Hindawi Publishing Corporation
Geometry
Volume 2013, Article ID 497191, 6 pages
http://dx.doi.org/10.1155/2013/497191

Research Article

Sufficient Conditions for Meromorphically p-Valent Starlikeness and Close-to-Convexity

B. A. Frasin,[1] Tariq Al-Hawary,[2] and M. Darus[2]

[1] *Department of Mathematics, Faculty of Science, Al al-Bayt University, P.O. Box 130095, Mafraq, Jordan*
[2] *School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia*

Correspondence should be addressed to M. Darus; maslina@ukm.my

Received 14 October 2012; Accepted 6 December 2012

Academic Editor: JinLin Liu

Making use of the linear operator I_λ^m defined by (Frasin 2012), we introduce the class $\mathbb{M}_{p,j}^m(\lambda,\mu,\alpha)$ of meromorphically p-valent functions in the punctured unit disk $\mathcal{U}^*$. Furthermore, we obtain some sufficient conditions for starlikeness and close-to-convexity for functions belonging to this class. Several corollaries and consequences of the main results are also considered.

1. Introduction and Definitions

Let $\Sigma_{p,j}$ denote the class of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{n=j}^{\infty}a_{n+p-1}z^{n+p-1},\quad (p,j\in\mathbb{N}:=\{1,2,3,\dots\}),\tag{1}$$

which are p-valent in the punctured unit disk $\mathcal{U}^*=\mathcal{U}\setminus\{0\}=\{z: z\in\mathbb{C};|z|<1\}$. A function $f(z)$ in $\Sigma_{p,j}$ is said to be meromorphically p-valent starlike of order α if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad (z\in\mathcal{U}^*),\tag{2}$$

for some α $(0\le\alpha<p)$. We denote by $\Sigma_{p,j}^*(\alpha)$ the class of all meromorphically p-valent starlike of order δ. Further, a function $f(z)$ in $\Sigma_{p,j}$ is said to be meromorphically p-valent convex of order α if and only if

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad (z\in\mathcal{U}^*),\tag{3}$$

for some α $(0\le\alpha<p)$. We denote by $\Sigma_{p,j}^k(\alpha)$ the class of all meromorphically p-valent convex of order δ. A function $f(z)$ belonging to $\Sigma_{p,j}$ is said to be meromorphically p-valent close-to-convex of order α if it satisfies

$$\operatorname{Re}\left(-\frac{f'(z)}{z^{-p-1}}\right)>\alpha,\quad (z\in\mathcal{U}^*),\tag{4}$$

for some α $(0\le\alpha<p)$. We denote by $\Sigma_{p,j}^c(\alpha)$ the subclass of $\Sigma_{p,j}$ consisting of functions which are meromorphically p-valent close-to-convex of order α in $\mathcal{U}^*$.

Many interesting families of analytic and multivalent functions were considered by earlier authors in Geometric Functions Theory (cf. e.g., [1–4]). Some subclasses of $\Sigma_{p,j}=\Sigma$ when $p=j=1$ were considered by (e.g.) Miller [5], Pommerenke [6], Clunie [7], Owa et al. [8], and Royster [9]. Furthermore, several subclasses of $\Sigma_{p,j}=\Sigma_p$ when $j=1$ were studied by (amongst others) Mogra et al. [10], Uralegaddi and Ganigi [11], Cho et al. [12], Aouf [13, 14], and Uralegaddi and Somanatha [15].

For a function f in $\Sigma_{p,j}$, Frasin [16] introduced and studied the following differential operator:

$$\begin{aligned}I^0f(z)&=f(z),\\ I_\lambda^1f(z)&=(1-\lambda)f(z)+\lambda zf'(z)+\frac{\lambda(p+1)}{z^p},\quad \lambda\ge0,\\ I_\lambda^2f(z)&=(1-\lambda)I_\lambda^1f(z)+\lambda z\left(I_\lambda^1f(z)\right)'+\frac{\lambda(p+1)}{z^p},\end{aligned}\tag{5}$$

and for $m = 1, 2, 3, \ldots$

$$I_\lambda^m f(z) = (1-\lambda) I_\lambda^{m-1} f(z) + \lambda z\left(I_\lambda^{m-1} f(z)\right)' + \frac{\lambda(p+1)}{z^p} = \frac{1}{z^p} + \sum_{n=j}^{\infty}[1+\lambda(p+n-2)]^m a_{n+p-1} z^{n+p-1}. \tag{6}$$

Note that for $\lambda = p = j = 1$, we have the operator $I^m f(z)$ introduced and studied by Frasin and Darus [17].

