Hindawi Publishing Corporation
ISRN Geometry
Volume 2013, Article ID 391721, 6 pages
http://dx.doi.org/10.1155/2013/391721

Research Article

Postulation of a Union $X \subset \mathbb{P}^r$, $r \geq 4$, of a Given Zero-Dimensional Scheme and Several General Lines

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Received 10 October 2013; Accepted 5 November 2013

Academic Editors: E. Previato and F. Shi

We study the Hilbert function of the union X of a given zero-dimensional scheme $Z \subset \mathbb{P}^r$, $r \geq 4$, and t general lines of $\mathbb{P}^r$, proving, for example, that if $t \gg \deg(Z)$, then X has maximal rank.

1. Introduction

A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if for all integers $t > 0$, the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(X, \mathcal{O}_X(t))$ is either injective or surjective, that is, if either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, that is, if X imposes the "expected" number of conditions to the vector space of the homogeneous degree t polynomials in $r+1$ variables. Hartshorne and Hirschowitz proved that for all $t > 0$ and $r \geq 3$, a general union $X \subset \mathbb{P}^r$ of t general lines has maximal rank [1]. Carlini et al. considered several cases in which we allow unions of linear spaces with certain multiplicities (for zero-dimensional subspaces these are general unions of fat points) [2–4]. Let $L(r,t)$ denote the set of all unions $X \subset \mathbb{P}^r$ of t disjoint lines. In this paper (following [4]) we consider the case in which X is a disjoint union of a given zero-dimensional scheme Z and a general $Y \in L(r,t)$.

Fix a zero-dimensional scheme $Z \subset \mathbb{P}^r$, $r \geq 4$, and integers $c > 0$ and $t \geq 0$. Set $z := \deg(Z)$. In this note we study the Hilbert function of a general union of Z and t disjoint lines. We assume almost nothing on Z. We only prescribe the integer $z := \deg(Z)$ and an integer $c > 0$. Take a general $T \in L(r,t)$. Since Z is fixed, while T is general, we have $Z \cap T = \emptyset$, and hence $h^0(T \cup Z, \mathcal{O}_{T\cup Z}(x)) = (x+1)t + z$ for all $x \geq 0$. To study the Hilbert function of a general union of Z and t disjoint lines, we need the following integers $e_{z,r,k}$ and $f_{z,r,k}$, $k \geq 0$, defined by the following relations:

$$z + (k+1)\, e_{z,r,k} + f_{z,r,k} = \binom{k+r}{r}, \quad 0 \leq f_{z,r,k} \leq k. \tag{1}$$

The scheme $T \cup Z$ has maximal rank if and only if $h^1(\mathcal{I}_{T\cup Z}(k)) = 0$ for all integers k with $t \leq e_{z,r,k}$ and $h^0(\mathcal{I}_{T\cup Z}(k)) = 0$ for the minimal integer $k \geq c$ such that either $e_{z,r,k} > t$ or $e_{z,r,k} = t$ and $f_{z,r,k} = 0$.

We first prove the following result, which says that if we may handle general unions of Z and general lines with respect to homogeneous polynomials in two consecutive degrees (not small with respect to the integer $z := \deg(Z)$), then these information propagates to higher degree polynomials.

Theorem 1. *Fix a zero-dimensional scheme $Z \subset \mathbb{P}^r$, $r \geq 4$, and an integer $c > 0$. Set $z := \deg(Z)$. Assume $z \leq \binom{r+c}{r}$. If $r = 4$, assume $c \geq 8$. Let δ_1 be any nonnegative integer such that $h^0(\mathcal{I}_{Z\cup X}(c)) \leq f_{z,r,c} + \delta_1$ for a general $X \in L(r, e_{z,r,c})$. Let δ_2 be any nonnegative integer such that $h^0(\mathcal{I}_{Z\cup X}(c+1)) \leq f_{z,r,c+1} + \delta_2$ for a general $X \in L(r, e_{z,r,c+1})$. Set $\delta := \max\{\delta_1, \delta_2\}$.*

Fix an integer $y > e_{z,r,c+1}$, and let Y be a general element of $L(r,y)$.

(a) *For all integers $k \geq c$, either $h^0(\mathcal{I}_{Z\cup Y}(k)) \leq \delta$ or $h^1(\mathcal{I}_{Z\cup Y}(k)) \leq \delta$.*

(b) *Assume $f_{z,r,c} \leq f_{z,r,c+1}$. Then for all integers $k \geq c$, either $h^0(\mathcal{I}_{Z\cup Y}(k)) \leq \delta_1$ or $h^1(\mathcal{I}_{Z\cup Y}(k)) \leq \delta_1$.*

Then we go to a case in which we only assume something about the integer $z = \deg(Z)$ and prove the following result.

