Research Article

The Linear 2- and 4-Arboricity of Complete Bipartite Graph $K_{m,n}$

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A linear k -forest of an undirected graph G is a subgraph of G whose components are paths with lengths at most k . The linear k -arboricity of G , denoted by $la_k(G)$, is the minimum number of linear k -forests needed to decompose G . In case the lengths of paths are not restricted, we then have the linear arboricity of G , denoted by $la(G)$. In this paper, the exact value of the linear 2- and 4-arboricity of complete bipartite graph $K_{m,n}$ for some m and n is obtained.

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, and simple. Let $\mathbb{N}$ represent the set of natural numbers and let $[a, b]$ denote the set $\{n \in \mathbb{N} \mid a \leq n \leq b\}$. A graph G is m -partite ($m \geq 2$), if it is possible to partition the vertex set $V(G)$ into m independent sets $V_1, V_2, \dots, V_m$ (called partite sets) such that every edge of G joins the vertices in different partite sets. A complete m -partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset, which is denoted by $K_{n_1, n_2, \dots, n_m}$ if $|V_i| = n_i$ for $i \in [1, m]$. For $m = 2$, such graphs are called complete bipartite graphs and denoted by K_{n_1, n_2} . When $n_1 = n_2 = \dots = n_m = n$, we denote $K_{n_1, n_2, \dots, n_m}$ by $K_{m(n)}$, which is called balanced complete m -partite graph. For $m = 2$, such graphs are called balanced complete bipartite graphs and are denoted by $K_{n,n}$. On the other hand, a graph with order n , in which any pair of different vertices are adjacent, is a complete graph, denoted by K_n . Other notations and terminology in the paper are the same as in [1].

A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \dots, G_t$, then we say that $G_1, G_2, \dots, G_t$ decompose G or G can be decomposed into $G_1, G_2, \dots, G_t$. Furthermore, a linear k -forest is a forest whose components are paths of length at most k . The linear

k -arboricity of a graph G , denoted by $la_k(G)$, is the least number of linear k -forests needed to decompose G .

The notion of linear k -arboricity of a graph was first introduced by Habib and Peroche [2]. It is a natural generalization of edge coloring. Clearly, a linear 1-forest is induced by a *matching*, and $la_1(G)$ is the chromatic index $\chi'(G)$ of a graph G . Moreover, the linear k -arboricity $la_k(G)$ is also a refinement of the ordinary linear arboricity $la(G)$ (or $la_\infty(G)$) which is the case when every component of each forest is a path with no length constraint. By the way, the notion of linear arboricity was introduced earlier by Harary in [3].

In 1982, Habib and Péroche [4] proposed the following conjecture for an upper bound on $la_k(G)$.

Conjecture 1 (see [4]). *If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then*

$$la_k(G) \leq \begin{cases} \left\lceil \frac{\Delta(G) |V(G)|}{2 \lfloor k |V(G)| / (k+1) \rfloor} \right\rceil, & \text{when } \Delta(G) = |V(G)| - 1, \\ \left\lceil \frac{\Delta(G) |V(G)| + 1}{2 \lfloor k |V(G)| / (k+1) \rfloor} \right\rceil, & \text{when } \Delta(G) < |V(G)| - 1. \end{cases} \quad (1)$$

For $k = |V(G)| - 1$, this is Akiyama's conjecture [5].

Conjecture 2 (see [5]). Consider $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$.

So far, there have been a lot of results on the verification of Conjecture 1 in the literature, especially for graphs with particular structures, such as trees [2, 6, 7], regular graphs [8, 9], planar graphs [10], and complete graphs [11–15]. The linear arboricity and the linear k -arboricity of cubic graphs are studied in recent years, and for more details, please read the papers [11, 16–20]. The linear 2-arboricity, the linear 3-arboricity and the low bound of linear k -arboricity of balanced complete bipartite graph, and linear 3-arboricity of the balanced complete multipartite graph in [13, 15, 21, 22], respectively, are obtained. In 2010, Xue and Zuo obtained the linear $(n - 1)$ -arboricity of $K_{n(n)}$ in [23].

