, we have $\lambda > 0$ for θ large enough. Therefore, $\{\|u_n\|\}$ is bounded in X. Passing to a subsequence, if necessary, there exists $u \in X$ such that $u_n \rightharpoonup u$. Therefore, we have the embeddings

$$
\begin{aligned}
u_n &\rightharpoonup u \quad \text{in } X, \\
u_n &\longrightarrow u \quad \text{in } L^{\beta(x)}(\Omega), \\
u_n &\longrightarrow u \quad \text{a.e. in } \Omega.
\end{aligned} \tag{28}
$$

By (26), we have $\langle I'(u_n), u_n - u\rangle \to 0$. Thus

$$
\begin{aligned}
&\langle I'(u_n), u_n - u\rangle \\
&\quad = M_1\left(\int_\Omega \frac{|\nabla u_n|^{p_1(x)}}{p_1(x)} dx\right) \\
&\qquad \times \int_\Omega |\nabla u_n|^{p_1(x)-2} \nabla u_n (\nabla u_n - \nabla u)\, dx \\
&\qquad + M_2\left(\int_\Omega \frac{|\nabla u_n|^{p_2(x)}}{p_2(x)} dx\right) \\
&\qquad \times \int_\Omega |\nabla u_n|^{p_2(x)-2} \nabla u_n (\nabla u_n - \nabla u)\, dx \\
&\qquad - \int_\Omega f(x,u_n)(u_n - u)\, dx.
\end{aligned} \tag{29}
$$

From $(\mathbf{f}_1)$ and Proposition 1, it follows that

$$\begin{aligned}&\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right|\\&\le C_{10}\left|\int_\Omega |u_n|^{\beta(x)-2}u_n(u_n-u)\,dx\right|\\&\quad+C_9\left|\int_\Omega (u_n-u)\,dx\right|\\&\le C_{10}\left||u_n|^{\beta(x)-1}\right|_{\beta(x)/(\beta(x)-1)}\\&\quad\times|u_n-u|_{\beta(x)}+C_9\int_\Omega|u_n-u|\,dx.\end{aligned}\tag{30}$$

If we consider the relations given in (28), we get

$$\int_\Omega f(x,u_n)(u_n-u)\,dx\longrightarrow 0.\tag{31}$$

Hence,

$$\begin{aligned}&M_1\left(\int_\Omega\frac{|\nabla u_n|^{p_1(x)}}{p_1(x)}dx\right)\\&\quad\times\int_\Omega|\nabla u_n|^{p_1(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\\&+M_2\left(\int_\Omega\frac{|\nabla u_n|^{p_2(x)}}{p_2(x)}dx\right)\\&\quad\times\int_\Omega|\nabla u_n|^{p_2(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\longrightarrow 0.\end{aligned}\tag{32}$$

From $(\mathbf{M}_1)$, it follows that

$$\sum_{i=1}^2\int_\Omega|\nabla u_n|^{p_i(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\longrightarrow 0.\tag{33}$$

Furthermore, since $u_n\rightharpoonup u$ in X, we have

$$\sum_{i=1}^2\int_\Omega|\nabla u|^{p_i(x)-2}\nabla u(\nabla u_n-\nabla u)\,dx\longrightarrow 0.\tag{34}$$

From (33) and (34), we deduce that

$$\begin{aligned}&\sum_{i=1}^2\int_\Omega\left(|\nabla u_n|^{p_i(x)-2}\nabla u_n-|\nabla u|^{p_i(x)-2}\nabla u\right)\\&\quad\times(\nabla u_n-\nabla u)\,dx\longrightarrow 0.\end{aligned}\tag{35}$$

Next, we apply the following well-known inequality

$$\begin{aligned}&\left(|\xi_i|^{r_i-2}\xi_i-|\psi_i|^{r_i-2}\psi_i\right)\cdot(\xi_i-\psi_i)\\&\quad\ge 2^{-r_i}|\xi_i-\psi_i|^{r_i},\quad \xi_i,\psi_i\in\mathbb{R}^N,\end{aligned}\tag{36}$$

valid for all $r_i\ge 2$ $(i=1,2)$. From the relations (35) and (36), we infer that

$$\sum_{i=1}^2\int_\Omega|\nabla u_n-\nabla u|^{p_i(x)}dx\longrightarrow 0,\tag{37}$$

and, consequently, $u_n\to u$ in X. We are done. □

Lemma 9. *Suppose* $(\mathbf{M}_1)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, *and* $(\mathbf{AR})$ *hold. Then the following statements hold:*

(i) *there exist two positive real numbers* γ *and* a *such that* $I(u)\ge a>0$, $u\in X$ *with* $\|u\|=\gamma$;

(ii) *there exists* $u\in X$ *such that* $\|u\|>\gamma$, $I(u)<0$.