It easily verified from (6) that

$$\lambda z(I_\lambda^m f(z))' = I_\lambda^{m+1} f(z) - (1-\lambda) I_\lambda^m f(z) - \frac{\lambda(p+1)}{z^p},$$
$$\lambda z(I_\lambda^m f(z))'' = \left(I_\lambda^{m+1} f(z)\right)' - \left(I_\lambda^m f(z)\right)' + \frac{\lambda p(p+1)}{z^{p+1}}. \tag{7}$$

Making use of the above operator I_λ^m, we now introduce a new class of meromorphically and p-valent functions defined as follows.

Definition 1. A function $f(z) \in \Sigma_{p,j}$ is said to be a member of the class $\mathbb{M}_{p,j}^m(\lambda, \mu, \alpha)$ if and only if

$$\left|\frac{z^{p+1}(I_\lambda^m f(z))'}{(z^p I_\lambda^m f(z))^{\mu-1}} + p\right| < p - \alpha, \tag{8}$$

for some α $(0 \le \alpha < p)$; $\mu \ge 0$; $\lambda \ge 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for all $z \in \mathscr{U}^*$.

Note that condition (8) implies that

$$\operatorname{Re}\left(-\frac{z^{p+1}(I_\lambda^m f(z))'}{(z^p I_\lambda^m f(z))^{\mu-1}}\right) > \alpha. \tag{9}$$

Clearly, we have $\mathbb{M}_{p,j}^0(1, 2, \alpha) = \Sigma_{p,j}^*(\alpha)$ and $\mathbb{M}_{p,j}^0(1, 1, \alpha) = \Sigma_{p,j}^c(\alpha)$.

In this paper, we obtain some sufficient conditions for functions belonging to the class $\mathbb{M}_{p,j}^m(\lambda, \mu, \alpha)$. Several corollaries and consequences of the main results are also considered.

In order to derive our main results, we have to recall the following lemmas.

Lemma 2 (see [18]). *Let $w(z)$ be analytic in $\mathscr{U}$ and such that $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_o \in \mathscr{U}$, we have*

$$z_o w'(z_o) = k w(z_o), \tag{10}$$

where $k \ge 1$ is a real number.

Lemma 3 (see [19]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2 \times \mathscr{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathscr{U}$, and for all real x, y such that $y \le -n(1+x_2^2)/2$. If the function $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathscr{U}$ such that $\Phi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathscr{U}$, then $\operatorname{Re}\{q(z)\} > 0$.*

Lemma 4 (see [20]). *Let $q(z)$ be analytic in $\mathscr{U}$ with $q(0) = 1$. If there exists a point $z_0 \in \mathscr{U}$ such that*

$$\operatorname{Re}\{q(z)\} > 0, \quad (|z| < |z_0|),$$
$$\operatorname{Re}\{q(z_0)\} = 0, \quad q(z) \ne 0, \tag{11}$$

then

$$q(z_0) = ia, \qquad \frac{zq'(z_0)}{q(z_0)} = i\frac{k}{2}\left(a + \frac{1}{a}\right), \tag{12}$$

where $a \in \mathbb{R} \setminus \{0\}$ and $k \ge 1$.

2. Sufficient Conditions for Meromorphically p-Valent Starlikeness and Close-to-Convexity

Making use of Lemma 2, we first prove

Theorem 5. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\Bigg|p + 1 + \frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'} - 1 + \frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right] - (\mu - 1)\left[p + \frac{1}{\lambda}\left(\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)} - (1-\lambda) - \frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right)\right] - \gamma\left(\frac{z^{p+1}(I_\lambda^m f(z))'}{(z^p I_\lambda^m f(z))^{\mu-1}} + p\right)\Bigg| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \quad (z \in \mathscr{U}), \tag{13}$$

for some α $(0 \le \alpha < p)$; $\mu, \gamma \ge 0$; $\lambda > 0$, $p \in \mathbb{N}$, and $m \in \mathbb{N}_0$, then $f(z) \in \mathbb{M}_{p,j}^m(\lambda, \mu, \alpha)$.