As for a low bound on $la_k(G)$, since any vertex in a linear k -forest has degree at most 2 and a linear k -forest in a graph G has at most $\lfloor k \cdot |V(G)| / (k + 1) \rfloor$ edges, the following result is obvious.

Lemma 3 (see [15]). For any graph G with maximum degree $\Delta(G)$,

$$la_k(G) \geq \max \left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\lfloor k \cdot |V(G)| / (k + 1) \rfloor} \right\rceil \right\}. \quad (2)$$

It is clear that the following lemma holds.

Lemma 4. If $G = G_1 \cup G_2 \cup \dots \cup G_n$, then $la_k(G) \leq la_k(G_1) + la_k(G_2) + \dots + la_k(G_n)$.

In the following, we will study the linear 2- and 4-arboricity of the complete bipartite graph $K_{m,n}$.

2. The Linear 2-Arboricity of Complete Bipartite Graph $K_{m,n}$

Lemma 5. Consider $la_2(K_{4,2}) = 2$ and $la_2(K_{4,3}) = la_2(K_{5,2}) = 3$.

Proof. By Lemma 3, $la_2(K_{4,2}) \geq 2$, $la_2(K_{4,3}) \geq 3$, and $la_2(K_{5,2}) \geq 3$.

Let $K_{4,2} = (V_1, V_2)$, where $V_1 = \{v_i : i \in [1, 4]\}$ and $V_2 = \{u_1, u_2\}$. Clearly, $K_{4,2}$ can be decomposed into two linear 2-forests: $\{v_1u_1v_2, v_3u_2v_4\}$ and $\{v_1u_2v_2, v_3u_1v_4\}$. Thus $la_2(K_{4,2}) = 2$.

Let $K_{4,3} = (V'_1, V'_2)$, where $V'_1 = \{v_i : i \in [1, 4]\}$ and $V'_2 = \{u_j : j \in [1, 3]\}$. Clearly, $K_{4,3}$ can be decomposed into three linear 2-forests: $\{v_1u_1v_2, v_3u_2v_4\}$, $\{v_1u_2v_2, v_3u_3v_4\}$, and $\{v_1u_3v_2, v_3u_1v_4\}$. Thus $la_2(K_{4,3}) = 3$.

Let $K_{5,2} = (V''_1, V''_2)$, where $V''_1 = \{v_i : i \in [1, 5]\}$ and $V''_2 = \{u_1, u_2\}$. Clearly, $K_{5,2}$ can be decomposed into three linear 2-forests: $\{v_1u_1v_2, v_3u_2v_4\}$, $\{v_1u_2v_2, v_4u_1v_5\}$, and $\{v_3u_1, v_5u_2\}$. Thus $la_2(K_{5,2}) = 3$. $\square$

In the following, we mainly consider the complete bipartite graphs $K_{4n,2n}$, $K_{4n+1,2n}$, and $K_{4n,2n+1}$ in this section.

Let $G(V_1, V_2)$ be a bipartite graph with partite sets $V_1 = \{v_{10}, v_{11}, \dots, v_{1(r-1)}\}$ and $V_2 = \{v_{20}, v_{21}, \dots, v_{2(s-1)}\}$. Suppose

that $|V_2| = s \geq r = |V_1|$. In [15], Fu et al. defined the *bipartite difference* of an edge $v_{1p}v_{2q}$ in $G(V_1, V_2)$ as the value $(q - p) \pmod s$. It is not difficult to see that a set consisting of those edges in $G(V_1, V_2)$ with the same bipartite difference must be a matching. In particular, such a set is a perfect matching if $G(V_1, V_2)$ is a $K_{n,n}$. Furthermore, we can partition the edge set of $K_{n,n}$ into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_i is exactly the set of edges of bipartite difference i in $K_{n,n}$ for $i \in [0, n - 1]$.

Theorem 6. One has $la_2(K_{4n,2n}) = 2n$.

Proof. Let $G(V_1, V_2) = K_{4n,2n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 4n$ and $|V_2| = 2n$. Next, we partition V_1 and V_2 into n vertex sets V_{1i} and V_{2i} , respectively, such that $|V_{1i}| = 4$ and $|V_{2i}| = 2$ for $i \in [0, n - 1]$.