Theorem 2. *Fix positive integers r, z, and c such that $r \geq 4$ and a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $\deg(Z) = z$ and $h^1(\mathcal{I}_Z(c)) = 0$. Set $\tau := \binom{r+c}{r} - z$. If either $r \geq 5$ or $r = 4$ and $c \geq 10$, then set $\tau' := \tau$. If $r = 4$ and $c \leq 9$, then set $\tau' := \tau + 10 - c$. Fix any integer $t \geq (\binom{r+c+\tau'+1}{r} - z)/(c+\tau'+1)$. Let $X \subset \mathbb{P}^r$ be a general union of Z and t lines. Then X has maximal rank.*

If $z = \binom{r+c}{r}$, then we have the following result.

Proposition 3. *Fix integers $r \geq 4$ and $c > 0$. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\deg(Z) = \binom{r+c}{r}$ and $h^0(\mathcal{I}_Z(c)) = 0$. Then for each integer $t \geq 0$, a general union of Z and t lines has maximal rank.*

In the case $Z = (c+1)P$, we get the case $r \geq 4$ of [4]. We omit the proof of Proposition 3, because the proof of [4] works verbatim (e.g., quote [4, Lemma 5.1] instead of Lemmas 10, 12, and 13).

2. The Proofs

For all integers $z \geq 0$, $r \geq 4$, $c \geq 0$, $x \in \mathbb{Z}$, and $k \geq 0$, define the integers $e_{z,r,k,x}$ and $f_{z,r,k,x}$ by the relations

$$z + (k+1)\, e_{z,r,k,x} + f_{z,r,k,x} = \binom{r+k}{r} + x, \qquad 0 \leq f_{z,r,k,x} \leq k. \tag{2}$$

Notice that $e_{z,r,k} = e_{z,r,k,0}$ and $f_{z,r,k} = f_{z,r,k,0}$. For any $t \geq 0$, $c > 0$, the *critical value* of the triple (z, r, t) (resp., (z, r, c)) is the minimal integer $k > 0$ (resp., $k > c$) such that $z + (k+1)t \leq \binom{r+k}{k}$, that is, the minimal integer $k > 0$ (resp., $k > c$) such that $t \leq e_{z,r,k}$.

Taking the equation in (2) minus the same equation for the integer $k' := k - 1$, we get

$$e_{z,r,k-1,x} + (k+1)\left(e_{z,r,k,x} - e_{z,r,k-1,x}\right) + f_{z,r,k,x} - f_{z,r,k-1,x} = \binom{k+r-1}{r-1}. \tag{3}$$

Take Z, z, c, δ, and δ_1 as in Theorem 1. For each integer $k \geq c$ consider the following assertion A_k:

A_k: we have $h^0(\mathcal{I}_{Z\cup A}(k)) \leq f_{z,r,k} + \delta$ for a general $A \in L(r, e_{z,r,k})$.

Take any $A \in L(r, e_{z,r,k})$ such that $Z \cap A = \emptyset$. Since $h^0(Z \cup A, \mathcal{O}_{Z\cup A}(k)) = \deg(Z) + (k+1)\deg(A)$, we have $h^0(\mathcal{I}_{Z\cup A}(k)) = h^1(\mathcal{I}_{Z\cup A}(k)) + f_{z,r,k}$. Hence A_k is true if and only if $h^1(\mathcal{I}_{Z\cup A}(k)) \leq \delta$ for a general $A \in L(r, e_{z,r,k})$.

Call A'_k, the same assertion with δ_1 instead of δ.

Remark 4. By the definition of the integers δ_1 and δ, the assertions A_c, A'_c, and A_{c+1} are true.

For any hyperplane $H \subset \mathbb{P}^r$, and any closed subscheme $W \subset \mathbb{P}^r$, let $\mathrm{Res}_H(W)$ be the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_H$ as its ideal sheaf. If $W = Z \sqcup Y$ with $Z \cap H = \emptyset$ and Y is a reduced scheme, then $\mathrm{Res}_H(W) = Z \sqcup Y'$, where Y' is the union of the irreducible components of Y not contained in H.

Lemma 5. *Fix a hyperplane $H \subset \mathbb{P}^n$, $n \geq 3$, an integer $k > 0$ and a closed subscheme $W \subset \mathbb{P}^n$. If $h^0(\mathcal{I}_W(k)) > h^0(\mathcal{I}_{Res_H(W)}(k-1))$, then $h^0(\mathcal{I}_{W\cup\{P\}}(k)) = h^0(\mathcal{I}_W(k)) - 1$ for a general $P \in H$.*

Proof. The assumption implies that H is not contained in the base scheme of $|\mathcal{I}_W(k)|$. The long cohomology exact sequence associated to the following exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_H(W)}(k-1) \longrightarrow \mathcal{I}_W(k) \longrightarrow \mathcal{I}_{W\cap H,H}(k) \longrightarrow 0 \tag{4}$$

gives the following inequalities:

(1) $h^0(\mathcal{I}_W(k)) \leq h^0(\mathcal{I}_{\mathrm{Res}_H(W)}(k-1)) + h^0(H, \mathcal{I}_{W\cap H,H}(k))$;

(2) $h^1(\mathcal{I}_W(k)) \leq h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(k-1)) + h^1(H, \mathcal{I}_{W\cap H,H}(k))$.

