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Research Article

Some Inclusion Relationships of Certain Subclasses of p-Valent Functions Associated with a Family of Integral Operators

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By making use of the new integral operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$, we introduce and investigate several new subclasses of p-valent starlike, p-valent convex, p-valent close-to-convex, and p-valent quasi-convex functions. In particular, we establish some inclusion relationships associated with the aforementioned integral operators. Some of the results established in this paper would provide extensions of those given in earlier works.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the unit disc $U = \{z : z \in \mathbb{C}, \text{ and } |z| < 1\}$ and let $A(1) = A$.

A function $f \in A(p)$ is said to be in the class $S_p^*(\lambda)$ of p-valent starlike functions of order λ in U if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \lambda \quad (z \in U;\ 0 \le \lambda < p). \tag{2}$$

The class $S_p^*(\lambda)$ was introduced by Patil and Thakare [1].

Owa [2] introduced the class $K_p(\lambda)$ of p-valent convex of order λ in U if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda \quad (z \in U;\ 0 \le \lambda < p). \tag{3}$$

It is easy to observe from (2) and (3) that

$$f(z) \in K_p(\lambda) \Longleftrightarrow \frac{zf'(z)}{p} \in S_p^*(\lambda). \tag{4}$$

We denote by $S_p^* = S_p^*(0)$ and $K_p = K_p(0)$ where S_p^* and K_p are the classes of p-valently starlike functions and p-valently convex functions, respectively, (see Goodman [3]).

For a function $f \in A(p)$, we say that $f \in C_p(\eta, \lambda)$ if there exists a function $g \in S_p^*(\lambda)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \eta \quad (z \in U;\ 0 \le \lambda, \eta < p). \tag{5}$$

Functions in the class $C_p(\eta, \lambda)$ are called p-valent close-to-convex functions of order η and type λ. The class $C_p(\eta, \lambda)$ was studied by Aouf [4] and the class $C_1(\eta, \lambda)$ was studied by Libera [5].

Noor [6, 7] introduced and studied the classes $C_p^*(\eta, \lambda)$ and $C_1^*(\eta, \lambda)$ as follows.

A function $f \in A(p)$ is said to be in the class $C_p^*(\eta, \lambda)$ of quasi-convex functions of order η and type λ if there exists a function $g \in K_p(\lambda)$ such that

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \eta \quad (z \in U;\ 0 \le \lambda,\ \eta < p). \tag{6}$$

It follows from (5) and (6) that

$$f(z) \in C_p^*(\eta,\lambda) \Longleftrightarrow \frac{zf'(z)}{p} \in C_p(\eta,\lambda). \tag{7}$$

For functions $f \in A(p)$ given by (1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{8}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z). \tag{9}$$

For the function $f \in A(p)$, we introduced the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma} : A(p) \to A(p)$ as follows:

$$\begin{aligned}
\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z) &= \binom{p+\alpha+\beta-\gamma}{p+\beta-1} \frac{(\alpha-\gamma+1)}{z^\beta} \\
&\quad \times \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\,dt \\
&= \frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)\Gamma(\alpha-\gamma+1)} \frac{1}{z^\beta} \\
&\quad \times \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\,dt \\
&= z^p + \frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)} \\
&\quad \times \sum_{k=1}^{\infty} \left[\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)}\right] a_{k+p} z^{k+p} \\
&\quad (\beta > -p;\ \alpha \ge \gamma-1;\ \gamma > 0;\ p \in \mathbb{N};\ z \in U).
\end{aligned} \tag{10}$$

From (10), it is easy to verify that

$$\begin{aligned}
z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z)\right)' &= (\alpha+\beta+p-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z) \\
&\quad - (\alpha+\beta-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z).
\end{aligned} \tag{11}$$