Proof. Define the function $w(z)$ by

$$\frac{z^{p+1}(I_\lambda^m f(z))'}{(z^p I_\lambda^m f(z))^{\mu-1}} = -p + (\alpha - p) w(z). \tag{14}$$

Then $w(z)$ is analytic in $\mathscr{U}$ and $w(0) = 0$. It follows from (14) and the identities (7) that

$$\begin{aligned} & p+1+\frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right] \\ & \quad -(\mu-1)\left[p+\frac{1}{\lambda}\left(\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-(1-\lambda)-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right)\right] \\ & \quad -\gamma\left(\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}+p\right) \\ & \quad =\gamma(p-\alpha) w(z)+\frac{(p-\alpha) z w'(z)}{p+(p-\alpha) w(z)}. \end{aligned} \tag{15}$$

Suppose that there exists $z_0 \in \mathscr{U}$ such that

$$\max_{|z|<z_0} |w(z)| = |w(z_0)| = 1, \tag{16}$$

then from Lemma 2, we have (10). Therefore, letting $z_o w'(z_o) = ke^{i\theta}$ $(0 \le \theta < 2\pi)$, with $k \ge 1$, we obtain that

$$\begin{aligned} & \left|p+1+\frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z_0)\right)'}{\left(I_\lambda^m f(z_0)\right)'}-1+\frac{\lambda p(p+1)}{z_0^{p+1}\left(I_\lambda^m f(z_0)\right)'}\right]\right. \\ & \quad -(\mu-1)\left[p+\frac{1}{\lambda}\left(\frac{I_\lambda^{m+1} f(z_0)}{I_\lambda^m f(z_0)}-(1-\lambda)-\frac{\lambda(p+1)}{z_0^p I_\lambda^m f(z_0)}\right)\right] \\ & \quad \left.-\gamma\left(\frac{z_0^{p+1}\left(I_\lambda^m f(z_0)\right)'}{\left(z_0^p I_\lambda^m f(z_0)\right)^{\mu-1}}+p\right)\right| \\ & \quad =\left|\gamma(p-\alpha) w(z_0)+\frac{(p-\alpha) z w'(z_0)}{p+(p-\alpha) w(z_0)}\right| \\ & \quad \ge \operatorname{Re}\left\{\gamma(p-\alpha)+\frac{(p-\alpha) k}{p+(p-\alpha) w(z_0)}\right\} \\ & \quad >\gamma(p-\alpha)+\frac{p-\alpha}{2p-\alpha}=\frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \end{aligned} \tag{17}$$

which contradicts our assumption (13). Therefore we have $|w(z)| < 1$ in $\mathscr{U}$. Finally, we have

$$\left|\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}+p\right| = (p-\alpha)|w(z)| < p-\alpha, \quad (z \in \mathscr{U}), \tag{18}$$

that is, $f(z) \in \mathbb{M}_{p,j}^m(\lambda, \mu, \alpha)$. This completes the proof of the theorem. □

Next we prove the following.

Theorem 6. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\begin{aligned} & \operatorname{Re}\left\{\left(\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}\right)^2-\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}\right. \\ & \quad \times\left(1+\frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right]\right. \\ & \quad \left.\left.+\frac{1-\mu}{\lambda}\left[\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right]\right)\right\} \\ & \quad >\delta\left(\delta+\frac{(1-\mu)(1-\lambda)}{\lambda}+\frac{n}{2}\right)+p\left(\delta(\mu-2)-\frac{n}{2}\right), \end{aligned} \tag{19}$$

for some δ $(0 \le \delta < p)$; $\mu \ge 0$; $\lambda > 0$, $p \in \mathbb{N}$ and $m \in \mathbb{N}_0$, then $f(z) \in \mathbb{M}_{p,j}^m(\lambda, \mu, \delta)$.

