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Research Article

On α-Cogenerated Commutative Unital C^*-Algebras

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Gelfand-Naimark's theorem states that every commutative C^*-algebra is isomorphic to a complex valued algebra of continuous functions over a suitable compact space. We observe that for a completely regular space X, βX is dense-α-separable if and only if $C(X)$ is α-cogenerated if and only if every family of maximal ideals of $C(X)$ with zero intersection has a subfamily with cardinal number less than α and zero intersection. This gives a simple characterization of α-cogenerated commutative unital C^*-algebras via their maximal ideals.

1. Introduction

In this paper, by R we always mean a commutative ring with identity. Let F denote the reals or the complexes. For a completely regular (topological space) X, let $C(X)$ stand for the F-algebra of continuous maps $X \to F$. The reader is referred to [1] for undefined terms and notations. By βX, we mean the Stone-Čech compactification of X. We denote the ring of all bounded continuous functions by $C^*(X)$. It is well known that for every completely regular space X, we have $C^*(X) \cong C(\beta X)$ (see [1, 7.1]). This note is a continuation of [2], in which we showed that for a compact space X, the following are equivalent: X is dense-separable if and only if $C(X)$ is $\aleph_1$-cogenerated if and only if $C(X)$ is separable. Here, we will drop the compactness condition of the space X and improve our main result in [2]. Furthermore we generalize our results to any regular cardinal α.

Let α be a regular cardinal. A set A is said to be an α-set if $|A| < \alpha$. Following Motamedi in [3], we call a ring R α-cogenerated if for any set $\{A_i \mid i \in I\}$ of ideals of R with $\bigcap_{i\in I} A_i = (0)$ there exists an α-subset I_0 of I such that $\bigcap_{i\in I_0} A_i = (0)$ and α is the least regular cardinal with this property. Any left or right Artinian ring is $\aleph_0$-cogenerated. Any ring with countably many distinct ideals is α-cogenerated, where α is one of $\aleph_0$ or $\aleph_1$. In [2], it has been observed that $C[0,1]$, $C(\mathbb{K})$, where $\mathbb{K}$ is the Cantor perfect set and $C(\beta\mathbb{Q})$ are $\aleph_1$-cogenerated. We call a ring R *α-separable* if it has the following property: if $\{M_i\}_{i\in I}$ is a family of maximal ideals with $\bigcap_{i\in I} M_i = (0)$, then there exists an α-subset I_0 of I such that $\bigcap_{i\in I_0} M_i = (0)$. In this note $\aleph_1$-separable rings are also called separable. Every α-cogenerated ring is α-separable. However, the converse is not true. In [2], we give an example of a separable ring which is not $\aleph_1$-cogenerated.

The density of a space X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X; this cardinal number is denoted by $d(X)$ (see [4]). A space X is called *dense-α-separable* if every dense subset A of X has a dense-α-subset B, which implies that $d(A)$ and hence $d(X)$ are less than α. *Dense-separable* (or in our terminologies dense-$\aleph_1$-separable) spaces are of great interest. Dense-separable spaces were introduced and studied by Levy and McDowell in [5]. It is evident that every dense-separable space is separable and every second countable space is dense-separable. It is well known that $\mathbb{R}_l$ the Sorgenfrey line satisfies all the countability axioms but the second (see [6], page 195, example 3). Since every dense subset of $\mathbb{R}$ is also dense in $\mathbb{R}_l$, the Sorgenfrey line is dense-separable. In [5], it has been shown that $\beta\mathbb{Q}$ and $\beta\mathbb{Q} \setminus \mathbb{Q}$ are dense-separable.

2. Dense-α-Separable Spaces

Since it is easy to observe that $C(X)$ is $\aleph_0$-cogenerated if and only if X is a finite space, and in this case $C(X)$ is a finite direct product of F, we may suppose that in our discussion

X is an infinite space. Before proving our results, we need two easy (but useful) lemmas. Comparing these two lemmas with [2, Lemmas 1 and 2], one may observe that these two will improve and generalize [2, Lemmas 1 and 2]. In fact they give us enough space to use the full power of Mcknight's Theorem.

Lemma 1. *Let X be a completely regular space; then the following hold.*

(1) *Suppose Y is a subset of X; then Y is a dense subset of X if and only if $f \in C(X)$; $f|_Y = 0$ implies that $f \equiv 0$.*

(2) *Suppose Y is a subset of X; then Y is a dense subset of X if and only if $f \in C(X)$; $Y \subset \mathrm{int}_X Z(f)$ implies that $f \equiv 0$.*

Proof. Part 1. ($\Leftarrow$): let $x \in X$ and U_x an open set containing x. We must show that $U_x \cap Y \neq \emptyset$, and suppose on the contrary that $U_x \cap Y = \emptyset$; then $Y \subseteq X \setminus U_x$; by complete regularity of X, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(Y) = 0$, and this is a contradiction to our hypothesis.