Remark 1. Consider (i) For $\gamma = 1$,

$$\begin{aligned}
\mathfrak{R}_{\beta,p}^{\alpha,1} f(z) &= Q_{\beta,p}^{\alpha} f(z) \\
&= \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^\beta} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt \\
&= \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)\Gamma(\alpha)} \frac{1}{z^\beta} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt \\
&= z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \\
&\quad \times \sum_{k=1}^{\infty} \left[\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k)}\right] a_{k+p} z^{k+p} \\
&\quad (\beta > -p;\ \alpha \ge 0;\ p \in \mathbb{N};\ z \in U),
\end{aligned} \tag{12}$$

where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [8], and $Q_{\beta,1}^{\alpha} = Q_{\beta}^{\alpha}$, where the operator Q_{β}^{α} was introduced and studied by Jung et al. [9];

(ii) For $\alpha = \gamma = 1$ and $\beta = c$,

$$\begin{aligned}
\mathfrak{R}_{c,p}^{1,1} f(z) = J_{c,p}(z) &= \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\,dt \\
&= z^p + \sum_{k=1}^{\infty} \left(\frac{c+p}{c+k+p}\right) a_{k+p} z^{k+p} \\
&\quad (c > -p;\ p \in \mathbb{N};\ z \in U),
\end{aligned} \tag{13}$$

where $J_{c,p}(z)$ is the familiar integral operator, which was defined by Cho and Kim [10]. The operator $J_{c,1}(z) = J_c(z)$ was introduced by Bernardi [11] and we note that $J_{1,1}(z) = J(z)$ was introduced and studied by Libera [12] and Livingston [13].

The main object of this paper is to investigate the various inclusion relationships for each of the following subclasses of the normalized analytic function class $A(p)$ which are defined here by means of the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$ given by (10).

Definition 2. In conjunction with (2) and (10),

$$\begin{aligned}
S_{\alpha,\beta,\gamma,p}^*(\lambda) = \Big\{ f : f \in A(p), \mathfrak{R}_{\beta,p}^{\alpha,\gamma} f \in S_p^*(\lambda), \\
0 \le \lambda < p, p \in \mathbb{N} \Big\}.
\end{aligned} \tag{14}$$

Definition 3. In conjunction with (3) and (10),

$$\begin{aligned}
K_{\alpha,\beta,\gamma,p}(\lambda) = \Big\{ f : f \in A(p), \mathfrak{R}_{\beta,p}^{\alpha,\gamma} f \in K_p(\lambda), \\
0 \le \lambda < p, p \in \mathbb{N} \Big\}.
\end{aligned} \tag{15}$$

Definition 4. In conjunction with (5) and (10),

$$\begin{aligned}
C_{\alpha,\beta,\gamma,p}(\eta,\lambda) = \Big\{ f : f \in A(p), \mathfrak{R}_{\beta,p}^{\alpha,\gamma} f \in C_p(\eta,\lambda), \\
0 \le \lambda, \eta < p, p \in \mathbb{N} \Big\}.
\end{aligned} \tag{16}$$

Definition 5. In conjunction with (6) and (10),

$$\begin{aligned}
C_{\alpha,\beta,\gamma,p}^*(\eta,\lambda) = \Big\{ f : f \in A(p), \mathfrak{R}_{\beta,p}^{\alpha,\gamma} f \in C_p^*(\eta,\lambda), \\
0 \le \lambda, \eta < p, p \in \mathbb{N} \Big\}.
\end{aligned} \tag{17}$$