Proof. (i) Let $\|u\|<1$. Then by $(\mathbf{M}_1)$ and Proposition 2, we have

$$\begin{aligned}I(u)&\ge\frac{C_5}{\alpha(p_1^+)^\alpha}\|u\|^{\alpha p_1^+}\\&\quad+\frac{C_7}{\alpha(p_2^+)^\alpha}\|u\|^{\alpha p_2^+}-\int_\Omega F(x,u)\,dx\\&\ge\frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_M^+}-\int_\Omega F(x,u)\,dx,\end{aligned}\tag{38}$$

where $c^*=\min\{C_5,C_7\}$. Since $\alpha p_M^+<\beta^-\le\beta^+<p_M^*(x)$ for all $x\in\overline{\Omega}$, we have the continuous embeddings $X\hookrightarrow L^{\alpha p_M^+}(\Omega)$ and $X\hookrightarrow L^{\beta^+}(\Omega)\hookrightarrow L^{\beta^-}(\Omega)$, and also there are positive constants C_{11}, C_{12} and C_{13} such that

$$\begin{aligned}&|u|_{\alpha p_M^+}\le C_{11}\|u\|,\qquad |u|_{\beta^-}\le C_{12}\|u\|,\\&|u|_{\beta^+}\le C_{13}\|u\|,\quad\forall u\in X.\end{aligned}\tag{39}$$

Let $\varepsilon>0$ be small enough such that $\varepsilon C_{11}^{\alpha p_M^+}<(c^*/2\alpha(p_M^+)^\alpha)$. By the assumptions $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we have $F(x,t)\le\varepsilon|t|^{\alpha p_M^+}+C_\varepsilon|t|^{\beta(x)}$, for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$.

Then, for $\|u\|\le 1$ it follows that

$$\begin{aligned}I(u)&\ge\frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_M^+}\\&\quad-\varepsilon\int_\Omega|u|^{\alpha p_M^+}dx-C_\varepsilon\int_\Omega|u|^{\beta(x)}dx\\&\ge\frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_M^+}-\varepsilon C_{11}^{\alpha p_M^+}\|u\|^{\alpha p_M^+}\\&\quad-C_\varepsilon C_{12}^{\beta^-}\|u\|^{\beta^-}.\end{aligned}\tag{40}$$

Therefore, there exists two positive real numbers γ and a such that $I(u)\ge a>0$, for all $u\in X$ *with* $\|u\|=\gamma$.

(ii) From $(\mathbf{AR})$ it follows that $F(x,t)\ge c|t|^\theta$, for all $x\in\overline{\Omega}$ and $|t|\ge K$. In the other hand, when $|t|\ge K$ from $(\mathbf{M}_1)$ we obtain that

$$\begin{aligned}&\widehat{M}_1(t)\le\frac{C_6}{\alpha}t^\alpha\le\frac{C_6}{\alpha}t^{\alpha p_M^+},\\&\widehat{M}_2(t)\le\frac{C_8}{\alpha}t^\alpha\le\frac{C_8}{\alpha}t^{\alpha p_M^+}.\end{aligned}\tag{41}$$

Hence, for any fixed $\omega \in X \setminus \{0\}$ and $t > 1$ we have

$$\begin{aligned} I(t\omega) &\le \frac{C_6}{\alpha}\left(\int_\Omega \frac{|\nabla t\omega|^{p_1(x)}}{p_1(x)}dx\right)^{\alpha p_M^+} \\ &\quad + \frac{C_8}{\alpha}\left(\int_\Omega \frac{|\nabla t\omega|^{p_2(x)}}{p_2(x)}dx\right)^{\alpha p_M^+} \\ &\quad - \int_\Omega F(x, t\omega)\,dx \\ &\le \frac{C_6}{\alpha} t^{\alpha p_M^+}\left(\int_\Omega \frac{|\nabla \omega|^{p_1(x)}}{p_1(x)}dx\right)^{\alpha p_M^+} \\ &\quad + \frac{C_8}{\alpha} t^{\alpha p_M^+}\left(\int_\Omega \frac{|\nabla \omega|^{p_2(x)}}{p_2(x)}dx\right)^{\alpha p_M^+} \\ &\quad - t^\theta \int_\Omega F(x,\omega)\,dx, \end{aligned} \tag{42}$$

which implies $I(t\omega) \to -\infty (t \to +\infty)$. □

Proof of Theorem 7. From Lemmas 8 and 9 and the fact that $I(0) = 0$, I satisfies the Mountain-Pass theorem (see [20, 21]). Therefore, I has at least one nontrivial critical point; that is, (P) has a nontrivial weak solution. The proof is complete. □

In the following, we will prove the second main result of the present paper.

