Conference Paper

Stability and Error Analysis of the Semidiscretized Fractional Nonlocal Thermistor Problem

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A finite difference scheme is proposed for temporal discretization of the nonlocal time-fractional thermistor problem. Stability and error analysis of the proposed scheme are provided.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$ and let $Q_T = \Omega \times (0, T)$. In this work, we propose a finite difference scheme for the following nonlocal time-fractional thermistor problem:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} - \Delta u &= \frac{\lambda f(u)}{\left(\int_\Omega f(u) dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } S_T = \partial\Omega \times (0, T), \\ u(0) &= u_0, \quad \text{in } \Omega, \end{aligned} \quad (1)$$

where $\partial^\alpha u(x, t)/\partial t^\alpha$ denotes the Caputo fractional derivative of order α ($0 < \alpha < 1$), Δ is the Laplacian with respect to the spacial variables, f is supposed to be a smooth function prescribed next, and T is a fixed positive real. Here ν denotes the outward unit normal and $\partial/\partial \nu = \nu \cdot \nabla$ is the normal derivative on $\partial\Omega$. Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity f is strongly dependent on the temperature u . When $\alpha = 1$, (1) describes the diffusion of the

temperature with the presence of a nonlocal term. Constant λ is a dimensionless parameter, which can be identified with the square of the applied potential difference at the ends of the conductor. Function β is the positive thermal transfer coefficient. The given value u_0 is the temperature outside Ω . For the sake of simplicity, boundary conditions are chosen of homogeneous Neumann type. Mixed or more general boundary conditions which model the coupling of the thermistor to its surroundings appear naturally. $u(x)$ is the temperature inside the conductor, and $f(u)$ is the temperature dependent electrical conductivity. Recall that (1) is obtained from the so-called nonlocal thermistor problem by replacing the first-order time derivative with a fractional derivative of order α ($0 < \alpha < 1$). For more description about the history of thermistors and more detailed accounts of their advantages and applications in industry, refer to [1–4].

In recent years, it has been turned out that fractional differential equations can be used successfully to model many phenomena in various fields as fluids mechanics, viscoelasticity, chemistry, and engineering [5–8]. In [4], existence and uniqueness of a positive solution to a generalized nonlocal thermistor problem with fractional-order derivatives were proved. In this work, a finite difference method is proposed

for solving the time-fractional nonlocal thermistor system. Stability and error analysis for this scheme are presented showing that the temporal accuracy is of $2 - \alpha$ order.

2. Formulation and Statement of the Problem

We consider the time-fractional thermistor problem (1), which is obtained from

$$\frac{\partial u(x, t)}{\partial t} - \Delta u = \frac{\lambda f(u)}{\left(\int_{\Omega} f(u) dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0, T), \quad (2)$$

by replacing the first-order time derivative with a fractional derivative on Caputo sense as defined in [9] and given by

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \quad (3)$$

subject to the initial and homogenous boundary conditions and where α ($0 < \alpha < 1$) is the order of the time-fractional derivative. (1) covers (2) and extends it to general cases. The classical nonlocal thermistor problem (2) with the time derivative of integer order can be obtained by taking the limit $\alpha \rightarrow 1$ in (1). While the case $\alpha = 0$ corresponds to the steady state thermistor problem, in the case $0 < \alpha < 1$, the Caputo fractional derivative depends on and uses the information of the solutions at all previous time levels (non-Markovian process). In this case, the physical interpretation of fractional derivative is that it represents a degree of memory in the diffusing material [10].

In the analysis of the numerical method, we will assume that problem (1) has a unique and sufficiently smooth solution which can be established by assuming more hypotheses and regularity on the data (see [11]). In the sequel, we will assume the following assumptions:

(H1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a positive Lipschitzian continuous function;

(H2) there exist positive constants c and α such that for all $\xi \in \mathbb{R}$ we have

$$c \leq f(\xi) \leq c|\xi|^{\alpha+1} + c, \quad (4)$$

(H3) $u_0 \in L^\infty(\Omega)$.

