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Research Article

Infinite-Scroll Attractor Generated by the Complex Pendulum Model

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We report the finding of the simple nonlinear autonomous system exhibiting infinite-scroll attractor. The system is generated from the pendulum equation with complex-valued function. The proposed system is having infinitely many saddle points of index two which are responsible for the infinite-scroll attractor.

1. Introduction

A variety of natural systems show a chaotic (aperiodic) behaviour. Such systems depend sensitively on initial data, and one cannot predict the future of the solutions. There are various chaotic systems such as the Lorenz system [1], the Rossler system [2], the Chen system [3], and the Lü system [4] where the dependent variables are the real-valued functions. Though the chaos has been intensively studied over the past several decades, very few articles are devoted to study the complex dynamical systems. Ning and Haken [5] proposed a complex Lorenz system arising in lasers. Wang et al. [6] discussed the applications in genetic networks. Mahmoud and coworkers have studied complex Van der Pol oscillator [7], new complex system [8], complex Duffing oscillator [9], and so forth. Complex multiscroll attractors have a close relationship with complex networks also [10–12].

In this work, we propose a complex pendulum equation exhibiting infinite-scroll attractor. The chaotic phase portraits are plotted, and maximum Lyapunov exponents are given for the different values of the parameter.

2. The Model

The real pendulum equation is given by [13]

$$\ddot{x} = -a \sin(x), \tag{1}$$

where $a > 0$ is constant and x is a real valued function. We propose a complex version of (1) given by

$$\ddot{z} = -a \sin(z), \tag{2}$$

where $z(t) = x(t) + \imath y(t)$ is a complex-valued function. The system (2) gives rise to a coupled nonlinear system

$$\begin{aligned} \ddot{x} &= -a \sin(x) \cosh(y), \\ \ddot{y} &= -a \sinh(y) \cos(x). \end{aligned} \tag{3}$$

Using the new variables $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, and $x_4 = \dot{y}$, the system (3) can be written as the autonomous system of first-order ordinary differential equations given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a \sin(x_1) \cosh(x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -a \sinh(x_3) \cos(x_1). \end{aligned} \tag{4}$$

2.1. Symmetry. Symmetry about the x_1, x_2-axes (or x_3, x_4 axes), since $(x_1, x_2, x_3, x_4) \rightarrow (x_1, x_2, -x_3, -x_4)$ (or $(x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, x_3, x_4)$) do not change the equations.

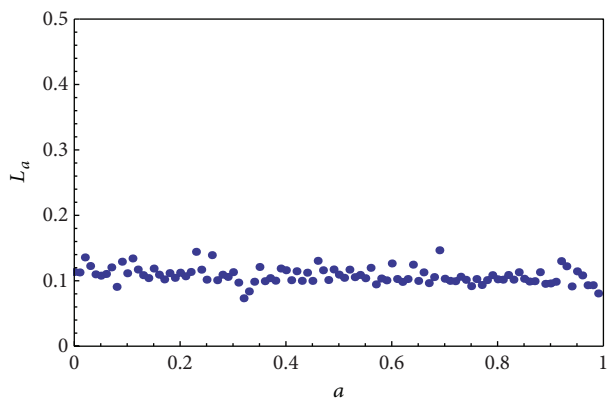

Figure 1: Maximum Lyapunov exponents.

2.2. Conservation. Consider the following:

$$\nabla \cdot F = \sum_{i=1}^{4} \frac{\partial \dot{x}_i}{\partial x_i} = 0. \tag{5}$$

System is conservative.

2.3. Equilibrium Points and Their Stability. It can be checked that the system (4) has infinitely many real equilibrium points given by $E_k = (k\pi, 0, 0, 0)$, where $k = 0, \pm 1, \pm 2, \ldots$. Jacobian matrix corresponding to the system (4) is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a\cos(x_1)\cosh(x_3) & 0 & -a\sin(x_1)\sinh(x_3) & 0 \\ 0 & 0 & 0 & 1 \\ a\sin(x_1)\sinh(x_3) & 0 & -a\cos(x_1)\cosh(x_3) & 0 \end{pmatrix}. \tag{6}$$

Since the eigenvalues of $J(E_{2k})$ are $\imath\sqrt{a}, \imath\sqrt{a}, -\imath\sqrt{a}$, and $-\imath\sqrt{a}$, the points E_{2k} are stable equilibrium points. The eigenvalues of $J(E_{2k+1})$ are $\sqrt{a}$, $\sqrt{a}$, $-\sqrt{a}$, and $-\sqrt{a}$.

An equilibrium point p is called a saddle point if the Jacobian matrix at p has at least one eigenvalue with negative real part (stable) and one eigenvalue with nonnegative real part (unstable). A saddle point is said to have index one (/two) if there is exactly one (/two) unstable eigenvalue/s. It is established in the literature [14–17] that scrolls are generated only around the saddle points of index two.

It is now clear that the system (4) has infinitely many saddle equilibrium points E_{2k+1}, $k = 0, \pm 1, \pm 2, \ldots$ of index two which gives rise to an infinite-scroll attractor.