Remark 6. Consider (I) For $\gamma = 1$, in the above definitions, we have

$$\begin{aligned}
S^*_{\alpha,\beta,1,p}(\lambda) &= S^*_{\alpha,\beta,p}(\lambda) \\
&= \left\{f : f \in A(p), Q^{\alpha}_{\beta,p} f \in S^*_p(\lambda), \right. \\
&\qquad \left. 0 \le \lambda < p, p \in \mathbb{N}\right\}, \\
K_{\alpha,\beta,1,p}(\lambda) &= K_{\alpha,\beta,p}(\lambda) \\
&= \left\{f : f \in A(p), Q^{\alpha}_{\beta,p} f \in K_p(\lambda), \right. \\
&\qquad \left. 0 \le \lambda < p, p \in \mathbb{N}\right\}, \\
C_{\alpha,\beta,1,p}(\eta,\lambda) &= C_{\alpha,\beta,p}(\eta,\lambda) \\
&= \left\{f : f \in A(p), Q^{\alpha}_{\beta,p} f \in C_p(\eta,\lambda), \right. \\
&\qquad \left. 0 \le \lambda, \eta < p, p \in \mathbb{N}\right\}, \\
C^*_{\alpha,\beta,1,p}(\eta,\lambda) &= C^*_{\alpha,\beta,p}(\eta,\lambda) \\
&= \left\{f : f \in A(p), Q^{\alpha}_{\beta,p} f \in C^*_p(\eta,\lambda), \right. \\
&\qquad \left. 0 \le \lambda, \eta < p, p \in \mathbb{N}\right\}.
\end{aligned} \tag{18}$$

(II) For $\alpha = \gamma = 1$ and $\beta = c$ $(c > -p)$, in the above definitions, we have

$$\begin{aligned}
S^*_{1,c,1,p}(\lambda) &= S^*_{c,p}(\lambda) \\
&= \left\{f : f \in A(p), J_{c,p} f(z) \in S^*_p(\lambda), \right. \\
&\qquad \left. 0 \le \lambda < p, p \in \mathbb{N}\right\}, \\
K_{1,c,1,p}(\lambda) &= K_{c,p}(\lambda) \\
&= \left\{f : f \in A(p), J_{c,p} f(z) \in K_p(\lambda), \right. \\
&\qquad \left. 0 \le \lambda < p, p \in \mathbb{N}\right\}, \\
C_{1,c,1,p}(\eta,\lambda) &= C_{c,p}(\eta,\lambda) \\
&= \left\{f : f \in A(p), J_{c,p} f(z) \in C_p(\eta,\lambda), \right. \\
&\qquad \left. 0 \le \lambda, \eta < p, p \in \mathbb{N}\right\}, \\
C^*_{1,c,1,p}(\eta,\lambda) &= C^*_{c,p}(\eta,\lambda) \\
&= \left\{f : f \in A(p), J_{c,p} f(z) \in C^*_p(\eta,\lambda), \right. \\
&\qquad \left. 0 \le \lambda, \eta < p, p \in \mathbb{N}\right\}.
\end{aligned} \tag{19}$$

(III) For $\gamma = p = 1$, in the above definitions, we have

$$\begin{aligned}
S^*_{\alpha,\beta,1,1}(\lambda) &= S^*_{\alpha,\beta}(\lambda) \\
&= \left\{f : f \in A, Q^{\alpha}_{\beta} f \in S^*(\lambda), 0 \le \lambda < 1\right\}, \\
K_{\alpha,\beta,1,1}(\lambda) &= K_{\alpha,\beta}(\lambda) \\
&= \left\{f : f \in A, Q^{\alpha}_{\beta} f \in K(\lambda), 0 \le \lambda < 1\right\}, \\
C_{\alpha,\beta,1,1}(\eta,\lambda) &= C_{\alpha,\beta}(\eta,\lambda) \\
&= \left\{f : f \in A, Q^{\alpha}_{\beta} f \in C(\eta,\lambda), 0 \le \lambda, \eta < 1\right\}, \\
C^*_{\alpha,\beta,1,1}(\eta,\lambda) &= C^*_{\alpha,\beta}(\eta,\lambda) \\
&= \left\{f : f \in A, Q^{\alpha}_{\beta} f \in C^*(\eta,\lambda), \right. \\
&\qquad \left. 0 \le \lambda, \eta < 1\right\},
\end{aligned} \tag{20}$$

where the classes $S^*_{\alpha,\beta}(\lambda)$, $K_{\alpha,\beta}(\lambda)$, $C_{\alpha,\beta}(\eta,\lambda)$, and $C^*_{\alpha,\beta}(\eta,\lambda)$ were introduced and studied by Gao et al. [14].