For each V_{1i} ($i \in [0, n - 1]$), we identify 4 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} , we identify 2 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a balanced complete bipartite graph $H = K_{n,n}$ with vertex sets $V'_1 = \{v_{1[i]} : i \in [0, n - 1]\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$. The edge set of $K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n - 1]$. Every edge in M_j corresponds to a subgraph of $K_{4n,2n}$ isomorphic to $K_{4,2}$. Since $la_2(K_{4,2}) = 2$ by Lemma 5, each M_j corresponds to two linear 2-forests in $K_{4n,2n}$. Thus $la_2(K_{4n,2n}) \leq 2n$ by Lemma 4. Applying Lemma 3, we have $la_2(K_{4n,2n}) \geq 2n$. Hence $la_2(K_{4n,2n}) = 2n$. $\square$

Corollary 7. One has $la_2(K_{4n,2n-1}) = la_2(K_{4n-1,2n}) = 2n$.

Proof. The results can be obtained by Lemma 3 and Theorem 6 immediately. $\square$

Theorem 8. $la_2(K_{4n,2n+1}) = 2n + 1$, where n is odd.

Proof. Let $G(V_1, V_2) = K_{2n+1,4n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 2n + 1$ and $|V_2| = 4n$. Next, we partition V_1 into $n + 1$ vertex sets V_{1j} ($j \in [0, n]$) and partition V_2 into n vertex sets V_{2i} ($i \in [0, n - 1]$), respectively, such that $|V_{1i}| = 2$, $|V_{2i}| = 4$ for $i \in [0, n - 1]$, and $|V_{1n}| = |\{u\}| = 1$.

For each V_{1i} ($i \in [0, n - 1]$), we identify 2 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} ($i \in [0, n - 1]$), we identify 4 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a complete bipartite graph $H = K_{n+1,n}$ with partite sets $V'_1 = \{v_{1[i]} : i \in [0, n - 1]\} \cup \{u\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$. Let $H_1(V, V'_2) = K_{n,n}$ be a balanced complete bipartite graph with partite sets $V = V'_1 \setminus \{u\} = \{v_{1[i]} : i \in [0, n - 1]\}$ and $V'_2 = \{v_{2[j]} : j \in [0, n - 1]\}$. Let $H_2 = K_{n,1}$ be a complete bipartite graph with vertex sets $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$ and $\{u\}$. In fact, $H_2 = K_{n,1}$ is a star S_{n+1} with the center u . Thus, $H = H_1 \cup H_2$. Since the edge set of $H_1(V, V'_2) = K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n - 1]$ and n is odd, the edge set of $H = K_{n+1,n}$ can be partitioned

into n pairwise disjoint edge sets $M'_0, M'_1, \dots, M'_{n-1}$ such that $M'_j = M_j \cup \{uv_{2[2j]}v_{1[j]}\}$ ($j \in [0, n-1]$) which consists of a 2-path $uv_{2[2j]}v_{1[j]}$ and $n-1$ isolated edges $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$), where $2j, i+j$ are taken modulo n .

Thus, each edge $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$) in M'_j corresponds to a subgraph isomorphic to $K_{4,2}$ and the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds to a subgraph isomorphic to $K_{4,3}$ with partition $(V_{2(2j)}, V_{1j} \cup \{u\})$ in $K_{2n+1, 4n}$. Let

$$V_{1j} = \{v_{1[j]0}, v_{1[j]1}\}, \quad (3)$$

$$V_{2(2j)} = \{v_{2[2j]0}, v_{2[2j]1}, v_{2[2j]2}, v_{2[2j]3}\},$$

where the indices $2j$ are taken modulo n .

After taking away the four edges $v_{2[2j]0}v_{1[j]0}, v_{2[2j]1}v_{1[j]0}, v_{2[2j]2}v_{1[j]1}$, and $v_{2[2j]3}v_{1[j]1}$ from each $K_{4,3}$ to which the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds, each M'_j corresponds to two linear 2-forests of $K_{4n+1, 2n}$ by Lemma 5. The edges that we take away form one linear 2-forest. Hence $la_2(K_{2n+1, 4n}) \leq 2n+1$ by Lemma 4. Since $la_2(K_{2n+1, 4n}) \geq 2n+1$ by Lemma 3, we have $la_2(K_{2n+1, 4n}) = 2n+1$. $\square$

Corollary 9. $la_2(K_{4n-1, 2n+1}) = 2n+1$, where n is odd.