We have $\mathrm{Res}_H(W \cup \{P\}) = \mathrm{Res}_H(W)$ and $h^0(H, \mathcal{I}_{(W\cap H)\cup\{P\}}(k)) = h^0(H, \mathcal{I}_{W\cap H}(k)) - 1$, because a general $P \in H$ is not in the base locus of $|\mathcal{I}_W(k)|$. Apply (4) first to W and then to $W \cup \{P\}$. □

As in [2–4], we will call *Casteluonovo's sequence* for any of the inequalities in the proof of Lemma 5.

Lemma 6. *Fix integers $n \geq 3$, $k > 0$, $e \geq 0$, and $f \geq 0$ such that $(k+1)(e+2f) \leq \binom{n+k}{n}$. Let $X \subset \mathbb{P}^n$ be a general union of e lines and f reducible conics. Then $h^1(\mathcal{I}_X(k)) = 0$.*

Proof. For each $P \in \mathrm{Sing}(X)$, let $C_P \subset X$ be the connected component of X containing P. Write $X = Y \sqcup \bigcup_{P\in\mathrm{Sing}(X)} C_P$. For each $P \in \mathrm{Sing}(X)$, let $N_P \subseteq \mathbb{P}^n$ be a general 3-dimensional linear space containing C_P. Let $E_P \subset N_P$ be the sundial with C_P as its support. Set $X' := Y \sqcup \bigcup_{P\in\mathrm{Sing}(X)} E_P$. Since X and each N_P are general, X' is a general union of e lines and f sundials. Hence X' has maximal rank [3]. Since $(k+1)(e+2f) \leq \binom{n+k}{n}$, we get $h^1(\mathcal{I}_{X'}(k)) = 0$. Fix $P \in \mathrm{Sing}(X)$. Let η be the nilpotent sheaf of E_P. We have an exact sequence

$$0 \longrightarrow \eta \longrightarrow \mathcal{O}_{E_P}(k) \longrightarrow \mathcal{O}_{C_P}(k) \longrightarrow 0. \tag{5}$$

Since η is supported by a single point, P, we have $h^1(E_P, \eta) = 0$. Hence (5) gives the surjectivity of the restriction map $H^0(E_P, \mathcal{O}_{E_P}(k)) \to H^0(C_P, \mathcal{O}_{C_P}(k))$. Looking at all the connected components of X and X', we get the surjectivity of the restriction map $H^0(X', \mathcal{O}_{X'}(k)) \to H^0(X, \mathcal{O}_X(k))$. Hence $h^1(\mathcal{I}_X(k)) \leq h^1(\mathcal{I}_{X'}(k)) = 0$. □

Lemma 7. *A_k is true for all integers $k \geq c$.*

Proof. By Remark 4 we may assume $k \geq c+2$. We use induction on k. Since A_{c+1} is true, we may assume that A_{k-1} is

true. Fix a hyperplane $H \subset \mathbb{P}^r$ such that $H \cap Z = \emptyset$. Fix a general $Y \in L(r, e_{z,r,k-1})$. We have $Y \cap Z = \emptyset$, $H \cap Y$ is a general union of $e_{z,r,k-1}$ points of H, and $h^0(\mathcal{I}_{Z\cup Y}(k)) \le f_{z,r,k-1} + \delta$.

(i) First assume $f_{z,r,k-1} \le f_{z,r,k}$. Let $E \subset H$ be a general union of $e_{z,r,k} - e_{z,r,k-1}$ lines; this is possible, because $e_{z,r,k} - e_{z,r,k-1} \ge 0$ (Lemma 10). By (3), we have $e_{z,r,k-1} + (k+1)(e_{z,r,k} - e_{z,r,k-1}) + f_{z,r,k} - f_{z,r,k-1} = \binom{k+r-1}{r-1}$. Since E has maximal rank in H [1] and $Y \cap H$ is general in H, we have $h^0(H, \mathcal{I}_{Y\cap H\cup F}(k)) = f_{z,r,k} - f_{z,r,k-1}$. Castelnuovo's sequence gives $h^0(\mathcal{I}_{Z\cup Y\cup E}(k)) \le f_{z,r,k}+\delta$, proving A_k in this case.