In order to establish our main results, we need the following lemma due to Miller and Mocanu [15].

Lemma 7 (see [15]). *Let Θ be a complex-valued function such that*

$$\Theta : D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}), \tag{21}$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that $\Theta(u, v)$ satisfies the following conditions:

(i) *$\Theta(u, v)$ is continuous in D;*

(ii) *$(1, 0) \in D$ and $\mathrm{Re}\{\Theta(1, 0)\} > 0$;*

(iii) *$\mathrm{Re}\{\Theta(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \le -(1/2)(1 + u_2^2)$.*

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots \tag{22}$$

be analytic in U such that $(q(z), z\, q'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\left\{\Theta\left(q(z), z\, q'(z)\right)\right\} > 0 \quad (z \in U), \tag{23}$$

then

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in U). \tag{24}$$

2. The Main Results

In this section, we give several inclusion relationships for analytic function classes, which are associated with the integral operator $\mathfrak{R}^{\alpha,\gamma}_{\beta,p}$. Unless otherwise mentioned, we assume throughout this paper that $\beta > -p$, $\alpha \ge \gamma - 1$, $\gamma > 0$, $p \in \mathbb{N}$, and $z \in U$.

Theorem 8. *Let $0 \le \lambda < p$. Then*

$$S^*_{\alpha,\beta,\gamma,p}(\lambda) \subset S^*_{\alpha+1,\beta,\gamma,p}(\lambda). \tag{25}$$

Proof. Let $f \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$ and set

$$\frac{z\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)\right)'}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)} - \lambda = (p-\lambda)\, q(z), \tag{26}$$

where $q(z)$ is given by (22). By using identity (11), we obtain

$$\frac{\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)} = \frac{\lambda + (p-\lambda)\, q(z) + (\alpha+\beta-\gamma+1)}{\alpha+\beta+p-\gamma+1}. \tag{27}$$

Differentiating (27) logarithmically with respect to z, we obtain

$$\begin{aligned}
&\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)} \\
&\quad = \frac{z\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)\right)'}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)} + \frac{(p-\lambda)\, zq'(z)}{(p-\lambda)\, q(z) + \lambda + \alpha + \beta - \gamma + 1} \\
&\quad = \lambda + (p-\lambda)\, q(z) + \frac{(p-\lambda)\, zq'(z)}{(p-\lambda)\, q(z) + \lambda + \alpha + \beta - \gamma + 1}.
\end{aligned} \tag{28}$$

We now choose $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, and define the function Θ by

$$\Theta(u,v) = (p-\lambda)\, u + \frac{(p-\lambda)\, v}{(p-\lambda)\, u + \lambda + \alpha + \beta - \gamma + 1}. \tag{29}$$

Then, clearly $\Theta(u,v)$ satisfies the following conditions:

(i) $\Theta(u,v)$ is continuous in $D = (\mathbb{C} \setminus \{(\lambda+\alpha+\beta-\gamma+1)/(\lambda-p)\}) \times \mathbb{C}$;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\Theta(1,0)\} = p - \lambda > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -(1/2)(1+u_2^2)$ we have

$$\begin{aligned}
\mathrm{Re}\{\Theta(iu_2, v_1)\} &= \mathrm{Re}\left\{\frac{(p-\lambda)\, v_1}{(p-\lambda)\, iu_2 + \lambda + \alpha + \beta - \gamma + 1}\right\} \\
&= \frac{(p-\lambda)(\lambda+\alpha+\beta-\gamma+1)\, v_1}{(p-\lambda)^2 u_2^2 + (\lambda+\alpha+\beta-\gamma+1)^2} \\
&\le -\frac{(p-\lambda)(1+u_2^2)(\lambda+\alpha+\beta-\gamma+1)}{2\left[(p-\lambda)^2 u_2^2 + (\lambda+\alpha+\beta-\gamma+1)^2\right]} \\
&< 0,
\end{aligned} \tag{30}$$

which shows that the function Θ satisfies the hypotheses of Lemma 7. Consequently, we easily obtain the inclusion relationship (25). □