Theorem 10. *Suppose* $(\mathbf{M}_1)$, $(\mathbf{AR})$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, *and* $(\mathbf{f}_3)$ *hold. Then I has a sequence of critical points $\{u_n\}$ such that $I(u_n) \to +\infty$ and* (P) *has infinite many pairs of solutions.*

Since X is a reflexive and separable Banach space, then there are $\{e_i\} \subset X$ and $\{e_i^*\} \subset X^*$ such that

$$\begin{gathered} X = \overline{\operatorname{span}\{e_i : i = 1, 2, \ldots\}}, \\ X^* = \overline{\operatorname{span}\{e_i^* : i = 1, 2, \ldots\}}, \\ \langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \end{gathered} \tag{43}$$

For convenience, we write $X_i = \operatorname{span}\{e_i\}$, $Y_k = \bigoplus_{i=1}^k X_i$, and $Z_k = \bigoplus_{i=k}^\infty X_i$.

Lemma 11. *If $\beta \in C_+(\overline{\Omega})$ such that $\beta(x) < p_M^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\delta_k := \sup\left\{|u|_{\beta(x)} : \|u\| = 1, u \in Z_k\right\}. \tag{44}$$

Then $\lim_{k\to\infty}\delta_k = 0$.

Since the proof of Lemma 11 is similar to that of Lemma 4.9 in [7], we omit it.

Proof of Theorem 10. By the assumptions $(\mathbf{M}_1)$, $(\mathbf{AR})$, and $(\mathbf{f}_1)$, I satisfies $(\mathbf{PS})$ condition and from $(\mathbf{f}_3)$ it is also an even functional. In the sequel, we will show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that

(i) $b_k := \inf_{u\in Z_k, \|u\|=r_k} I(u) \to +\infty (k \to \infty)$;

(ii) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} I(u) \le 0$.

Therefore, to obtain the results of Theorem 10 it is enough to apply Fountain theorem (see [21]).

(i) For any $u \in Z_k$ with $\|u\|$ big enough, we have

$$\begin{aligned} I(u) &\ge \frac{C_5}{\alpha(p_1^+)^\alpha}\|u\|^{\alpha p_1^-} + \frac{C_7}{\alpha(p_2^+)^\alpha}\|u\|^{\alpha p_2^-} \\ &\quad - C_{14}\int_\Omega |u|^{\beta(x)}dx - c_1 \\ &\ge \frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{14}|u|_{\beta(x)}^{\beta(\xi)} - c_2, \quad \text{where } \xi \in \Omega. \\ I(u) &\ge \frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{14} - c_2, \quad \text{if } |u|_{\beta(x)} \le 1, \\ I(u) &\ge \frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{14}\delta_k^{\beta^+}\|u\|^{\beta^+} - c_2, \quad \text{if } |u|_{\beta(x)} > 1, \\ I(u) &\ge \frac{c^*}{\alpha(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{14}\delta_k^{\beta^+}\|u\|^{\beta^+} - c_3 \\ &= \frac{c^*}{\alpha}\left(\frac{1}{(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{15}\delta_k^{\beta^+}\|u\|^{\beta^+}\right) - c_3. \end{aligned} \tag{45}$$

Set $\|u\| = r_k = (C_{15}\beta^+\delta_k^{\beta^+})^{1/(\alpha p_m^- - \beta^+)}$. Because $\delta_k \to 0$ and $\alpha p_m^- \le \alpha p_M^+ < \beta^+$, we have

$$\begin{aligned} I(u) &\ge \frac{c^*}{\alpha}\left(\frac{1}{(p_M^+)^\alpha}\|u\|^{\alpha p_m^-} - C_{15}\delta_k^{\beta^+}\|u\|^{\beta^+}\right) - c_3 \\ &= \frac{c^*}{\alpha}\left(\frac{1}{(p_M^+)^\alpha\left(C_{15}\beta^+\delta_k^{\beta^+}\right)^{\alpha p_m^-/(\alpha p_m^- - \beta^+)}}\right. \\ &\qquad \left. - C_{15}\delta_k^{\beta^+}\left(C_{15}\beta^+\delta_k^{\beta^+}\right)^{\beta^+/(\alpha p_m^- - \beta^+)}\right) - c_3 \\ &= \frac{c^*}{\alpha}\left(\frac{1}{(p_M^+)^\alpha} - \frac{1}{(\beta^+)^\alpha}\right) \\ &\quad \times \left(C_{15}\beta^+\delta_k^{\beta^+}\right)^{\alpha p_m^-/(\alpha p_m^- - \beta^+)} - c_3 \longrightarrow +\infty \quad (k \longrightarrow \infty). \end{aligned} \tag{46}$$

(ii) From $(\mathbf{AR})$, we have $F(x,t) \ge C_{16}|t|^\theta - C_{17}$. Because $\theta > (p_M^+)^\alpha$ and $\dim Y_k = k$, it is obvious that $I(u) \to -\infty$ as $\|u\| \to \infty$ for $u \in Y_k$. □

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