It can be shown (e.g., see [12, 13]) that the quantity

$$\|v\|_* = \left(\|v\|_2^2 + \alpha_0 \left\| \frac{dv}{dx} \right\|_2^2 \right)^{1/2}, \quad (5)$$

where α_0 is given next, defines a norm on $H^1(\Omega)$ which is equivalent to the $\|\cdot\|_{H^1(\Omega)}$ norm.

3. Time Discretization:

A Finite Difference Scheme

We introduce a finite difference approximation to discretize the time-fractional derivative. Let $\delta = T/N$ be the length of

each time step, for some large N . $t_k = k\delta$, $k = 0, 1, \dots, N$. We use the following formulation: for all $0 \leq k \leq N-1$,

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x, t_{j+1}) - u(x, t_j)}{\delta} \\ &\quad \times \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + r_\delta^{k+1}, \end{aligned} \quad (6)$$

where r_δ^{k+1} is the truncation error. It can be seen from [14] that the truncation error verifies

$$r_\delta^{k+1} \lesssim c_u \delta^{2-\alpha}, \quad (7)$$

where c_u is a constant depending only on u . On the other hand, by change of variables, we have

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x, t_{j+1}) - u(x, t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x, t_{j+1}) - u(x, t_j)}{\delta} \int_{t_{k-j}}^{t_{k+1-j}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x, t_{k+1-j}) - u(x, t_{k-j})}{\delta} \int_{t_j}^{t_{j+1}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^k \frac{u(x, t_{k+1-j}) - u(x, t_{k-j})}{\delta^\alpha} \\ &\quad \times \{(j+1)^{1-\alpha} - (j)^{1-\alpha}\}. \end{aligned} \quad (8)$$

Let us denote $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, \dots, k$, and define the discrete fractional differential operator L_t^α by

$$L_t^\alpha u(x, t_{k+1}) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^k b_j \frac{u(x, t_{k+1-j}) - u(x, t_{k-j})}{\delta^\alpha}. \quad (9)$$

Then (6) becomes

$$\frac{\partial^\alpha u(x, t_{k+1})}{\partial t^\alpha} = L_t^\alpha u(x, t_{k+1}) + r_\delta^{k+1}. \quad (10)$$

Using this approximation, it yields the following finite difference scheme to (1): for $k = 1, \dots, N-1$,

$$L_t^\alpha u^{k+1}(x) - \Delta u^{k+1} = \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \quad \text{in } \Omega, \quad (11)$$

where $u^{k+1}(x)$ are approximations to $u(x, t_{k+1})$. Scheme (11) can be reformulated in the form

$$\begin{aligned}
 & b_0 u^{k+1} - \Gamma(2-\alpha) \delta^\alpha \Delta u^{k+1} \\
 &= b_0 u^k - \sum_{j=1}^k b_j \{u^{k+1-j} - u^{k-j}\} \\
 &\quad + \Gamma(2-\alpha) \delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \\
 &= b_0 u^k - \sum_{j=0}^{k-1} b_{j+1} u^{k-j} + \sum_{j=1}^k b_j u^{k-j} \\
 &\quad + \Gamma(2-\alpha) \delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \\
 &= b_0 u^k + \sum_{j=0}^{k-1} (b_j - b_{j+1}) u^{k-j} \\
 &\quad + \Gamma(2-\alpha) \delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2}.
 \end{aligned} \tag{12}$$

To complete the semidiscrete problem, we consider the boundary conditions

$$\frac{\partial u^{k+1}}{\partial n} = 0, \tag{13}$$

and the initial condition $u^0 = u_0$, noting that

$$b_j > 0, \quad j = 0, 1, \dots, k,$$

$$1 = b_0 > b_1 > \dots > b_k, \quad b_k \longrightarrow 0 \text{ as } k \longrightarrow \infty,$$

$$\sum_{j=1}^k (b_j - b_{j+1}) + b_{k+1} = (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k = 1. \tag{14}$$

If we set

$$\alpha_0 = \Gamma(2-\alpha) \delta^\alpha, \tag{15}$$

then (12) can be rewritten into

$$\begin{aligned}
 u^{k+1} - \alpha_0 \Delta u^{k+1} &= (1 - b_1) u^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) u^{k-j} \\
 &\quad + b_k u^0 + \alpha_0 \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2},
 \end{aligned} \tag{16}$$

for all $k \geq 1$. When $k = 0$, scheme (12) reads

$$u^1 - \alpha_0 \Delta u^1 = u^0 + \alpha_0 \frac{\lambda f(u^1)}{\left(\int_{\Omega} f(u^1) dx\right)^2}. \tag{17}$$