Proof. Define the function $q(z)$ by

$$\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}} = -\delta + (\delta - p) q(z). \tag{20}$$

Then, we see that $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathscr{U}$. Differentiating both sides of (20) with respect z logarithmically, we get

$$\begin{aligned} & 1+\frac{z\left(I_\lambda^m f(z)\right)''}{\left(I_\lambda^m f(z)\right)'}+(1-\mu)\frac{z\left(I_\lambda^m f(z)\right)'}{I_\lambda^m f(z)} \\ & \quad =\frac{(p-\delta) z q'(z)}{\delta+(p-\delta) q(z)}+p(\mu-2). \end{aligned} \tag{21}$$

Using the identities (7) in (21), we find that

$$\begin{aligned} & 1+\frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right] \\ & \quad +\frac{1-\mu}{\lambda}\left[\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right] \\ & \quad =\frac{(p-\delta) z q'(z)}{\delta+(p-\delta) q(z)}+p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}. \end{aligned} \tag{22}$$

From (20) and (22), we immediately get

$$\begin{aligned}
&\left(\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}\right)^2-\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}\\
&\quad\times\left(1+\frac{1}{\lambda}\left[\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right]\right.\\
&\quad\left.+\frac{1-\mu}{\lambda}\left[\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right]\right)\\
&=(p-\delta)zq'(z)+(p-\delta)^2q^2(z)+(p-\delta)q(z)\\
&\quad\times\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}+2\delta\right)\\
&\quad+\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}\right)\delta+\delta^2\\
&=\Phi\left(q(z),zq'(z);z\right),
\end{aligned}\tag{23}$$

where

$$\begin{aligned}
\Phi(r,s;t)=&\ (p-\delta)s+(p-\delta)^2r^2\\
&+(p-\delta)r\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}+2\delta\right)\\
&+\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}\right)\delta+\delta^2.
\end{aligned}\tag{24}$$

For all real x, y satisfying $y\le -n(1+x_2^2)/2$, we have

$$\begin{aligned}
\operatorname{Re}\Phi(ix,y;z)&=(p-\delta)y-(p-\delta)^2x^2\\
&\quad+\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}\right)\delta+\delta^2\\
&\le-\frac{n}{2}(p-\delta)-(p-\delta)\left[\frac{n}{2}+p-\delta\right]x^2\\
&\quad+\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}\right)\delta+\delta^2\\
&\le\left(p(\mu-2)+\frac{(1-\mu)(1-\lambda)}{\lambda}\right)\delta+\delta^2-\frac{n}{2}(p-\delta)\\
&=\delta\left(\delta+\frac{(1-\mu)(1-\lambda)}{\lambda}+\frac{n}{2}\right)+p\left(\delta(\mu-2)-\frac{n}{2}\right).
\end{aligned}\tag{25}$$

Let $\Omega=\{w:\ \operatorname{Re}w>\delta(\delta+(1-\mu)(1-\lambda)/\lambda+n/2)+p(\delta(\mu-2)-n/2)\}$. Then $\Phi(q(z),zq'(z);z)\in\Omega$, and $\Phi(ix,y;z)\notin\Omega$, for all real x and $y\le -n(1+x_2^2)/2$, $z\in\mathcal{U}$. By using Lemma 3, we have $\operatorname{Re}q(z)>0$, that is, $f(z)\in\mathbb{M}_{p,j}^m(\lambda,\mu,\delta)$. □

Finally, we prove the next theorem.

Theorem 7. *If $f(z)\in\Sigma_{p,j}$ satisfies*

$$\begin{aligned}
\operatorname{Re}\Bigg\{&(\mu-1)\left(\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right)\\
&-\left(\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right)\Bigg\}\\
&<\lambda-p\lambda(\mu-2)-(1-\mu)(1-\lambda)+\frac{\lambda(p-\delta)}{2\delta}
\end{aligned}\tag{26}$$

for some δ $(p/2\le\delta<p)$; $\mu\ge0$; $\lambda>0$, $p\in\mathbb{N}$, and $m\in\mathbb{N}_0$, then $f(z)\in\mathbb{M}_{p,j}^m(\lambda,\mu,\delta)$.