(ii) Now assume $f_{z,r,k-1} > f_{z,r,k}$. In this case we have $e_{z,r,k} - e_{z,r,k-1} \ge 2(f_{z,r,k-1} - f_{z,r,k})$ (Lemma 13). Let $F_1 \subset H$ be a general union of $f_{z,r,k-1} - f_{z,r,k}$ sundials of H and $e_{z,r,k}-e_{z,r,k-1}-2(f_{z,r,k-1}-f_{z,r,k})$ lines. By [2], F_1 has maximal rank in H. Lemma 10 for the integer $k' := k-1$ gives $e_{z,r,k-1} \ge k-1$. Hence $e_{z,r,k-1} \ge f_{z,r,k-1} - f_{z,r,k}$. Hence (3) gives $h^1(H, \mathcal{I}_{F_1}(k)) = 0$. Set $F := (F_1)_{\mathrm{red}}$ and $S := \mathrm{Sing}(F)$. Let U be the union of the degree one connected components of F. Let $F_2 \subset \mathbb{P}^r$ be a general union of disjoint lines and sundials with F has its reduction. Since $(Y \cup F_2) \cap Z = \emptyset$ and $Y \cup F_2$ is a degeneration of a family of disjoint unions of $e_{z,r,k}$ lines, the semicontinuity theorem for cohomology ([5, Theorem III.12.8]) implies that it is sufficient to prove that $h^0(\mathcal{I}_{Z\cup Y\cup F_2}(k)) \le f_{z,r,k} + \delta$. We have $\mathrm{Res}_H(Z\cup Y\cup F_2) = Z\cup Y\cup S$. The set S is a general union of $f_{z,r,k-1}-f_{z,r,k-1}$ points of S. If S were general points of $\mathbb{P}^r$, then obviously $h^0(\mathcal{I}_{Z\cup Y\cup S}(k)) = \max\{0, h^0(\mathcal{I}_{Z\cup Y}(k)) - \#(S)\} \le f_{z,r,k} + \delta$. To apply $f_{z,r,k-1} - f_{z,r,k-1}$ times Lemma 5 and get the inequality $h^0(\mathcal{I}_{Z\cup Y\cup S}(k-1)) \le f_{z,r,k} + \delta$, we need to check the restriction on the residual scheme in its statement. Assume that this lemma cannot be applied and that A_k fails in this case; that is, $h^0(\mathcal{I}_{Z\cup Y\cup S}(k-1)) > f_{z,r,k}+\delta$. We would get $h^0(\mathcal{I}_{Z\cup Y}(k-2)) > f_{z,r,k} + \delta$. If $k-2 = c$, we get $h^0(\mathcal{I}_{Z\cup Y'}(c) > \delta)$ for a general $Y' \in L(r,t)$ (contrary to the definition of δ), because $e_{z,r,k-1} \ge 0$ by Lemma 10. Now assume $k \ge c+3$, and let $Y_1 \subset Y$ a union of $e_{z,r,k-2}$ lines. By A_{k-2}, we have $h^0(\mathcal{I}_{Z\cup Y_1}(k-2)) \le f_{z,r,k-2} + \delta$. Any two points of $\mathbb{P}^r$ are contained in a line. Hence $h^0(\mathcal{I}_{Z\cup Y}(k-2)) \le \max\{0, f_{z,r,k-2} + \delta - 2(\deg(Y) - \deg(Y_1))\}$. Lemma 10 gives $2(\deg(Y) - \deg(Y_1)) \ge 2(k-1) \ge f_{z,r,k-2}$. Hence $h^0(\mathcal{I}_{Z\cup Y\cup S}(k)) = \max\{0, h^0(\mathcal{I}_{Z\cup Y}(k)) - \#(S)\} \le f_{z,r,k} + \delta$.

We have $H \cap (Z \cup Y \cup F_2) = (Y \cap H) \cup F$. Let η be the nilpotent sheaf of F_1. Since η is supported by the finite set S, we have $h^1(F_1, \eta \otimes L) = 0$ for each line bundle L on F_1. The exact sequence

$$0 \longrightarrow \eta \longrightarrow \mathcal{O}_{F_1}(k) \longrightarrow \mathcal{O}_F(k) \longrightarrow 0 \qquad (6)$$

gives the surjectivity of the restriction map $H^0(F_1, \mathcal{O}_{F_1}(k)) \to H^0(F, \mathcal{O}_F(k))$. Hence $h^1(H, \mathcal{I}_F(k)) \le h^1(H, \mathcal{I}_{F_1}(k)) = 0$. Hence $h^0(H, \mathcal{I}_F(k)) = \binom{k+r-1}{r-1} - (k+1)(e_{z,r,k} - e_{z,r,k}) + f_{z,r,k} - f_{z,r,k-1}$. Since $Y \cap H$ is formed by general points of H, (3) gives $h^0(H, \mathcal{I}_{(Z\cup Y\cup F_2)\cap H}(k)) = 0$. Castelnuovo's sequence gives $h^0(\mathcal{I}_{Z\cup Y\cup F_2}(k)) \le f_{z,r,k} + \delta$, concluding the proof of A_k. □

Lemma 8. *Assume $f_{z,r,c} \le f_{z,r,c+1}$. Then A'_k is true for all $k \ge c$.*

Proof. A'_c is true (Remark 4). Now assume $k = c+1$. To copy step (i) of the proof of Lemma 7, it is sufficient to have $e_{z,r,c+1} \ge e_{z,r,c}$. Use Lemma 12. The case $k \ge c+2$ is done as in the proof of Lemma 7. □