When $k = 1$, scheme (12) becomes

$$u^2 - \alpha_0 \Delta u^2 = (1 - b_1) u^1 + b_1 u^0 + \alpha_0 \frac{\lambda f(u^2)}{\left(\int_{\Omega} f(u^2) dx\right)^2}. \tag{18}$$

We define the error term r^{k+1} by

$$r^{k+1} = \alpha_0 \left\{ \frac{\partial^\alpha u(x, t_{k+1})}{\partial t^\alpha} - L_t^\alpha u(x, t_{k+1}) \right\}. \tag{19}$$

Then we get from (7) that

$$|r^{k+1}| = \Gamma(2-\alpha) \delta^\alpha |r_\delta^{k+1}| \leq c_u \delta^2. \tag{20}$$

3.1. Existence of the Semidiscrete Scheme

Definition 1. We say that u^{k+1} is a weak solution of (11) if

$$\begin{aligned}
 \langle u^{k+1}, v \rangle + \alpha_0 \int_{\Omega} \nabla u^{k+1} \nabla v dx \\
 = (f^k, v) + \frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2},
 \end{aligned} \tag{21}$$

where $f^k = (1 - b_1) u^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) u^{k-j} + b_k u^0$.

At each time step, we solve a discretized fractional thermistor problem.

Theorem 2. Let hypotheses (H1)–(H3) be satisfied; then there exists at least a weak solution u^k of (12), such that

$$u^k \in H^1(\Omega). \tag{22}$$

Existence and uniqueness results follow from general results of elliptic problems [3, 4, 13]. From now on, we denote by c a generic constant which may not be the same at different occurrences.

3.2. A Priori Estimates. We search a priori estimates for solutions.

Lemma 3. There exists a positive constant c independent of k , such that

$$\|u^{k+1}\|_{H^1(\Omega)} \leq c. \tag{23}$$

Proof. We prove this result by recurrence. First, when $k = 0$, we have, for $v \in H_0^1(\Omega)$,

$$\begin{aligned}
 \int_{\Omega} u^1 v dx + \alpha_0 \int_{\Omega} \nabla u^1 \nabla v dx \\
 = \int_{\Omega} u^0 v dx + \frac{\lambda \alpha_0}{\left(\int_{\Omega} f(u^1) dx\right)^2} (f(u^1), v).
 \end{aligned} \tag{24}$$

Notice that $u^0 \in L^\infty(\Omega) \subset L^2(\Omega)$. Taking $v = u^1$, we have

$$\begin{aligned} \|u^1\|_2^2 + \alpha_0 \|\nabla u^1\|_2^2 &= \int_{\Omega} u^0 u^1 dx + \frac{\lambda \alpha_0 \int_{\Omega} f(u^1) u^1 dx}{\left(\int_{\Omega} f(u^1) dx\right)^2} \\ &\leq \|u^1\|_2 \|u^0\|_2 + c \alpha_0 \int_{\Omega} f(u^1) u^1 dx \quad (25) \\ &\leq c \|u^1\|_2 + c \alpha_0 \int_{\Omega} f(u^1) u^1 dx, \end{aligned}$$

or

$$\begin{aligned} \int_{\Omega} f(u^1) u^1 dx &\leq c \int_{\Omega} (|u|^{\alpha+1} + c) |u^1| dx \\ &\leq c \|u^1\|_1 \leq c \|u^1\|_2 \leq c \|u^1\|_{H^1(\Omega)} \quad (26) \\ &\leq \frac{1}{2} \|u^1\|_{H^1(\Omega)}^2 + c. \end{aligned}$$