Theorem 9. *Let $0 \le \lambda < p$, $p \in \mathbb{N}$. Then*

$$K_{\alpha,\beta,\gamma,p}(\lambda) \subset K_{\alpha+1,\beta,\gamma,p}(\lambda). \tag{31}$$

Proof. Let $f \in K_{\alpha,\beta,\gamma,p}(\lambda)$. Then, from Definition 3, we have

$$\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f \in K_p(\lambda), \quad (0 \le \lambda < p;\ p \in \mathbb{N}). \tag{32}$$

Furthermore, in view of the relationship (4), we find that

$$\frac{z}{p}\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)\right)' \in S^*_p(\lambda), \tag{33}$$

that is, that

$$\mathfrak{R}^{\alpha,\gamma}_{\beta,p}\left(\frac{zf'(z)}{p}\right) \in S^*_p(\lambda). \tag{34}$$

Thus, by using Definition 2 and Theorem 8, we have

$$\frac{zf'(z)}{p} \in S^*_p(\alpha,\beta,\gamma,\lambda) \subset S^*_p(\alpha+1,\beta,\gamma,\lambda), \tag{35}$$

which implies that

$$K_{\alpha,\beta,\gamma,p}(\lambda) \subset K_{\alpha+1,\beta,\gamma,p}(\lambda). \tag{36}$$

The proof of Theorem 9 is thus completed. □

Theorem 10. *Let $0 \le \lambda, \eta < p$. Then*

$$C_{\alpha,\beta,\gamma,p}(\eta,\lambda) \subset C_{\alpha+1,\beta,\gamma,p}(\eta,\lambda). \tag{37}$$

Proof. Let $f \in C_{\alpha,\beta,\gamma,p}(\eta,\lambda)$. Then there exists a function $\Psi \in S^*_p(\lambda)$ such that

$$\mathrm{Re}\left\{\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)\right)'}{\Psi(z)}\right\} > \eta \quad (0 \le \lambda, \eta < p,\ z \in U). \tag{38}$$

We put

$$\mathfrak{R}^{\alpha,\gamma}_{\beta,p} g(z) = \Psi(z), \tag{39}$$

so that we have

$$g \in S^*_{\alpha,\beta,\gamma,p}(\lambda), \quad \mathrm{Re}\left\{\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p} f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p} g(z)}\right\} > \eta \quad (z \in U). \tag{40}$$

We next put

$$\frac{z\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} f(z)\right)'}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p} g(z)} = \eta + (p-\eta)\, q(z), \tag{41}$$

where $q(z)$ is given by (22). Thus, by using identity (11), we obtain

$$\begin{aligned}\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)} &= \frac{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(zf'(z)\right)}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)} \\ &= \left(z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}\left(zf'(z)\right)\right]' \right. \\ &\quad \left. + (\alpha+\beta-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}\left(zf'(z)\right)\right) \\ &\quad \times \left(z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\right]' \right. \\ &\quad \left. + (\alpha+\beta-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\right)^{-1} \\ &= \left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}\left(zf'(z)\right)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)} \right. \\ &\quad \left. + (\alpha+\beta-\gamma+1)\frac{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}\left(zf'(z)\right)}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)}\right) \\ &\quad \times \left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)} \right. \\ &\quad \left. + (\alpha+\beta-\gamma+1)\right)^{-1}.\end{aligned} \tag{42}$$

Since $g \in S_{\alpha,\beta,\gamma,p}^{*}(\lambda)$, then from Theorem 8 we have $g \in S_{\alpha+1,\beta,\gamma,p}^{*}(\lambda)$, so that we can put

$$\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)} = \lambda + (p-\lambda)G(z), \tag{43}$$

where

$$\begin{aligned} G(z) &= g_1(x,y) + ig_2(x,y), \\ \operatorname{Re}(G(z)) &= g_1(x,y) > 0 \quad (z \in U). \end{aligned} \tag{44}$$