Proof. Define the function $q(z)$ by

$$-\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}=\delta+(p-\delta)q(z).\tag{27}$$

Then, we see that $q(z)$ is analytic in $\mathcal{U}$ with $q(0)=1$. From (22) it follows that and

$$\begin{aligned}
&(\mu-1)\left(\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}-\frac{\lambda(p+1)}{z^p I_\lambda^m f(z)}\right)\\
&\quad-\left(\frac{\left(I_\lambda^{m+1} f(z)\right)'}{\left(I_\lambda^m f(z)\right)'}-1+\frac{\lambda p(p+1)}{z^{p+1}\left(I_\lambda^m f(z)\right)'}\right)\\
&\quad=\lambda-p\lambda(\mu-2)-(1-\mu)(1-\lambda)-\frac{\lambda(p-\delta)zq'(z)}{\delta+(p-\delta)q(z)}.
\end{aligned}\tag{28}$$

If there exists a point $z_0\in\mathcal{U}$ such that

$$\begin{aligned}
&\operatorname{Re}\{q(z)\}>0,\quad (|z|<|z_0|),\\
&\operatorname{Re}\{q(z_0)\}=0,\quad q(z)\ne0.
\end{aligned}\tag{29}$$

Then applying Lemma 4, we have

$$q(z_0)=ia,\qquad \frac{zq'(z_0)}{q(z_0)}=i\frac{k}{2}\left(a+\frac{1}{a}\right),\tag{30}$$

where $a \in \mathbb{R} \setminus \{0\}$ and $k \geq 1$. Thus, from (28) and (30) we get

$$\begin{aligned}&(\mu-1)\left(\frac{I_\lambda^{m+1} f(z_0)}{I_\lambda^m f(z_0)} - \frac{\lambda(p+1)}{z_0^p I_\lambda^m f(z_0)}\right)\\&\quad-\left(\frac{\left(I_\lambda^{m+1} f(z_0)\right)'}{\left(I_\lambda^m f(z_0)\right)'} - 1 + \frac{\lambda p(p+1)}{z_0^{p+1}\left(I_\lambda^m f(z_0)\right)'}\right)\\&= \lambda - p\lambda(\mu-2) - (1-\mu)(1-\lambda) - \frac{\lambda(p-\delta) z_0 q'(z_0)}{\delta + (p-\delta) q(z_0)}\\&= \lambda - p\lambda(\mu-2) - (1-\mu)(1-\lambda) + \frac{k\lambda(p-\delta)\left(1+a^2\right)}{2(\delta + i(p-\delta)a)}.\end{aligned} \tag{31}$$

Therefore, we have

$$\begin{aligned}&\operatorname{Re}\left\{(\mu-1)\left(\frac{I_\lambda^{m+1} f(z_0)}{I_\lambda^m f(z_0)} - \frac{\lambda(p+1)}{z_0^p I_\lambda^m f(z_0)}\right)\right.\\&\quad\left.-\left(\frac{\left(I_\lambda^{m+1} f(z_0)\right)'}{\left(I_\lambda^m f(z_0)\right)'} - 1 + \frac{\lambda p(p+1)}{z_0^{p+1}\left(I_\lambda^m f(z_0)\right)'}\right)\right\}\\&= \lambda - p\lambda(\mu-2) - (1-\mu)(1-\lambda) + \frac{k\lambda(p-\delta)\left(1+a^2\right)\delta}{2\left(\delta^2 + (p-\delta)^2 a^2\right)}\\&\geq \lambda - p\lambda(\mu-2) - (1-\mu)(1-\lambda) + \frac{k\lambda(p-\delta)}{2\delta}\\&\geq \lambda - p\lambda(\mu-2) - (1-\mu)(1-\lambda) + \frac{\lambda(p-\delta)}{2\delta}.\end{aligned} \tag{32}$$

This contradicts our assumption. Thus, we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathcal{U}$, that is,

$$\operatorname{Re}\left(-\frac{z^{p+1}\left(I_\lambda^m f(z)\right)'}{\left(z^p I_\lambda^m f(z)\right)^{\mu-1}}\right) > \delta. \tag{33}$$

□

3. Special Cases and Consequences

Among the various interesting and important consequences of Theorems 5–7, we mention now some of the corollaries relating to the classes $\Sigma_{p,j}^*(\alpha)$, and $\Sigma_{p,j}^c(\delta)$, which are deducible from the main results.