Then

$$\|u^1\|_2^2 + \alpha_0 \|\nabla u^1\|_2^2 \leq c. \quad (27)$$

Hence, since the standard H^1 -norm and the norm $\|\cdot\|_*$ defined by (5) are equivalent, we have

$$\|u^1\|_{H^1(\Omega)} \leq c. \quad (28)$$

Suppose now that we have

$$\|u^j\|_{H^1(\Omega)} \leq c, \quad j = 1, 2, \dots, k, \quad (29)$$

and prove that $\|u^{k+1}\|_{H^1(\Omega)} \leq c$. Multiplying (16) by $v = u^{k+1}$ and using the fact that $f^k \in H^1(\Omega)$, we obtain

$$\begin{aligned} \|u^{k+1}\|_2^2 + \alpha_0 \|\nabla u^{k+1}\|_2^2 &= \int_{\Omega} f^k u^{k+1} dx + \frac{\lambda \alpha_0 \int_{\Omega} f(u^{k+1}) u^{k+1} dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \\ &\leq \|f^k\|_2 \|u^{k+1}\|_2 + \frac{\lambda \alpha_0 \int_{\Omega} f(u^{k+1}) u^{k+1} dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \quad (30) \\ &\leq c \|u^{k+1}\|_2 + \frac{\lambda \alpha_0 \int_{\Omega} f(u^{k+1}) u^{k+1} dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2}. \end{aligned}$$

Following the same as for the case $j = 1$ with respect to the nonlocal term $\lambda \alpha_0 \int_{\Omega} f(u^{k+1}) u^{k+1} dx / (\int_{\Omega} f(u^{k+1}) dx)^2$, we then have

$$\|u^{k+1}\|_2^2 + \alpha_0 \|\nabla u^{k+1}\|_2^2 \leq c. \quad (31)$$

Hence,

$$\|u^{k+1}\|_{H^1(\Omega)} \leq c. \quad (32)$$

□

4. Stability and Error Analysis

4.1. Stability Result. The weak formulation of (16) is for all $k \geq 1$ and $v \in H^1(\Omega)$:

$$\begin{aligned} (u^{k+1}, v) + \alpha_0 (-\Delta u^{k+1}, v) &= (1 - b_1)(u^k, v) \\ &\quad + \sum_{j=1}^{k-1} (b_j - b_{j+1})(u^{k-j}, v) \\ &\quad + b_k(u^0, v) \\ &\quad + \lambda \alpha_0 \left(\frac{f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2}, v \right). \end{aligned} \quad (33)$$

We have the following unconditional stability result.

Theorem 4. *The semidiscretized problem is stable in the sense that for all $\delta > 0$ it holds*

$$\|u^{k+1}\|_{H^1(\Omega)} \leq \|u^0\|_2 + c. \quad (34)$$

Proof. We prove this result by recurrence. First, when $k = 0$, we have, for $v \in H^1(\Omega)$,

$$(u^1, v) + \alpha_0 (-\Delta u^1, v) = (u^0, v) + \alpha_0 \frac{\lambda (f(u^1), v)}{\left(\int_{\Omega} f(u^1) dx\right)^2}. \quad (35)$$

On other terms

$$\begin{aligned} \int_{\Omega} u^1 v dx + \alpha_0 \int_{\Omega} \nabla u^1 \nabla v dx &= \int_{\Omega} u^0 v dx + \alpha_0 \frac{\lambda \int_{\Omega} f(u^{k+1}) v dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2}. \end{aligned} \quad (36)$$

Taking $v = u^1$ in (36), we have

$$\begin{aligned} \int_{\Omega} |u^1|^2 dx + \alpha_0 \int_{\Omega} |\nabla u^1|^2 dx &= \int_{\Omega} u^0 u^1 dx + \alpha_0 \frac{\lambda \int_{\Omega} f(u^{k+1}) u^1 dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2}. \end{aligned} \quad (37)$$

In a similar way, we have

$$\alpha_0 \frac{\lambda \int_{\Omega} f(u^{k+1}) u^1 dx}{\left(\int_{\Omega} f(u^{k+1}) dx\right)^2} \leq c \|u^{k+1}\|_{H^1(\Omega)}. \quad (38)$$

We also have

$$\begin{aligned} \int_{\Omega} u^0 u^1 dx &\leq \|u^0\|_2 \|u^1\|_2 \leq \|u^0\|_2 \|u^1\|_2 \\ &\leq \|u^0\|_2 \|u^1\|_{H^1(\Omega)}. \end{aligned} \quad (39)$$