Lemma 9. *Fix Z as in Theorem 1 and integers $k > c$ and $t > e_{z,r,k}$. Let Y be a general element of $L(r,t)$. Then $h^0(\mathcal{I}_{Z\cup Y}(k)) \le \delta$.*

Proof. It is sufficient to do the case $t = e_{z,r,k} + 1$. Fix a hyperplane $H \subset \mathbb{P}^r$. First assume $k \ge c+2$. Take a general $A \in L(r, e_{z,r,k-1})$. We have $A \cap Z = \emptyset$. A_{k-1} gives $h^0(\mathcal{I}_{A\cup Z}(k-1)) = f_{z,r,k-1}$. Let $F_1 \subset \mathbb{P}^r$ be a general union of $e_{z,r,k} + 1 - 2f_{z,r,k-1}$ lines of H and $f_{z,r,k-1}$ sundials whose support in contained in H. We conclude as in step (ii) of the proof of Lemma 7. □

Proof of Theorem 1. First assume $z + (k+1)y \ge \binom{r+k}{r}$. Since $k \ge c$ and $h^0(\mathcal{I}_{Z\cup Y}(c)) \le \delta$ for a general $Y \in L(r,t)$ and $y \ge t$, we may assume $k > c$. In this range the assumption is satisfied if and only if either $y > e_{z,r,k}$ or $y = e_{z,r,k}$ and $f_{z,r,k}$, because $e_{z,r,u}$ is a strictly increasing function of u (Lemma 10). Hence $h^0(\mathcal{I}_{Z\cup Y}(c)) \le \delta$ either by Lemma 9 (case $f_{z,r,k} > 0$) or by A_k (case $f_{z,r,k} = 0$).

Now assume $z + (k+1)y \le \binom{r+k}{r}$ (we allow the equality). By the semicontinuity theorem for cohomology ([5, Theorem III.12.8]), it is sufficient to find $A \in L(r,y)$ such that $A \cap Z = \emptyset$ and, $h^1(\mathcal{I}_{Z\cup A}(k)) \le \delta$. By (2) and the assumption $y > e_{z,r,c+1}$, we have $k \ge c+2$ and $y \le e_{z,r,k}$. Fix a general $W \subset L(r, e_{z,r,k})$. Let $U \subseteq W$ be the union of y of the lines of W. By A_k, we have $h^1(\mathcal{I}_{Z\cup W}(k)) \le \delta$. Since $Z\cup U$ is a union of some of the connected components of $Z \cup W$, we have $h^1(\mathcal{I}_{Z\cup U}(k)) \le \delta$.

Now we check part (b). Since $f_{z,r,c} \le f_{z,r,c+1}$, Lemma 8 gives A'_k for all $k > c$. Then we get Lemma 9 with δ_1 instead of δ. Then we continue as in the proof of part (a). □

Proof of Theorem 2. Fix a hyperplane $H \subset \mathbb{P}^r$ such that $H \cap Z = \emptyset$. If either $r \ge 5$ or $r = 4$ and $c \ge 10$ for each integer $k \ge c$, set $x_k = \max\{0, -\tau+k-c-1\}$. If $r = 4$ and $c \le 9$, set $x_k := -\tau$. Hence $x_c = x_{c+1} = -\tau$. If $r = 4$ and $c \le 9$, we have $x_k = -\tau$ for all $k \le 10$. Consider the following statement A''_k, $k \ge c$:

A''_k: we have $h^0(\mathcal{I}_{Z\cup A}(k)) = f_{z,r,k,x_k}$ for a general $A \in L(r, e_{z,r,k,x_k})$.

Take any $A \in L(r, e_{z,r,k,x_k})$ such that $Z \cap A = \emptyset$. We have $h^0(\mathcal{I}_{Z\cup A}(k)) = h^1(\mathcal{I}_{Z\cup A}(k)) + f_{z,r,k,x_k}$. Hence A''_k is true if and only if $h^1(\mathcal{I}_{Z\cup A}(k)) = 0$ for a general $A \in L(r, e_{z,r,k,x_k})$.

(a) In this step we prove A''_c and A''_{c+1}. A''_c is true, because we assumed that $h^1(\mathcal{I}_Z(c)) = 0$. To check A''_{c+1}, we first notice that $x_c = x_{c+1}$, and hence we may apply (3) for the integer $x := x_c$. Since $f_{z,r,c,x_c} = 0$, we

have $f_{z,r,c+1,x_{c+1}} \geq f_{z,r,c,x_c}$. Since $e_{z,r,c,x_c} = 0$, Lemma 15 gives $e_{z,r,c+1,x_c} \geq e_{z,r,c,x_c}$. Hence the construction in step (i) of the proof of Lemma 7 works verbatim.