Firstly, if we let $m = 0$, $\mu = 2$, and $\lambda = 1$ in Theorems 5–7, we get the following sufficient conditions for meromorphically p-valent starlike functions.

Corollary 8. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\begin{aligned}&\left|1 + \frac{zf''(z)}{f'(z)} - (1+\gamma)\frac{zf'(z)}{f(z)} - \gamma p\right|\\&\quad< \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha},\end{aligned} \tag{34}$$

for some α $(0 \leq \alpha < p)$; $p \in \mathbb{N}$ and $\gamma \geq 0$,then $f(z) \in \Sigma_{p,j}^*(\alpha)$.

Corollary 9. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\begin{aligned}&\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{2zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)}\right)\right\}\\&\quad> \delta\left(\delta + \frac{n}{2}\right) - \frac{np}{2},\end{aligned} \tag{35}$$

for some δ $(0 \leq \delta < p)$; $p \in \mathbb{N}$ and $\gamma \geq 0$, then $f(z) \in \Sigma_{p,j}^*(\delta)$.

Corollary 10. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{p-\delta}{2\delta}, \tag{36}$$

for some δ $(p/2 \leq \delta < p)$; $p \in \mathbb{N}$, then $f(z) \in \Sigma_{p,j}^*(\delta)$.

Setting $m = 0$ and $\mu = \lambda = 1$ in Theorems 5–7, we get the following sufficient conditions for meromorphically p-valent close-to-convex functions.

Corollary 11. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\begin{aligned}&\left|p + 1 + \frac{zf''(z)}{f'(z)} - \gamma\left(z^{p+1} f'(z) + p\right)\right|\\&\quad< \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha},\end{aligned} \tag{37}$$

for some α $(0 \leq \alpha < p)$; $p \in \mathbb{N}$ and $\gamma \geq 0$, then $f(z) \in \Sigma_{p,j}^c(\alpha)$.

Corollary 12. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\begin{aligned}&\operatorname{Re}\left\{\left(z^{p+1} f'(z)\right)\left(z^{p+1} f'(z) - 1 - \frac{zf''(z)}{f'(z)}\right)\right\}\\&\quad> (\delta - p)\left(\delta + \frac{n}{2}\right),\end{aligned} \tag{38}$$

for some δ $(0 \leq \delta < p)$; $p \in \mathbb{N}$, then $f(z) \in \Sigma_{p,j}^c(\delta)$.

Corollary 13. *If $f(z) \in \Sigma_{p,j}$ satisfies*

$$\operatorname{Re}\left\{-\frac{zf''(z)}{f'(z)}\right\} < 1 + p + \frac{p-\delta}{2\delta}, \tag{39}$$

for some δ $(p/2 \leq \delta < p)$; $p \in \mathbb{N}$, then $f(z) \in \Sigma_{p,j}^c(\delta)$.

Setting $p = j = 1$ in Corollary 10, we have

Corollary 14. *If $f(z) \in \Sigma$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{1-\delta}{2\delta}, \tag{40}$$

for some δ $(1/2 \le \delta < 1)$, *then* $f(z) \in \Sigma^{*}(\delta)$. *In particular, if* $f(z) \in \Sigma$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}<\frac{3}{2}, \tag{41}$$

then $f(z)$ *is meromorphically starlike of order* $1/2$.

Setting $p = j = 1$ in Corollary 13, we have the following.

Corollary 15. *If* $f(z) \in \Sigma$ *satisfies*

$$\operatorname{Re}\left\{-\frac{zf''(z)}{f'(z)}\right\}<2+\frac{1-\delta}{2\delta}, \tag{42}$$

for some δ $(1/2 \le \delta < 1)$, *then* $f(z) \in \Sigma^{c}(\delta)$. *In particular, if* $f(z) \in \Sigma$ *satisfies*

$$\operatorname{Re}\left\{-\frac{zf''(z)}{f'(z)}\right\}<\frac{5}{2}, \tag{43}$$

then $f(z)$ *is meromorphically close-to-convex of order* $1/2$.

Remark 16. (i) If we put $\gamma = p = j = 1$ in Corollaries 8 and 11, we get Corollaries 5 and 1, respectively, proved by Goyal and Prajapat [21].