Proof. The results can be obtained by Lemma 3 and Theorem 8 immediately. $\square$

Theorem 10. $la_2(K_{4n+1, 2n}) = 2n+1$, where n is odd.

Proof. The proof is similar to Theorem 8. Let $G(V_1, V_2) = K_{4n+1, 2n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 4n+1$ and $|V_2| = 2n$. Next, we partition V_2 into n vertex sets V_{2i} ($i \in [0, n-1]$) and partition V_1 into $n+1$ vertex sets V_{1i} ($i \in [0, n]$), respectively, such that $|V_{2i}| = 2$, $|V_{1i}| = 4$ for $i \in [0, n-1]$, and $|V_{1n}| = |\{u\}| = 1$.

For each V_{1i} ($i \in [0, n-1]$), we identify 4 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} ($i \in [0, n-1]$), we identify 2 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a complete bipartite graph $H = K_{n+1, n}$ with partite sets $V'_1 = \{v_{1[i]} : i \in [0, n-1]\} \cup \{u\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n-1]\}$. Let $H_1(V, V'_2) = K_{n,n}$ be a balanced complete bipartite graph with vertex sets $V = V'_1 \setminus \{u\} = \{v_{1[i]} : i \in [0, n-1]\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n-1]\}$. Let $H_2 = K_{n,1}$ be a complete bipartite graph with partite sets $V'_2 = \{v_{2[i]} : i \in [0, n-1]\}$ and $\{u\}$. In fact, $H_2 = K_{n,1}$ is a star S_{n+1} with the center u . Thus, $H = H_1 \cup H_2$. Since the edge set of $H_1(V, V'_2) = K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n-1]$ and n is odd, the edge set of $H = K_{n+1, n}$ can be partitioned into n pairwise disjoint edge sets $M'_0, M'_1, \dots, M'_{n-1}$ such that $M'_j = M_j \cup \{uv_{2[2j]}\}$ ($j \in [0, n-1]$) which consists of a 2-path $uv_{2[2j]}v_{1[j]}$ and $n-1$ isolated edges $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$), where $2j$ and $i+j$ are taken modulo n .

Thus, each edge $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$) in M'_j corresponds to a subgraph isomorphic to $K_{4,2}$ and the

2-path $uv_{2[2j]}v_{1[j]}$ corresponds to a subgraph isomorphic to $K_{5,2}$ with partition $(V_{1j} \cup \{u\}, V_{2(2j)})$ in $K_{4n+1, 2n}$. Let

$$V_{2(2j)} = \{v_{2[2j]0}, v_{2[2j]1}\}, \quad (4)$$

$$V_{1j} = \{v_{1[j]0}, v_{1[j]1}, v_{1[j]2}, v_{1[j]3}\},$$

where the indices $2j$ are taken modulo n .

After taking away the two edges $v_{1[j]0}v_{2[2j]1}, v_{1[j]1}v_{2[2j]3}$ from each $K_{5,2}$ to which the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds, each M'_j corresponds to two linear 2-forests of $K_{4n+1, 2n}$ by Lemma 5. The edges that we take away form one linear 2-forest. Hence $la_2(K_{4n+1, 2n}) \leq 2n+1$ by Lemma 4. Since $la_2(K_{4n+1, 2n}) \geq 2n+1$ by Lemma 3, we obtain $la_2(K_{4n+1, 2n}) = 2n+1$. $\square$

3. The Linear 4-Arboricity of Complete Bipartite Graph $K_{m,n}$

Lemma 11. One has $la_4(K_{6,4}) = 3$.