(b) In this step we prove A''_k for all $k \geq c$. Since the cases $k = c, c+1$ are true by step (a), we may assume that $k \geq c+2$ and that A''_{k-1} is true. Taking the difference of (2) with $k' := k-1$ and $x := x'$ and integers k, x, we get

$$\begin{aligned} e_{z,r,k-1,x} &+ (k+1)\left(e_{z,r,k,x} - e_{z,r,k-1,x'}\right) \\ &+ f_{z,r,k,x} - f_{z,r,k-1,x'} + x - x' \\ &= \binom{k+r-1}{r-1}. \end{aligned} \tag{7}$$

We apply (7) with $x' = x_{k-1}$ and $x = x_k$. Hence either $x' = x$ (case $k \geq \tau+2+c$) or $x' = x-1$. In both cases, we have $0 \leq x - x' \leq 1$. If $x = x'$, then we may apply the numerical Lemmas 10, 12, and 13 used in the proof of Theorem 1. We need different numerical lemmas if $x \neq x'$, that is, if $x_{k-1} < 0$ and $x_k = x_{k-1} + 1$.

Fix a general $Y \in L(r, e_{z,r,k-1,x_{k-1}})$. We have $Z \cap Y = \emptyset$, $h^1(\mathcal{I}_{Z\cup Y}(k-1)) = 0$, $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1}$, and $Y \cap H$ is a general union of $e_{z,r,k-1,x_{k-1}}$ points of H. Since $h^1(\mathcal{I}_{Z\cup Y}(k-1)) = 0$, $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1}$, we have $h^0(\mathcal{I}_{Z\cup Y\cup\{P\}}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1} - 1$ for a general $P \in \mathbb{P}^r$ (and even a general $P \in H$ by Lemma 5 if $f_{z,r,k-1,x_{k-1}} - x_{k-1} - 1 \geq 0$). This is always the case if $x_{k-1} < 0$, that is, if $x_{k-1} \neq x_k$.

(b.1) First assume $f_{z,r,k-1,x_{k-1}} \leq f_{z,r,k,x_k}$. Since $0 \leq x_k - x_{k-1} \leq 1$, Lemma 15 gives $e_{z,r,k,x_k} \geq e_{z,r,k-1,x_{k-1}} + 2$. Let $E_1 \subset H$ be a general union of $e_{z,r,k,x_k} - e_{z,r,k-1,x_{k-1}} - 2$ lines. Take a general reducible conic $T \subset H$. Let $T' \subset \mathbb{P}^r$ be a general sundial of $\mathbb{P}^r$, and let $T'' \subset H$ be a general sundial of H with T as its support. Set $F := E_1 \cup T$. Since $Z \cup Y \cup E_1 \cup T'$ is a specialization of a union of Z and $\deg(Y \cup E_1 \cup T')$ lines, it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup Y\cup E_1\cup T'}(k)) = 0$. Castelnuovo's sequence gives $h^1(\mathcal{I}_{Z\cup Y\cup E}(k)) = 0$. Lemma 14 gives $e_{z,r,k-1} > 0$. By (7) to apply Lemma 5, we need that $Y \cap H$ has as least one point; this is the reason why this proof does not work if $k = c+1$.

(b.2) Now assume $f_{z,r,k-1,x_{k-1}} > f_{z,r,k,x_k}$. Since $0 \leq x_k - x_{k-1} \leq 1$, Lemma 16 gives $e_{z,r,k,x_k} - e_{z,r,k-1,x_{k-1}} \geq 2(f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k}) + 2$. We repeat step (ii) of the proof of Lemma 7 taking inside H instead of F a general union $\widetilde{F}$ of $e_{z,r,k,x_k} - e_{z,r,k-1,x_{k-1}} - 2(f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k} + 1)$ lines, $f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k} + 1$ reducible conics. Then we take the general sundials (in H and in $\mathbb{P}^r$) with these conics as their supports. To apply Lemma 6, we need $e_{z,r,k-1,x_{k-1}} \geq f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k} + 1$. This inequality is true by Lemma 15.

(c) Fix an integer $t \geq (\binom{r+c+\tau'+1}{r} - z)/(c+\tau'+1)$. Let k be the critical value for the triple (r, c, t), i.e. the minimal positive integer such that $z + (k+1)t \leq \binom{k+r}{r}$. Since $t \geq (\binom{r+c+\tau'+1}{r} - z)/(c+\tau'+1)$, either $k \geq c+\tau'+1$ or $k = c+\tau'$ and $f_{z,c,r,x_k} = 0$. To prove Theorem 2 for the integer t it is sufficient to prove $h^1(\mathcal{I}_{Z\cup Y}(k)) = 0$ and $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = 0$ for a general $Y \in L(r,t)$. Since $L(r,t)$ is irreducible, the semicontinuity theorem for cohomology says that to prove Theorem 2 for the integer t it is sufficient to prove the existence of $A \in L(r,t)$ and $B \in L(r,t)$ such that $A \cap Z = B \cap Z = \emptyset$, $h^1(\mathcal{I}_{Z\cup A}(k)) = 0$ and $h^0(\mathcal{I}_{Z\cup YB}(k-1)) = 0$. Since $t \geq (\binom{r+c+\tau'+1}{r} - z)/(c+\tau'+1)$, we have $k \geq \tau'+1$. Hence $x_k = x_{k-1} = 0$ Since $x_k = 0$ the existence of A is proved as. □