2.4. Chaos. Maximum Lyapunov exponents (MLEs) for the system (4) are plotted in Figure 1. The positive MLEs indicate that the system is chaotic for $a > 0$. Figures 2(a)–2(d) show the chaotic time series for $a = 0.3$. For the same values of a, Figures 3(a) and 3(b) represent multiscroll attractor in (x_1, x_3) plane and in (x_1, x_2, x_3) space, respectively.

3. Conclusions

We have generalized the real function in the pendulum equation to a complex one and studied the chaotic behaviour.

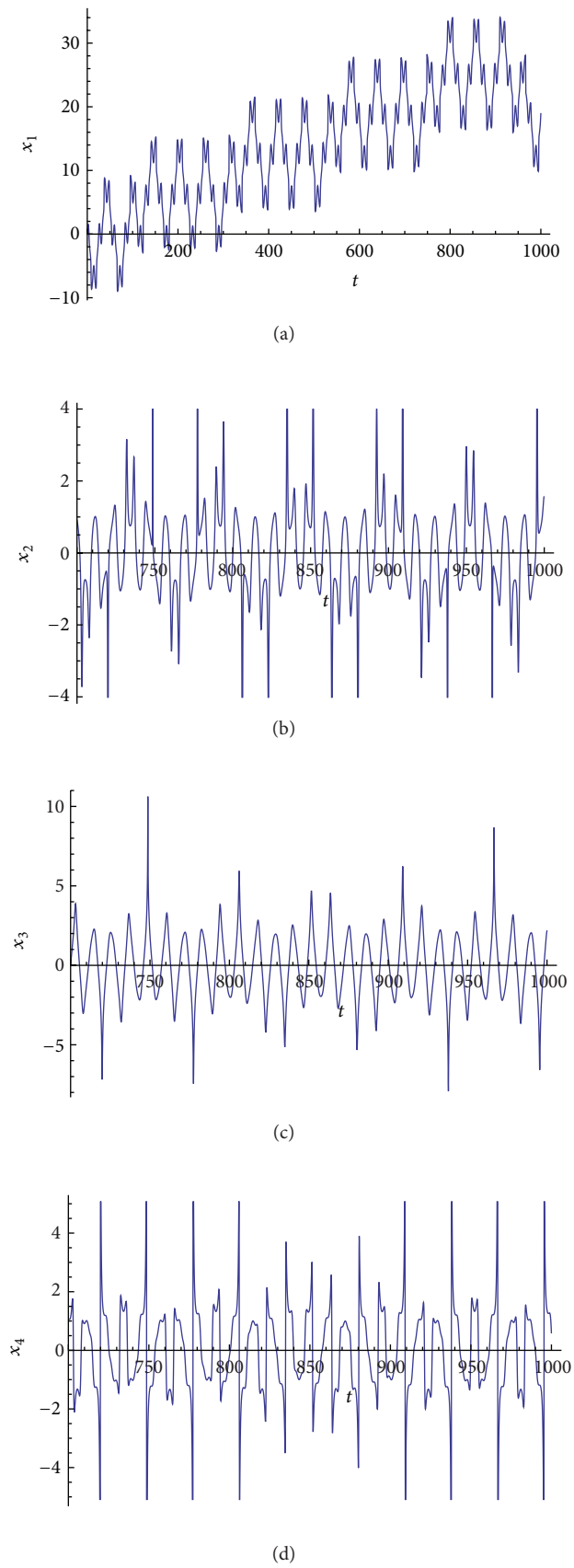

Figure 2: (a) Time series x_1, (b) time series x_2, (c) time series x_3, and (d) time series x_4.

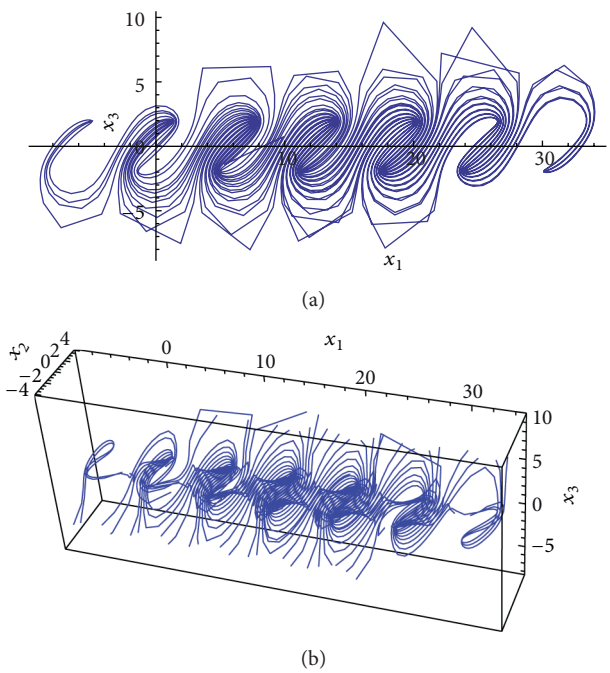

Figure 3: (a) Phase portrait, and (b) phase portrait in space.

The new system is equivalent to a system of four first-order ordinary differential equations. There are infinitely many saddle points of index two for this system which are responsible for the infinite-scroll chaotic attractor. Such example of infinite-scroll attractor will help researchers in the field of chaos to study the properties of such systems in detail.

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