Proof. By Lemma 3, $la_4(K_{6,4}) \geq 3$. In the following, $K_{6,4}$ can be partitioned into three linear 4-forests: let $K_{6,4} = (V_1, V_2)$, where $V_1 = \{v_i : i \in [1, 6]\}$ and $V_2 = \{u_j : j \in [1, 4]\}$. Clearly, $K_{6,4}$ can be decomposed into three linear 4-forests $\{v_1u_1v_2u_2v_3, v_4u_3v_5u_4v_6\}$, $\{v_2u_3v_1u_4v_3, v_4u_1v_6u_2v_5\}$, and $\{v_1u_2v_4u_4v_2, v_5u_1v_3u_3v_6\}$ (see Figure 1).

Thus $la_4(K_{6,4}) = 3$. $\square$

Lemma 12. $la_4(K_{6,5}) = 4$.

Proof. By Lemma 3, $la_4(K_{6,5}) \geq 4$. In the following, we partition the edge set of $K_{6,5}$ into four linear 4-forests: let $K_{6,5} = (V_1, V_2)$, where $V_1 = \{v_i : i \in [1, 6]\}$ and $V_2 = \{u_j : j \in [1, 5]\}$. Clearly, $K_{6,5}$ can be decomposed into four linear 4-forests $\{v_1u_1v_2u_2, v_3u_3v_4u_4, v_5u_5v_6\}$, $\{v_1u_5v_2, u_1v_3u_2v_4, u_3v_5u_4v_6\}$, $\{v_1u_3v_2u_4, v_3u_5v_4, u_1v_5u_2v_6\}$, and $\{u_2v_1u_4v_3, v_4u_1v_6u_3\}$ (see Figure 2).

Thus $la_4(K_{6,5}) = 4$. $\square$

Lemma 13. $la_4(K_{7,4}) = 4$.

Proof. By Lemma 3, $la_4(K_{7,4}) \geq 4$. In the following, we partition the edge set of $K_{7,4}$ into four linear 4-forests: let $K_{7,4} = (V_1, V_2)$, where $V_1 = \{v_i : i \in [1, 7]\}$ and $V_2 = \{u_j : j \in [1, 4]\}$. $K_{7,4}$ can be decomposed into four linear 4-forests $\{v_1u_1v_2, v_3u_2v_4, v_5u_3v_6u_4v_7\}$, $\{v_1u_4v_2, v_3u_1v_4u_3v_7, v_5u_2v_6\}$, $\{v_1u_3v_2, v_3u_4v_4, v_6u_1v_7u_2\}$, and $\{v_1u_2v_2, u_1v_5u_4, v_3u_3\}$ (see Figure 3).

Thus $la_4(K_{7,4}) = 4$. $\square$

In the following, we mainly consider the complete bipartite graphs $K_{6n, 4n}$, $K_{6n+1, 4n}$, and $K_{6n, 4n+1}$.

Theorem 14. One has $la_4(K_{6n, 4n}) = 3n$.

Proof. Let $G(V_1, V_2) = K_{6n, 4n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 6n$ and $|V_2| = 4n$.

FIGURE 1: The decomposition of linear 4-forests of $K_{6,4}$. FIGURE 2: The decomposition of linear 4-forests of $K_{6,5}$.

Next, we partition V_1 and V_2 into n vertex sets V_{1i} and V_{2i} , respectively, such that $|V_{1i}| = 6$ and $|V_{2i}| = 4$ for $i \in [0, n-1]$.

For each V_{1i} ($i \in [0, n-1]$), we identify 6 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} , we identify 4 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a balanced complete bipartite graph $H = K_{n,n}$ with vertex sets $V'_1 = \{v_{1[i]} : i \in [0, n-1]\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n-1]\}$. The edge set of $K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n-1]$. Every edge in M_j corresponds to a subgraph of $K_{6n,4n}$ isomorphic to $K_{6,4}$. Since $la_4(K_{6,4}) = 3$ by Lemma 11, each M_j corresponds to three linear 4-forests in $K_{6n,4n}$. Thus $la_4(K_{6n,4n}) \leq 3n$ by Lemma 11. Applying Lemma 3, we have

$$la_4(K_{6n,4n}) \geq \left\lceil \frac{6n \cdot 4n}{[4 \cdot (6n + 4n) / 5]} \right\rceil = 3n. \quad (5)$$

Hence $la_4(K_{6n,4n}) = 3n$. $\square$

Corollary 15. Consider $la_4(K_{6n,4n-1}) = la_4(K_{6n-1,4n}) = 3n$.