We then obtain by (5) and (36) that

$$\|u^1\|_{H^1(\Omega)}^2 \leq (\|u^0\|_2 + c) \|u^1\|_{H^1(\Omega)}. \quad (40)$$

Dividing both sides of the previous inequality (40) by $\|u^1\|_{H^1(\Omega)}$, we get

$$\|u^1\|_{H^1(\Omega)} \leq \|u^0\|_2 + c. \quad (41)$$

Suppose now that we have

$$\|u^j\|_{H^1(\Omega)} \leq \|u^0\|_2 + c, \quad j = 1, 2, \dots, k, \quad (42)$$

and prove that $\|u^{k+1}\|_{H^1(\Omega)} \leq \|u^0\|_2 + c$. Choosing $v = u^{k+1}$ in (33), we obtain

$$\begin{aligned} & (u^{k+1}, u^{k+1}) + \alpha_0 (-\Delta u^{k+1}, u^{k+1}) \\ &= (1 - b_1) (u^k, u^{k+1}) \\ &+ \sum_{j=1}^{k-1} (b_j - b_{j+1}) (u^{k-j}, u^{k+1}) + b_k (u^0, u^{k+1}) \\ &+ \alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right). \end{aligned} \quad (43)$$

Then using the recurrence hypothesis (42), we obtain

$$\begin{aligned} \|u^{k+1}\|_{H^1(\Omega)}^2 &\leq (1 - b_1) \|u^k\|_2 \|u^{k+1}\|_2 \\ &+ \sum_{j=1}^{k-1} (b_j - b_{j+1}) \|u^{k-j}\|_2 \|u^{k+1}\|_2 \\ &+ b_k \|u^0\|_2 \|u^{k+1}\|_2 \\ &+ \alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right) \\ &\leq \left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} \\ &\times (\|u^0\|_2 + c) \|u^{k+1}\|_2 \\ &+ \alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right) \end{aligned}$$

$$\begin{aligned} &\leq \left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} \\ &\times (\|u^0\|_2 + c) \|u^{k+1}\|_{H^1(\Omega)} \\ &+ \alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right) \\ &\leq (\|u^0\|_2 + c) \|u^{k+1}\|_{H^1(\Omega)} \\ &+ \alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right), \end{aligned} \quad (44)$$

since $(1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k = 1$. Similarly to the case $k = 0$, we have

$$\alpha_0 \left(\frac{\lambda f(u^{k+1})}{(\int_{\Omega} f(u^{k+1}) dx)^2}, u^{k+1} \right) \leq c \|u^{k+1}\|_{H^1(\Omega)}. \quad (45)$$

Then

$$\|u^{k+1}\|_{H^1(\Omega)} \leq \|u^0\|_2 + c. \quad (46)$$

We have the following error analysis for the solution of the semidiscretized problem. $\square$

Theorem 5. Let u be the exact solution of (1) and let $(u^j)_j$ be the time-discrete solution with the initial condition $u^0(x) = u(x, 0)$. Then one has the following error estimates:

(a)

$$\|u(t_j) - u^j\|_{H^1(\Omega)} \leq c_{u,\alpha} T^\alpha \delta^{2-\alpha}, \quad j = 1, \dots, N, \quad (47)$$

where $0 \leq \alpha < 1$ and $c_{u,\alpha} = c_u / (1 - \alpha)$; c_u is a constant depending on u .

(b) when $\alpha \rightarrow 1$,

$$\|u(t_j) - u^j\|_{H^1(\Omega)} \leq c_u T \delta, \quad j = 1, \dots, N. \quad (48)$$

Proof. Let $e^k = u(x, t_k) - u^k(x)$ the difference between the exact solution of (1) and u^k the solution of the time-discrete problem. Obviously $e^0 = 0$.