3. Numerical Lemmas

Lemma 10. *Assume $r \geq 4$. Fix an integer $c > 0$. One has $e_{z,r,k} \geq e_{z,r,k-1} + k$ for all $k \geq c+2$.*

Proof. Assume $e_{z,r,k} \leq e_{z,r,k-1} + k - 1$. From (3), we get

$$e_{z,r,k-1} + (k+1)(k-1) + f_{z,r,k} - f_{z,r,k-1} \geq \binom{r+k-1}{r-1}. \tag{8}$$

From (2) for the integer $k' := k-1$, we get

$$e_{z,r,k-1} = \frac{\left(\binom{r+k-1}{r} - z - f_{z,r,k-1}\right)}{k}. \tag{9}$$

Since $f_{z,r,k} \leq k$ and $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, from (11) and (12), we get

$$k\left(k^2+k-1\right) - (k-1) f_{z,r,k-1} \geq (r-1)\binom{r+k-1}{r} + z. \tag{10}$$

Set $\mu(r,k) := (r-1)\binom{r+k-1}{r} - k(k^2+k-1)$. Since $f_{z,r,k-1} \geq 0$ and $z > 0$, to get a contradiction, it is sufficient to prove that $\mu(r,k) \geq 0$. First assume $r = 4$. We have $8\mu(4,k) = (k+3)(k+2)(k+1)(k-1) - 8k(k^2+k-1)$. Obviously $8\mu(4,k) \geq 0$ if $k \geq 6$. We have $\mu(4,5) = 210 - 5\cdot 29$, $\mu(4,4) = 105 - 76 > 0$, and $\mu(4,3) = 45 - 33 > 0$.

We have $\mu(r+1,k) - \mu(r,k) = k\binom{r+k-1}{r} - \binom{r+k}{r+1} = \binom{r+k}{r+1}\{[k(r+1)/(r+k)] - 1\} > 0$. By induction on r, we get the lemma for all $k \geq 5$. □

Lemma 11. *Fix integers $r \geq 4$, $c > 0$, $z \geq 0$, and $x \leq 0$. Then $e_{z,c,k,x} \geq e_{z,c,k-1,x} + k$ for all $k \geq 2$.*

Proof. Assume $e_{z,c,k,x} \leq e_{z,c,k-1,x} + k - 1$. From (3), we get

$$e_{z,c,k-1,x} + (k+1)(k-1) + f_{z,c,k,x} - f_{z,c,k-1,x} \geq \binom{r+k-1}{r-1}. \tag{11}$$

From (2) for the integer $k' := k-1$, we get

$$e_{z,c,k-1,x} = \frac{\left(\binom{r+k-1}{r} + x - z - f_{z,c,k-1,x}\right)}{k}. \tag{12}$$

Since $f_{z,c,k,x} \le k$, $(c+1)t \le \binom{r+c}{r}$, $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, $z \ge 0$, and $x \le 0$, from (11) and (12), we get

$$k(k+1)(k-1) - (k-1) f_{z,c,k-1} \ge (r-1)\binom{r+k-1}{r}. \tag{13}$$

Set $\phi(r,k) := (r-1)\binom{r+k-1}{r} - k(k+1)(k-1)$. Since $f_{z,c,k-1} \ge 0$, to get a contradiction it is sufficient to prove that $\phi(r,k) > 0$. First assume $r = 4$. We have $8\phi(4,k) = (k+3)(k+2)(k+1)k - 8k(k+1)(k-1) = (k+1)k(k^2-3k+8)$. Hence $\phi(4,k) > 0$ for all $k \ge 2$. Now assume $r > 4$. We have $\phi(r+1,k) - \phi(r,k) = k\binom{r+k-1}{r} - \binom{r+k}{r+1} = \binom{r+k}{r+1}\{[k(r+1)/(r+k)] - 1\} > 0$. By induction on r, we get the lemma. □

Lemma 12. *Assume $r \ge 4$. Fix integers c, z such that $c > 0$ and $0 \le z \le \binom{r+c}{c}$. One has $e_{z,r,k} \ge e_{z,r,k-1}$ for all $k > c$.*

Proof. By Lemma 10, it would be sufficient to do the case $k = c+1$. Assume $e_{z,r,k} \le e_{z,r,k-1} - 1$. From (3) and (12), we get

$$\binom{r+k-1}{r} - k(k+1) - (k-1) f_{z,r,k-1} \ge k\binom{r+k-1}{r-1} \tag{14}$$

which is obviously false. □

Lemma 13. *Assume $r \ge 4$, $k > c > 0$, and $f_{z,r,k} < f_{z,r,k-1}$. If $r = 4$, assume $k \ge 9$. Then $e_{z,r,k} - e_{z,r,k-1} \ge 2(f_{z,r,k-1} - f_{z,r,k})$.*