Then

$$\begin{aligned}\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)} &= \left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}\left(zf'(z)\right)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)} \right. \\ &\quad \left. + (\alpha+\beta-\gamma+1)\left(\eta+(p-\eta)q(z)\right)\right) \\ &\quad \times \left((p-\lambda)G(z)+\lambda+\alpha+\beta-\gamma+1\right)^{-1}.\end{aligned} \tag{45}$$

We thus find from (41) that

$$z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z)\right)' = \mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\left[\eta+(p-\eta)q(z)\right]. \tag{46}$$

Differentiating both sides of (46) with respect to z, we obtain

$$\begin{aligned}&\frac{z\left[z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z)\right)'\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)} \\ &\quad = (p-\eta)zq'(z) + \frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}g(z)}\left[\eta+(p-\eta)q(z)\right] \\ &\quad = (p-\eta)zq'(z) \\ &\qquad + \left[\lambda+(p-\lambda)G(z)\right]\left[\eta+(p-\eta)q(z)\right].\end{aligned} \tag{47}$$

By substituting (47) into (45), we have

$$\begin{aligned}\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)} - \eta &= (p-\eta)q(z) \\ &\quad + \frac{(p-\eta)zq'(z)}{(p-\lambda)G(z)+\lambda+\alpha+\beta-\gamma+1}.\end{aligned} \tag{48}$$

The remainder of our proof of Theorem 10 is much akin to that of Theorem 8. We, therefore, choose to omit the details involved. □

Theorem 11. *Let $0 \le \lambda, \eta < p$. Then*

$$C_{\alpha,\beta,\gamma,p}^{*}(\eta,\lambda) \subset C_{\alpha+1,\beta,\gamma,p}^{*}(\eta,\lambda). \tag{49}$$

Proof. Just as we derived Theorem 9 as a consequence of Theorem 8 by using the equivalence (4), we can also prove Theorem 11 by using Theorem 10 in conjunction with the equivalence (7). □

Our main results in Theorems 8–11, can thus be applied with a view to deducing the following corollaries.

Taking $\gamma = 1$ in Theorems 8–11 above, we obtain the following corollary.

Corollary 12. *Let $0 \le \lambda, \eta < p$. Then*

$$\begin{aligned} S_{\alpha,\beta,p}^{*}(\lambda) &\subset S_{\alpha+1,\beta,p}^{*}(\lambda), \\ K_{\alpha,\beta,p}(\lambda) &\subset K_{\alpha+1,\beta,p}(\lambda), \\ C_{\alpha,\beta,p}(\eta,\lambda) &\subset C_{\alpha+1,\beta,p}(\eta,\lambda), \\ C_{\alpha,\beta,p}^{*}(\eta,\lambda) &\subset C_{\alpha+1,\beta,p}^{*}(\eta,\lambda). \end{aligned} \tag{50}$$

Remark 13. Taking $p = 1$ in Corollary 12, we obtain the results obtained by Gao et al. [14, Theorems 1-4].

Taking $\alpha = \gamma = 1$ and $\beta = c$ $(c > -p)$ in Theorems 8–11, we obtain the following corollary.

Corollary 14. *Let $0 \le \lambda, \eta < p$. Then*

$$\begin{aligned} S^*_{1,c,p}(\lambda) &\subset S^*_{2,c,p}(\lambda), \\ K_{1,c,p}(\lambda) &\subset K_{2,c,p}(\lambda), \\ C_{1,c,p}(\eta,\lambda) &\subset C_{2,c,p}(\eta,\lambda), \\ C^*_{1,c,p}(\eta,\lambda) &\subset C^*_{2,c,p}(\eta,\lambda). \end{aligned} \tag{51}$$

Remark 15. Taking $p = 1$ in Corollary 14, we obtain the results obtained by Gao et al. [14, Corollary 1–4].