($\Rightarrow$): since $\mathbb{R}$ is a T_1-space and $\{0\}$ is closed in $\mathbb{R}$, hence $f^{-1}\{0\}$ is closed in X. Since $Y \subseteq f^{-1}\{0\}$, we conclude that $X = \overline{Y} \subseteq f^{-1}\{0\}$. Hence $f \equiv 0$.

Part 2. It is enough to show the necessary part: suppose that $\overline{Y} \neq X$; hence there exists $z \in X \setminus \overline{Y}$ and a function $f : X \to [0,1]$ such that $f(z) = 1$ and $f(Y) = \{0\}$. In as much as Y and z are contained in disjoint zero sets, there exists a function g such that $\overline{Y} \subseteq \mathrm{int}\ Z(g)$ and $g(z) \neq 0$ (see [1, 1.15]). This is a contradiction. □

Lemma 2. *For $A \subseteq \beta X$, $O^A = (0)$ if and only if $M^A = (0)$.*

Proof. Suppose that $O^A = (0)$; then by the previous lemma $\overline{A} = \beta X$. Now suppose that $f \in M^A$. There is a positive unit u in $C(X)$ such that $\overline{f} = uf$ and $-1 \leq \overline{f} \leq 1$ (see [1, 1E.1]). Since $A \subseteq \mathrm{cl}_{\beta X}\ Z(f) \subseteq Z(\overline{f}^{\beta})$ and $\overline{A} = \beta X$, we have $\overline{f}^{\beta} = 0$. Hence $\overline{f} \equiv 0$ and this implies that $f \equiv 0$. □

In the next theorem, which is the main result of this note, we have generalized [2, Theorem 3] by removing the compactness hypothesis and also replacing $\aleph_0$ with an arbitrary (regular) cardinal. Since βX, by its very definition, is compact and $\beta X = X$ whenever X is compact, the earlier form of our result is just a special case of the new one. On the other hand since βX always exists, we can (always, i.e., for an arbitrary completely regular space X) judge when the ring $C(X)$ is α-cogenerating and also α-separating by looking at βX.

Remark 3. Before stating our main theorem, we need some useful facts. Let X be a completely space and $A \subseteq \beta X$. Suppose that $O^A = \{f \in C(X) \mid A \subseteq \mathrm{int}_{\beta X}\ \mathrm{cl}_{\beta X}\ Z(f)\}$ and $M^A = \{f \in C(X) \mid A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$. Lemma 1 (Lemma 2, resp.) shows if $M^A = (0)$ ($O^A = (0)$, resp.), then A is dense in X and vice versa. Dietrich Jr. in [7] has shown that for every ideal I of $C(X)$, there exists $A \subseteq \beta X$ such that $O^A \subseteq I \subseteq M^A$. By Mcknight Theorem [7, Theorem 1.3], the set A is $\bigcap_{f \in I}\ \mathrm{cl}_{\beta X} Z(f)$. Dietrich Jr. has also proved that [7, Lemma 1.6] $\bigcap Z[O^A] = \bigcap Z[M^A] = A$ and if $O^A \subseteq M^B$, then $\overline{B} \subseteq \overline{A}$.

Theorem 4. *Let X be an infinite completely regular space. The following are equivalent:*

(1) *βX is dense-α-separable;*

(2) *$C(X)$ is α-cogenerated;*

(3) *$C(X)$ is α-separable.*

Proof. (1)$\Rightarrow$(2): let $\bigcap_{j \in J} I_j = (0)$. For each $j \in J$, there exists $A_j \subseteq \beta X$ such that $O^{A_j} \subseteq I_j \subseteq M^{A_j}$. By the previous observations from [7] we have

$$O^{\cup A_j} = \bigcap_{j \in J} O^{A_j} \subseteq \bigcap_{j \in J} I_j \subseteq \bigcap_{j \in J} M^{A_j} = M^{\cup A_j}, \tag{1}$$

but $\bigcap_{j \in J} I_j = (0)$; therefore $O^{\cup_{j \in J} A_j} = \bigcap_{j \in J} O^{A_j} = (0)$, and hence by Lemma 1, $\bigcup_{j \in J} A_j$ is dense in βX. Since βX is a dense-α-separable space, there exists an α-subset B of $\bigcup_{j \in J} A_j$, such that $\overline{B} = X$. Hence there exists an α-set $J_0 \subset J$ such that $\bigcup_{j \in J_0} A_j$ is dense in βX; that is, $O^{\bigcup_{j \in J_0} A_j} = (0)$. Now by Lemma 2, $\bigcap_{j \in J_0} M^{A_j} = (0)$, and this latter observation in its turn shows that $\bigcap_{j \in J_0} I_j = (0)$.