Proof. Assume $e_{z,r,k} - e_{z,r,k-1} \le 2(f_{z,r,k-1} - f_{z,r,k}) - 1$. From (3), we get

$$e_{z,r,k-1} + (2k+1)(f_{z,r,k-1} - f_{z,r,k}) - 1 - k \ge \binom{r+k-1}{r-1}. \tag{15}$$

Since $e_{z,r,k-1} \le [\binom{k+r-1}{r} - z - f_{z,r,k-1}]/k$, $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, and $f_{z,r,k} \ge 0$, from (15), we get

$$(2k^2+k-1) f_{z,r,k-1} - k(k+1) \ge (r-1)\binom{r+k-1}{r} + z. \tag{16}$$

Since $f_{z,r,k-1} \le k$, we get

$$2k(k+1)(k-1) \ge (r-1)\binom{k+r-1}{r} + z. \tag{17}$$

Set $\alpha(r,k) := (r-1)\binom{k+r-1}{r} - k(2k^2-2)$. Since $z > 0$, to get a contradiction and hence to prove the lemma, it is sufficient to prove that $\alpha(r,k) > 0$. First assume $r = 4$. We have $8\alpha(4,k) = (k+3)(k+2)(k+1)k - 8k(2k^2-2) = k(k+1)[(k+3)(k+2) - 16k + 16]$. Hence $\alpha(4,k) \ge 0$ if and only if $k \ge 9$.

Now assume $r > 4$. We have $\alpha(r,k) - \alpha(r-1,k) = k\binom{r+k-2}{r-1} - \binom{k+r-2}{r} = \binom{k+r-2}{r}[(rk/(k-1) - 1] > 0$ for all $k \ge 2$. We have $\alpha(5,3) = 3\cdot 35 - 21 - 48 > 0$, and $\alpha(5,k)$ is an increasing function of k. Hence $\alpha(r,k) \ge 0$ if either $r \ge 5$ or $k \ge 9$. □

Lemma 14. *Assume $r \ge 4$. Fix integers k, c, z, x, and x' such that $k > c > 0$, $0 \le z \le \binom{r+c}{c}$, $x' < 0$, and $x = x' + 1$. One has $e_{z,r,k,x} \ge e_{z,r,k-1,x'} + 2$.*

Proof. Assume $e_{z,r,k,x} \le e_{z,r,k-1,x'} + 1$. By (7), we get

$$e_{z,r,k-1,x'} + f_{z,r,k,x} - f_{z,r,k-1} + 1 \ge \binom{r+k-1}{r-1}. \tag{18}$$

We have $f_{z,r,k,x} - f_{z,r,k-1} \le k$. Since $x' \le 0$, we have $e_{z,r,k-1,x'} \le e_{z,r,k-1} \le \binom{r+k-1}{r}/k$. Hence

$$\binom{r+k-1}{r} + k(2k+2) \ge k\binom{r+k-1}{r-1}, \tag{19}$$

Since $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, $r \ge 4$, and $k \ge 2$, we get a contradiction. □

Lemma 15. *Take the setup of Theorem 2. For each integer $y > c$ with $x_y < 0$, one has $e_{z,r,y,x_y} \ge 2y$.*

Proof. By Lemma 14, it is sufficient to do the case $y = c+1$. In this case, we have $x_y = \binom{r+c}{r} - z$. Assume $e_{z,r,c+1,x_y} \le 2c+1$. Since $f_{z,r,c+1,x_y} \le c+1$, we get $(c+2)(2c+1) + c + 1 \ge \binom{r+c+1}{r}$. The right hand side of this inequality is an increasing function of r. For $r = 4$, this inequality fails for all $c \ge 2$, because $24(2c^2+6c+3) < (c+5)(c+4)(c+3)(c+2)$ for all $c \ge 2$. □

Lemma 16. *Take the setup of Theorem 2. Assume $r \ge 4$, $k > c > 0$, $x_{k-1} < 0$, and $f_{z,r,k,x_k} < f_{z,r,k-1,x_{k-1}}$. Then $e_{z,r,k,x_k} - e_{z,r,k-1,x_{k-1}} \ge 2(f_{z,r,k-1} - f_{z,r,k}) + 2$.*

Proof. Assume $e_{z,r,k,x_k} - e_{z,r,k-1,x_{k-1}} \le 2(f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k}) + 1$. Since $x_{k-1} < 0$, we have $e_{z,r,k-1,x_{k-1}} < \binom{r+k-1}{r}/k$.

From (7) with $x_k = x_{k-1} + 1$, we get

$$\begin{aligned} \frac{\binom{r+k-1}{r}}{k} &+ (2k+1)\left(f_{z,r,k-1,x_{k-1}} - f_{z,r,k,x_k}\right) \\ &+ (k+1) - 1 \\ &> \binom{r+k-1}{r-1}. \end{aligned} \tag{20}$$

Since $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, we get

$$2k^2(k+1) > (r-1)\binom{r+k-1}{r}. \tag{21}$$

This inequality is false if either $r = 4$ and $k \ge 11$ or $r \ge 5$ and $k \ge 2$. □

Acknowledgment

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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