Proof. The results can be obtained by Lemma 3 and Theorem 14 immediately. $\square$

Theorem 16. $la_4(K_{4n+1,6n}) = 3n + 1$, where n is odd.

Proof. Let $G(V_1, V_2) = K_{4n+1,6n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 4n + 1$ and $|V_2| = 6n$. Next, we partition V_1 into $n + 1$ vertex sets V_{1i} ($i \in [0, n]$) and partition V_2 into n vertex sets V_{2j} , respectively, such that $|V_{1j}| = 4$, $|V_{2j}| = 6$ for $j \in [0, n-1]$, and $|V_{1n}| = |\{u\}| = 1$.

For each V_{1i} ($i \in [0, n-1]$), we identify 4 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} ($i \in [0, n-1]$), we identify 6 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a complete bipartite graph $H = K_{n+1,n}$ with partite sets $V'_1 = \{v_{1[i]} : i \in [0, n-1]\} \cup \{u\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n-1]\}$. Let $H_1(V, V'_2) = K_{n,n}$ be a balanced complete bipartite graph with partite sets $V = V'_1 \setminus \{u\} = \{v_{1[i]} : i \in [0, n-1]\}$ and V'_2 . Let $H_2 = K_{n,1}$ be a complete bipartite graph

with vertex sets V'_2 and $\{u\}$. In fact, $H_2 = K_{n,1}$ is a star S_{n+1} with the center u . Thus, $H = H_1 \cup H_2$. Since the edge set of $H_1(V, V'_2) = K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n-1]$ and n is odd, the edge set of $H = K_{n+1,n}$ can be partitioned into n pairwise disjoint edge sets $M'_0, M'_1, \dots, M'_{n-1}$ such that $M'_j = M_j \cup \{uv_{2[2j]}\}$ ($j \in [0, n-1]$) which consists of a 2-path $uv_{2[2j]}v_{1[j]}$ and $n-1$ isolated edges $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$), where $2j, i+j$ are taken modulo n .

Thus, each edge $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j-1] \cup [j+1, n-1]$) in M'_j corresponds to a subgraph isomorphic to $K_{6,4}$ and the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds to a subgraph isomorphic to $K_{6,5}$ with partition $(V_{2(2j)}, V_{1j} \cup \{u\})$ in $K_{6n,4n+1}$, where $2j$ is taken modulo n . Let

$$V_{1j} = \{v_{1[j]0}, v_{1[j]1}, v_{1[j]2}, v_{1[j]3}\}, \quad (6)$$

$$V_{2(2j)} = \{v_{2[2j]0}, v_{2[2j]1}, \dots, v_{2[2j]5}\},$$

where the indices $2j$ are taken modulo n .

After taking away the six edges $v_{2[2j]0}v_{1[j]1}, v_{2[2j]0}v_{1[j]3}, v_{2[2j]2}v_{1[j]3}, v_{2[2j]3}v_{1[j]0}, v_{2[2j]5}v_{1[j]0}$, and $v_{2[2j]5}v_{1[j]2}$ from each $K_{6,5}$ to which the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds, each M'_j corresponds to three linear 4-forests of $K_{6n,4n+1}$ by Lemma 12. The edges that we take away form one linear 4-forest (see Figure 2). Hence $la_4(K_{6n,4n+1}) \leq 3n + 1$ by Lemma 11. Since

$$la_4(K_{6n,4n+1}) \geq \left\lceil \frac{6n(4n+1)}{[4 \cdot (6n + 4n + 1) / 5]} \right\rceil = 3n + 1 \quad (7)$$

by Lemma 3, we have $la_4(K_{6n,4n+1}) = 3n + 1$. $\square$

Corollary 17. $la_4(K_{6n-1,4n+1}) = 3n + 1$, where n is odd.

Proof. The results can be obtained by Lemma 3 and Theorem 16 immediately. $\square$

FIGURE 3: The decomposition of linear 4-forests of $K_{7,4}$.

Theorem 18. $la_4(K_{6n+1,4n}) = 3n + 1$, where n is odd.