3. A Set of Integral-Preserving Properties

In this section, we present several integral-preserving properties of the analytic function classes introduced here. In order to obtain the integral-preserving properties involving the integral operator $J_{c,p}$ defined by (13).

Theorem 16. *Let c be any real number and $c > -p$. If $f(z) \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$, then $J_{c,p}(z) \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Proof. From (13), we have

$$z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}(z)\right)' = (c+p)\,\mathfrak{R}^{\alpha,\gamma}_{\beta,p} - c\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}(z). \tag{52}$$

Let $f \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$ and set

$$\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)} - \lambda = (p-\lambda)\,q(z), \tag{53}$$

where $q(z)$ is given by (22). By using identity (52), we obtain

$$\frac{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)} = \frac{\lambda + (p-\lambda)\,q(z) + c}{c+p}. \tag{54}$$

Differentiating (54) logarithmically with respect to z, we obtain

$$\begin{aligned} \frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)} &= \frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)} \\ &\quad + \frac{(p-\lambda)\,zq'(z)}{(p-\lambda)\,q(z)+\lambda+c} \\ &= \lambda + (p-\lambda)\,q(z) + \frac{(p-\lambda)\,zq'(z)}{(p-\lambda)\,q(z)+\lambda+c}. \end{aligned} \tag{55}$$

We now choose $u = q(z) = u_1 + iu_2$, and $v = zq'(z) = v_1 + iv_2$, and define the function Θ by

$$\Theta(u,v) = (p-\lambda)\,u + \frac{(p-\lambda)\,v}{(p-\lambda)\,u+\lambda+c}. \tag{56}$$

It is easy to see that the function $\Theta(u,v)$ satisfies the conditions of Lemma 7, and the remaining part of the proof of Theorem 16 is similar to that of Theorem 8. □

Taking $\gamma = 1$ in Theorem 16, we obtain the following corollary.

Corollary 17. *Let c be any real number and $c > -p$. If $f(z) \in S^*_{\alpha,\beta,p}(\lambda)$, then $J_{c,p}(z) \in S^*_{\alpha,\beta,p}(\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Theorem 18. *Let c be any real number and $c > -p$. If $f(z) \in K_{\alpha,\beta,\gamma,p}(\lambda)$, then $J_{c,p}(z) \in K_{\alpha,\beta,\gamma,p}(\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Proof. By applying Theorem 16 in conjunction with the equivalence (4), it follows that

$$\begin{aligned} & f(z) \in K_{\alpha,\beta,\gamma,p}(\lambda) \\ &\Longleftrightarrow \frac{z}{p}f'(z) \in S^*_{\alpha,\beta,\gamma,p}(\lambda) \\ &\Longrightarrow J_{c,p}\left(\frac{z}{p}f'(z)\right) \in S^*_{\alpha,\beta,\gamma,p}(\lambda) \\ &\Longleftrightarrow \frac{z}{p}\left(J_{c,p}f(z)\right)' \in S^*_{\alpha,\beta,\gamma,p}(\lambda) \\ &\Longrightarrow J_{c,p}f(z) \in K_{\alpha,\beta,\gamma,p}(\lambda), \end{aligned} \tag{57}$$

which proves Theorem 18. □

Taking $\gamma = 1$ in Theorem 18, we obtain the following corollary.

Corollary 19. *Let c be any real number and $c > -p$. If $f(z) \in K_{\alpha,\beta,p}(\lambda)$, then $J_{c,p}(z) \in K_{\alpha,\beta,p}(\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Theorem 20. *Let c be any real number and $c > -p$. If $f(z) \in C_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, then $J_{c,p}(z) \in C_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Proof. Let $f \in C_{\alpha,\beta,\gamma,p}(\eta,\lambda)$. Then there exists a function $\Psi \in S^*_p(\lambda)$ such that

$$\operatorname{Re}\left\{\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)'}{\Psi(z)}\right\} > \eta \quad (0 \le \lambda, \eta < p,\ z \in U). \tag{58}$$