(a) We will prove the result by induction. We begin with the first case when $0 \leq \alpha < 1$. For $j = 1$, by gathering

equations corresponding to exact and discrete solutions, the error equation reads

$$\begin{aligned}
 & (e^1, v) + \alpha_0 \int_{\Omega} \nabla e^1 \nabla v \, dx \\
 &= (e^0, v) + (r^1, v) + \alpha_0 \left(\frac{\lambda f(u(x, t_2))}{\left(\int_{\Omega} f(u(x, t_2)) \, dx\right)^2}, v \right) \\
 &\quad - \alpha_0 \left(\frac{\lambda f(u^2)}{\left(\int_{\Omega} f(u^2) \, dx\right)^2}, v \right) \\
 &= (r^1, v) + \alpha_0 \left(\frac{\lambda f(u(x, t_2))}{\left(\int_{\Omega} f(u(x, t_2)) \, dx\right)^2}, v \right) \\
 &\quad - \alpha_0 \left(\frac{\lambda f(u^2)}{\left(\int_{\Omega} f(u^2) \, dx\right)^2}, v \right).
 \end{aligned} \tag{49}$$

Choosing $v = e^1$ in the previous equation, it yields that

$$\begin{aligned}
 & \|e^1\|_2^2 + \alpha_0 \|\nabla e^1\|_2^2 \\
 &\leq \|r^1\|_2 \|e^1\|_2 + \alpha_0 \left(\frac{\lambda f(u(x, t_2))}{\left(\int_{\Omega} f(u(x, t_2)) \, dx\right)^2}, e^1 \right) \\
 &\quad - \alpha_0 \left(\frac{\lambda f(u^2)}{\left(\int_{\Omega} f(u^2) \, dx\right)^2}, e^1 \right).
 \end{aligned} \tag{50}$$

To continue the proof, we will need the following lemma which is used in the sequel. $\square$

Lemma 6. Let u_i , $i = 1, 2$, be two weak solutions of (1). Assume that (H1)–(H3) hold. Then one has

$$\left(\frac{\lambda f(u_1)}{\left(\int_{\Omega} f(u_1) \, dx\right)^2}, w \right) - \left(\frac{\lambda f(u_2)}{\left(\int_{\Omega} f(u_2) \, dx\right)^2}, w \right) \leq c \|w\|_2^2, \tag{51}$$

where $w = u_1 - u_2$ and ε , c , and c_ε are positive constants.

Proof. We have

$$\begin{aligned}
 & \frac{f(u_1)}{\left(\int_{\Omega} f(u_1) \, dx\right)^2} - \frac{f(u_2)}{\left(\int_{\Omega} f(u_2) \, dx\right)^2} \\
 &= \frac{1}{\left(\int_{\Omega} f(u_1) \, dx\right)^2} (f(u_1) - f(u_2)) \\
 &\quad + \left(\frac{1}{\left(\int_{\Omega} f(u_1) \, dx\right)^2} - \frac{1}{\left(\int_{\Omega} f(u_2) \, dx\right)^2} \right) f(u_2).
 \end{aligned} \tag{52}$$

If we multiply by w and integrate over Ω , we get

$$\begin{aligned}
 & \left(\frac{f(u_1)}{\left(\int_{\Omega} f(u_1) \, dx\right)^2} - \frac{f(u_2)}{\left(\int_{\Omega} f(u_2) \, dx\right)^2}, w \right) \\
 &\leq c \|w\|_2^2 + \frac{\int_{\Omega} (f(u_2) - f(u_1)) \, dx}{\left(\int_{\Omega} f(u_1) \, dx\right)^2} \frac{\int_{\Omega} (f(u_2) - f(u_1)) \, dx}{\left(\int_{\Omega} f(u_2) \, dx\right)^2} \\
 &\quad \times (f(u_2), w) \\
 &\leq c \|w\|_2^2 + c \|w\|_2 \|w\|_1 \\
 &\leq c \|w\|_2^2.
 \end{aligned} \tag{53}$$

The proof of Lemma 6 is now completed.