Proof. The proof is similar to Theorem 16. Let $G(V_1, V_2) = K_{6n+1,4n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1| = 6n + 1$ and $|V_2| = 4n$. Next, we partition V_2 into n vertex subsets V_{2i} ($i \in [0, n - 1]$) and partition V_1 into $n + 1$ vertex subsets V_{1i} ($i \in [0, n]$), respectively, such that $|V_{2i}| = 4$, $|V_{1i}| = 6$ for $i \in [0, n - 1]$, and $|V_{1n}| = |\{u\}| = 1$.

For each V_{1i} ($i \in [0, n - 1]$), we identify 6 vertices in V_{1i} and denote a new vertex by $v_{1[i]}$. For each V_{2i} ($i \in [0, n - 1]$), we identify 4 vertices in V_{2i} and denote a new vertex by $v_{2[i]}$. Then we obtain a complete bipartite graph $H = K_{n+1,n}$ with partite sets $V'_1 = \{v_{1[i]} : i \in [0, n - 1]\} \cup \{u\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$. Let $H_1(V, V'_2) = K_{n,n}$ be a balanced complete bipartite graph with vertex sets $V = V'_1 \setminus \{u\} = \{v_{1[i]} : i \in [0, n - 1]\}$ and $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$. Let $H_2 = K_{n,1}$ be a complete bipartite graph with partite sets $V'_2 = \{v_{2[i]} : i \in [0, n - 1]\}$ and $\{u\}$. In fact, $H_2 = K_{n,1}$ is a star S_{n+1} with the center u . Thus, $H = H_1 \cup H_2$. Since the edge set of $H_1(V, V'_2) = K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \dots, M_{n-1}$ such that M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n - 1]$ and n is odd, the edge set of $H = K_{n+1,n}$ can be partitioned into n pairwise disjoint edge sets $M'_0, M'_1, \dots, M'_{n-1}$ such that $M'_j = M_j \cup \{uv_{2[2j]}\}$ ($j \in [0, n - 1]$) which consists of a 2-path $uv_{2[2j]}v_{1[j]}$ and $n - 1$ isolated edges $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j - 1] \cup [j + 1, n - 1]$), where $2j, i + j$ are taken modulo n .

Thus, each edge $v_{1[i]}v_{2[i+j]}$ ($i \in [0, j - 1] \cup [j + 1, n - 1]$) in M'_j corresponds to a subgraph isomorphic to $K_{6,4}$ and the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds to a subgraph isomorphic to $K_{7,4}$ with partition $(V_{1j} \cup \{u\}, V_{2(2j)})$ in $K_{6n+1,4n}$, where $2j$ is taken modulo n . Let

$$\begin{aligned} V_{2(2j)} &= \{v_{2[2j]0}, v_{2[2j]1}, v_{2[2j]2}, v_{2[2j]3}\}, \\ V_{1j} &= \{v_{1[j]0}, v_{1[j]1}, \dots, v_{1[j]5}\}, \end{aligned} \quad (8)$$

where the indices $2j$ are taken modulo n .

After taking away five edges $v_{1[j]0}v_{2[2j]1}, v_{1[j]1}v_{2[2j]1}, v_{1[j]2}v_{2[2j]2}, v_{1[j]4}v_{2[2j]0}$, and $v_{1[j]4}v_{2[2j]3}$ from each $K_{7,4}$ to which the 2-path $uv_{2[2j]}v_{1[j]}$ corresponds, where the indices $2j$ are taken modulo n , each M'_j corresponds to three linear 4-forests of $K_{6n+1,4n}$ by Lemma 13. The edges that we take away form

one linear 4-forest (see Figure 3). Hence $la_4(K_{6n+1,4n}) \leq 3n + 1$ by Lemma 11. Since

$$la_4(K_{6n+1,4n}) \geq \left\lceil \frac{(6n+1)4n}{[4 \cdot (6n+1+4n)/5]} \right\rceil = 3n + 1 \quad (9)$$

by Lemma 3, we obtain $la_4(K_{6n+1,4n}) = 3n + 1$. $\square$

Corollary 19. $la_4(K_{6n+1,4n-1}) = 3n + 1$, where n is odd.

Proof. The results can be obtained by Lemma 3 and Theorem 18 immediately. $\square$

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