(ii) If we put $p = j = n = 1$ and $\delta = 0$ in Corollaries 9 and 12, we get Corollaries 8 and 4, respectively, proved by Goyal and Prajapat [21].

Acknowledgment

The work here was fully supported by LRGS/TD/2011/UKM/ICT/03/02.

References

[1] M. K. Aouf, "A generalization of meromorphic multivalent functions with positive coefficients," *Mathematica Japonica*, vol. 35, no. 4, pp. 609–614, 1990.

[2] S. R. Kulkarni, U. H. Naik, and H. M. Srivastava, "A certain class of meromorphically p-valent quasi-convex functions," *Panamerican Mathematical Journal*, vol. 8, no. 1, pp. 57–64, 1998.

[3] H. M. Srivastava and S. Owa, *Current Topics in Analytic Function Theory*, World Scientific Publishing Co, River Edge, NJ, USA, 1992.

[4] H. M. Srivastava, H. M. Hossen, and M. K. Aouf, "A unified presentation of some classes of meromorphically multivalent functions," *Computers & Mathematics with Applications*, vol. 38, no. 11-12, pp. 63–70, 1999.

[5] J. Miller, "Convex meromorphic mappings and related functions," *Proceedings of the American Mathematical Society*, vol. 25, pp. 220–228, 1970.

[6] Ch. Pommerenke, "On meromorphic starlike functions," *Pacific Journal of Mathematics*, vol. 13, pp. 221–235, 1963.

[7] J. Clunie, "On meromorphic schlicht functions," *Journal of the London Mathematical Society*, vol. 34, pp. 215–216, 1959.

[8] S. Owa, H. E. Darwish, and M. K. Aouf, "Meromorphic multivalent functions with positive and fixed second coefficients," *Mathematica Japonica*, vol. 46, no. 2, pp. 231–236, 1997.

[9] W. C. Royster, "Meromorphic starlike multivalent functions," *Transactions of the American Mathematical Society*, vol. 107, pp. 300–308, 1963.

[10] M. L. Mogra, T. R. Reddy, and O. P. Juneja, "Meromorphic univalent functions with positive coefficients," *Bulletin of the Australian Mathematical Society*, vol. 32, no. 2, pp. 161–176, 1985.

[11] B. A. Uralegaddi and M. D. Ganigi, "A certain class of meromorphically starlike functions with positive coefficients," *Pure and Applied Mathematika Sciences*, vol. 26, no. 1-2, pp. 75–81, 1987.

[12] N. E. Cho, S. H. Lee, and S. Owa, "A class of meromorphic univalent functions with positive coefficients," *Kobe Journal of Mathematics*, vol. 4, no. 1, pp. 43–50, 1987.

[13] M. K. Aouf, "On a class of meromorphic multivalent functions with positive coefficients," *Mathematica Japonica*, vol. 35, no. 4, pp. 603–608, 1990.

[14] M. K. Aouf, "On a certain class of meromorphic univalent functions with positive coefficients," *Rendiconti di Matematica e delle sue Applicazioni Serie VII*, vol. 11, no. 2, pp. 209–219, 1991.

[15] B. A. Uralegaddi and C. Somanatha, "New criteria for meromorphic starlike univalent functions," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 137–140, 1991.

[16] B. A. Frasin, "Differential operator of meromorphic p-valent functions," *Tamkang Journal of Mathematics*, vol. 43, pp. 417–421, 2012.

[17] B. A. Frasin and M. Darus, "On certain meromorphic functions with positive coefficients," *Southeast Asian Bulletin of Mathematics*, vol. 28, no. 4, pp. 615–623, 2004.

[18] I. S. Jack, "Functions starlike and convex of order α," *Journal of the London Mathematical Society*, vol. 3, pp. 469–474, 1971.

[19] S. S. Miller and P. T. Mocanu, "Differential subordinations and inequalities in the complex plane," *Journal of Differential Equations*, vol. 67, no. 2, pp. 199–211, 1987.

[20] M. Nunokawa, "On properties of non-Carathéodory functions," *Proceedings of the Japan Academy, Series A*, vol. 68, no. 6, pp. 152–153, 1992.

[21] S. P. Goyal and J. K. Prajapat, "A new class of meromorphic multivalent functions involving certain linear operator," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 25, no. 2, pp. 167–176, 2009.