Now, we continue the proof of Theorem 5. Using (50), it follows that

$$\begin{aligned}
 & \|e^1\|_2^2 + \alpha_0 \|\nabla e^1\|_2^2 \leq \|r^1\|_2 \|e^1\|_2 + c \|e^1\|_2^2 \\
 &\leq (c + \varepsilon) \|e^1\|_2^2 + c_\varepsilon \|r^1\|_2^2.
 \end{aligned} \tag{54}$$

Then, by (5), we have

$$\|e^1\|_{H^1(\Omega)}^2 \leq (c + \varepsilon) \|e^1\|_2^2 + c_\varepsilon \|r^1\|_2^2. \tag{55}$$

It follows that

$$(1 - (c + \varepsilon)) \|e^1\|_{H^1(\Omega)}^2 \leq c_\varepsilon \|r^1\|_2^2. \tag{56}$$

For a good choice of ε and using (20) and $b_0 = 1$, we obtain

$$\|u(t_1) - u^1\|_1 \leq c_u b_0^{-1} \delta^2. \tag{57}$$

Then point (a) is verified for $j = 1$. Suppose now that we have proven (a) for all $k = 1, \dots, j$, and prove it also for $k = j + 1$. We have

$$\begin{aligned}
 & (e^{k+1}, v) + \alpha_0 (-\Delta e^{k+1}, v) \\
 &= (1 - b_1) (e^k, v) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) (e^{k-j}, v) \\
 &\quad + b_k (e^0, v) + (r^{k+1}, v) \\
 &\quad + \alpha_0 \left(\frac{\lambda f(u(x, t_{k+1}))}{\left(\int_{\Omega} f(u(x, t_{k+1})) \, dx\right)^2}, v \right) \\
 &\quad - \alpha_0 \left(\frac{\lambda f(u^{k+1})}{\left(\int_{\Omega} f(u^{k+1}) \, dx\right)^2}, v \right).
 \end{aligned} \tag{58}$$

Taking $v = e^{k+1}$ in (58) and using Lemma 6, we then have

$$\begin{aligned} \|e^{k+1}\|_2^2 + \alpha_0 \|\nabla e^{k+1}\|_2^2 &\leq (1 - b_1) \|e^k\|_2 \|e^{k+1}\|_2 \\ &\quad + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \|e^{k-j}\|_2 \|e^{k+1}\|_2 \\ &\quad + b_k \|e^0\|_2 \|e^{k+1}\|_2 \\ &\quad + \|r^{k+1}\|_2 \|e^{k+1}\|_2 + c \|e^{k+1}\|_{H^1(\Omega)}^2. \end{aligned} \quad (59)$$

Using the induction assumption and the fact that $b_k^{-1}/b_{k+1}^{-1} < 1$ for a positive integer k , we have

$$\begin{aligned} \|e^{k+1}\|_2^2 + \alpha_0 \|\nabla e^{k+1}\|_2^2 &\leq \left\{ (1 - b_1) b_{k-1}^{-1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) b_{k-j-1}^{-1} \right\} \\ &\quad \times c_u \delta^2 \|e^{k+1}\|_2 + c \|e^{k+1}\|_{H^1(\Omega)}^2 \\ &\leq \left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} \\ &\quad \times c_u b_{k-1}^{-1} \delta^2 \|e^{k+1}\|_{H^1(\Omega)} + c \|e^{k+1}\|_{H^1(\Omega)}^2. \end{aligned} \quad (60)$$

We then have

$$\begin{aligned} \|e^{k+1}\|_2^2 + \alpha_0 \|\nabla e^{k+1}\|_2^2 &\leq c_u b_{k-1}^{-1} \delta^2 \|e^{k+1}\|_{H^1(\Omega)} \\ &\quad + c \|e^{k+1}\|_{H^1(\Omega)}^2, \end{aligned} \quad (61)$$

since $(1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k = 1$. Then

$$\|e^{k+1}\|_{H^1(\Omega)}^2 \leq c_u b_{k-1}^{-1} \delta^2 \|e^{k+1}\|_{H^1(\Omega)} + c \|e^{k+1}\|_{H^1(\Omega)}^2. \quad (62)$$

By using Young's inequality, we get

$$\|e^{k+1}\|_{H^1(\Omega)}^2 \leq (c + \varepsilon) \|e^{k+1}\|_{H^1(\Omega)}^2 + (c_\varepsilon c_u b_{k-1}^{-1} \delta^2)^2. \quad (63)$$

Hence,

$$(1 - (c + \varepsilon)) \|e^{k+1}\|_{H^1(\Omega)}^2 \leq (c_\varepsilon c_u b_{k-1}^{-1} \delta^2)^2. \quad (64)$$