(2)$\Rightarrow$(3): it is evident.

(3)$\Rightarrow$(1): let D be a dense subset of βX. Then by Lemma 1, $\bigcap_{t \in D} O^t = O^D = (0)$. Now by Lemma 2, $M^D = (0)$. Since $C(X)$ is α-separable, there is an α-subset A of D such that $M^A = (0) = O^A$, and again by Lemma 1, this shows that A is dense in βX and hence dense in D. □

Observe that when X is finite, $C(X)$ is artinian and hence $\aleph_0$-cogenerated. If $C(X)$ is separable, then X is dense-separable. A ring R is called von-Neumann regular if for every $a \in R$, there exists $b \in R$ such that $aba = a$. I. Kaplansky has shown that every ideal in a commutative von-Neumann regular ring can be written as the intersection of some family of maximal ideals. Hence, a commutative von-Neumann regular ring is α-separable if and only if it is α-cogenerated. This implies that for any p-space X, $C(X)$ is α-separable if and only if $C(X)$ is α-cogenerated. A ring is called right V-ring (after Villamayor) if every right simple R-module is injective. It is well known that over right V-rings, every submodule of a right R-module M can be written as the intersection of a family of maximal submodules of M. Hence over a right V-ring, a right R-module is α-cogenerated if and only if it is α-separable. Let $\{J_i\}_{i \in I}$ be a family of ideals of R; we have $\bigcap_{i \in I} M_n(J_i) = M_n(\bigcap_{i \in I} J_i)$. Hence as far as one is concerned with two sided ideals of $M_n(R)$, one obtains that $M_n(R)$ is α-separable (α-cogenerated, resp.) when R is α-separable (α-cogenerated, resp.).

Corollary 5. *The following are equivalent:*

(1) *βX is dense-α-separable;*

(2) *$C(\beta X)$ is α-cogenerated;*

(3) *$C(\beta X)$ is α-separable;*

(4) $C(X)$ *is* α*-cogenerated;*

(5) $C(X)$ *is* α*-separable;*

(6) $C^*(X)$ *is* α*-cogenerated;*

(7) $C^*(X)$ *is* α*-separable.*

Proof. It is well known that $C(\beta X) \cong C^*(X)$, and by Theorem 4 the verification is immediate. □

Corollary 6. *If X is either (1) separable metric or (2) separable and ordered, then $C(X)$ is $\aleph_1$-cogenerated.*

Proof. By [5, Corollary 3.2.], for these two cases βX is dense-separable. Now by Theorem 3 the proof is thorough. □

When $C(X)$ is separable, then X is also separable. However, the converse is not true. In [5, example 5.3], a separable compact space Y has been introduced which is not dense-separable; for the space Y, $C(Y)$ is not separable. Otherwise $Y = \beta Y$ should be dense-separable which is not the case. Based on these observations we have the following.

Example 7. There exists a separable space Y, such that $C(Y)$ is not separable.

However, the converse is true when we have a much stronger property as we observe in the next proposition.

Proposition 8. *Let R be a commutative ring. Then R is α-separable if and only if $X = \mathrm{Max}(R)$ is dense-α-separable.*

Proof. Let X be dense-α-separable and $\mathscr{A} = \{M_i\}_{i\in I}$ a family of maximal ideals with zero intersection. This family then will be dense in X. By dense-α-separability of X, $\mathscr{A}$ has an α-subset with zero intersection. Let R be an α-separable ring and $\mathscr{A}$ a dense subspace of X. By definition $\mathscr{A}$ has a zero intersection and by α-separability of R, it has an α-subspace which is dense in X. □

According to Gelfand-Naimark's theorem every commutative C^*-algebra with identity is isomorphic to $C(X, \mathbb{C})$, where X is a suitable compact Hausdorff space. Based on Theorem 3 and Gelfand-Naimark's theorem, we have the following.

Corollary 9. *Let A be a commutative C^*-algebra with identity and α an arbitrary regular cardinal. Then the following are equivalent:*

(1) *A is α-separable;*

(2) *A is α-cogenerated.*

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