We put

$$\mathfrak{R}^{\alpha,\gamma}_{\beta,p}g(z) = \Psi(z), \tag{59}$$

so that we have

$$g \in S^*_{\alpha,\beta,\gamma,p}(\lambda), \quad \operatorname{Re}\left\{\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}g(z)}\right\} > \eta \quad (z \in U). \tag{60}$$

We next put

$$\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}J_{c,p}g(z)} = \eta + (p-\eta)\,q(z), \tag{61}$$

where $q(z)$ is given by (22). Thus, by using identity (52), we obtain

$$\begin{aligned}\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} &= \frac{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} \\ &= \left(z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)\right]' \right. \\ &\quad \left. +c\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)\right) \\ &\quad \times\left(z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\right]' \right. \\ &\quad \left. +c\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\right)^{-1} \\ &= \left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} \right. \\ &\quad \left. +c\frac{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)}\right) \\ &\quad \times\left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)}+c\right)^{-1}.\end{aligned} \tag{62}$$

Since $g \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$, then from Theorem 16 we have $J_{c,p}g(z) \in S^*_{\alpha,\beta,\gamma,p}(\lambda)$, so that we can put

$$\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} = \lambda + (p-\lambda)G(z), \tag{63}$$

where

$$G(z) = g_1(x,y) + ig_2(x,y), \quad \mathrm{Re}(G(z)) = g_1(x,y) > 0 \quad (z \in U). \tag{64}$$

Then

$$\begin{aligned}\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} &= \left(\frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(z\left(J_{c,p}f(z)\right)'\right)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} \right. \\ &\quad \left. + c\left(\eta + (p-\eta)q(z)\right)\right) \\ &\quad \times\left((p-\lambda)G(z)+\lambda+c\right)^{-1}.\end{aligned} \tag{65}$$

We thus find from (61) that

$$z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}f(z)\right)' = \mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\left[\eta + (p-\eta)q(z)\right]. \tag{66}$$

Differentiating both sides of (66) with respect to z, we obtain

$$\begin{aligned}&\frac{z\left[z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}f(z)\right)'\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)} \\ &\quad = (p-\eta)zq'(z) \\ &\qquad + \frac{z\left[\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)\right]'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}J_{c,p}g(z)}\left[\eta + (p-\eta)q(z)\right] \\ &\quad = (p-\eta)zq'(z) \\ &\qquad + \left[\lambda + (p-\lambda)G(z)\right]\left[\eta + (p-\eta)q(z)\right].\end{aligned} \tag{67}$$

By substituting (67) into (65), we have

$$\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)} - \eta = (p-\eta)q(z) + \frac{(p-\eta)zq'(z)}{(p-\lambda)G(z)+\lambda+c}. \tag{68}$$

The remainder of our proof of Theorem 20 is much akin to that of Theorem 8. We, therefore, choose to omit the details involved. □

Taking $\gamma = 1$ in Theorem 20, we obtain the following corollary.

Corollary 21. *Let c be any real number and $c > -p$. If $f(z) \in C_{\alpha,\beta,p}(\eta,\lambda)$, then $J_{c,p}(z) \in C_{\alpha,\beta,p}(\eta,\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Theorem 22. *Let c be any real number and $c > -p$. If $f(z) \in C^*_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, then $J_{c,p}(z) \in C^*_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

Proof. Just as we derived Theorem 18 as a consequence of Theorem 16 by using the equivalence (4), we can also prove Theorem 22 by using Theorem 20 in conjunction with the equivalence (7). □

Taking $\gamma = 1$ in Theorem 22, we obtain the following corollary.

Corollary 23. *Let c be any real number and $c > -p$. If $f(z) \in C^*_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, then $J_{c,p}(z) \in C^*_{\alpha,\beta,\gamma,p}(\eta,\lambda)$, where $J_{c,p}f(z)$ is defined by* (13).

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