For a suitable choice of ε and dividing both sides by $\|e^{k+1}\|_{H^1(\Omega)}$, we get

$$\|e^{k+1}\|_{H^1(\Omega)} \leq c_u b_{k-1}^{-1} \delta^2. \quad (65)$$

One can show easily that

$$k^{-\alpha} b_{k-1}^{-1} \leq \frac{1}{1 - \alpha}, \quad k = 1, \dots, N. \quad (66)$$

Hence, we have, for all k , such that $k\delta \leq T$,

$$\begin{aligned} \|u(t_k) - u^k\|_{H^1(\Omega)} &\leq c_u b_{k-1}^{-1} \delta^2 = c_u k^{-\alpha} b_{k-1}^{-1} k^\alpha \delta^2 \\ &\leq c_u \frac{1}{1 - \alpha} (k\delta)^\alpha \delta^{2-\alpha} \\ &\leq \frac{c_u}{1 - \alpha} T^\alpha \delta^{2-\alpha}. \end{aligned} \quad (67)$$

(b) We are now interested in the case $\alpha \rightarrow 1$. We will derive again the following estimation by induction:

$$\|u(t_j) - u^j\|_1 \leq c_u j \delta^2, \quad j = 1, 2, \dots, N. \quad (68)$$

The previous inequality is obvious for $j = 1$. Suppose now that (68) holds for all $j = 1, 2, \dots, k$, and we need to prove that it holds also for $j = k + 1$. Similarly to the previous case, by combining the corresponding equations of the exact and discrete solutions and taking $v = e^{k+1}$ as a test function, it yields that

$$\begin{aligned} \|e^{k+1}\|_2^2 + \alpha_0 \|\nabla e^{k+1}\|_2^2 &\leq (1 - b_1) \|e^k\|_2 \|e^{k+1}\|_2 \\ &\quad + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \|e^{k-j}\|_2 \|e^{k+1}\|_2 \\ &\quad + b_k \|e^0\|_2 \|e^{k+1}\|_2 + \|r^{k+1}\|_2 \|e^{k+1}\|_2 \\ &\quad + c \|e^{k+1}\|_{H^1(\Omega)}^2 \\ &\leq \left\{ (1 - b_1) (c_u k \delta^2) \right. \\ &\quad \left. + \sum_{j=1}^{k-1} (b_j - b_{j+1}) (c_u (k - j) \delta^2) + c_u \delta^2 \right\} \|e^{k+1}\|_2 \\ &\quad + c \|e^{k+1}\|_{H^1(\Omega)}^2 \\ &\leq \left\{ (1 - b_1) \frac{k}{k+1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{k-j}{k+1} + \frac{1}{k+1} \right\} \\ &\quad \times c_u (k+1) \delta^2 \|e^{k+1}\|_2 \\ &\quad + c \|e^{k+1}\|_{H^1(\Omega)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) - (1 - b_1) \frac{1}{k+1} \right. \\
&\quad \left. - \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{j+1}{k+1} + \frac{1}{k+1} \right\} \\
&\quad \times c_u (k+1) \delta^2 \|e^{k+1}\|_2^2 + c \|e^{k+1}\|_{H^1(\Omega)}^2.
\end{aligned} \tag{69}$$

Notice that

$$\begin{aligned}
&(1 - b_1) \frac{1}{k+1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{j+1}{k+1} + b_k \\
&\geq \frac{1}{k+1} \left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} \\
&= \frac{1}{k+1}.
\end{aligned} \tag{70}$$

Then, similar to the earlier development, we have

$$\begin{aligned}
&(1 - (c + \varepsilon)) \|e^{k+1}\|_{H^1(\Omega)}^2 \\
&\leq \left(\left\{ (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} c_\varepsilon c_u (k+1) \delta^2 \right)^2 \\
&= (c_\varepsilon c_u (k+1) \delta^2)^2.
\end{aligned} \tag{71}$$

It follows, for an ε well chosen such that $1 - (c + \varepsilon) > 0$, that

$$\|e^{k+1}\|_{H^1(\Omega)} \leq c_u (k+1) \delta^2. \tag{72}$$

Then the estimate (b) is proved. This completes the proof of the theorem. $\square$

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