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Research Article

Class of Multivalent Analytic Functions Defined by a Linear Operator

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Making use of the linear operator $J_p^m(\lambda, l)$ defined in (Prajapat, 2012), we introduce the class $\mathbb{B}_p^m(\lambda, l, \mu, \alpha)$ of analytic and p-valent functions in the open unit disk $\mathcal{U}$. Furthermore, we obtain some sufficient conditions for starlikeness and close-to-convexity and some angular properties for functions belonging to this class. Several corollaries and consequences of the main results are also considered.

1. Introduction and Definitions

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. In particular, we set $\mathcal{A}_1(1) =: \mathcal{A}$. A function $f(z) \in \mathcal{A}_p(n)$ is said to be in the class $\mathcal{S}_p^*(n, \alpha)$ of p-valently starlike of order α in $\mathcal{U}$ if and only if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathcal{U}\ ;\ 0 \le \alpha < p). \tag{2}$$

Furthermore, a function $f(z) \in \mathcal{A}_p(n)$ is said to be in the class $\mathcal{C}_p(n, \alpha)$ of p-valently close-to-convex of order α in $\mathcal{U}$ if and only if it satisfies the inequality

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha, \quad (z \in \mathcal{U};\ \le \alpha < p). \tag{3}$$

In particular, we write $\mathcal{S}_1^*(1, 0) =: \mathcal{S}^*$ and $\mathcal{C}_1(1, 0) =: \mathcal{C}$, where $\mathcal{S}^*$ and $\mathcal{C}$ are the usual subclasses of $\mathcal{A}$ consisting of functions which are starlike and close-to-convex in $\mathcal{U}$, respectively.

In [1], Prajapat define a generalized multiplier transformation operator $J_p^m(\lambda, l)$ as follows:

$$\begin{aligned}
J_p^{-m}(\lambda, l) f(z) &= \frac{p+l}{\lambda} z^{p-(p+l)/\lambda} \\
&\quad \times \int_0^z t^{(p+l)/\lambda - p - 1} J_p^{-(m-1)}(\lambda, l) f(t)\, dt, \\
&\qquad (z \in \mathcal{U}), \\
&\ \ \vdots \\
J_p^{-2}(\lambda, l) f(z) &= \frac{p+l}{\lambda} z^{p-(p+l)/\lambda} \\
&\quad \times \int_0^z t^{(p+l)/\lambda - p - 1} J_p^{-1}(\lambda, l) f(t)\, dt, \\
&\qquad (z \in \mathcal{U}), \\
J_p^{-1}(\lambda, l) f(z) &= \frac{p+l}{\lambda} z^{p-(p+l)/\lambda} \int_0^z t^{(p+l)/\lambda - p - 1} f(t)\, dt, \\
&\qquad (z \in \mathcal{U}), \\
J_p^{0}(\lambda, l) f(z) &= f(z), \\
J_p^{1}(\lambda, l) f(z) &= \frac{\lambda}{p+l} z^{1+p-(p+l)/\lambda} \left(z^{(p+l)/\lambda - p} f(z)\right)', \\
&\qquad (z \in \mathcal{U}),
\end{aligned}$$

$$J_p^2(\lambda,l)\,f(z) = \frac{\lambda}{p+l} z^{1+p-(p+l)/\lambda} \times \left(z^{(p+l)/\lambda-p} J_p^1(\lambda,l)\,f(z)\right)', \quad (z\in\mathcal{U}),$$

$$\vdots$$

$$J_p^m(\lambda,l)\,f(z) = \frac{\lambda}{p+l} z^{1+p-(p+l)/\lambda} \times \left(z^{(p+l)/\lambda-p} J_p^{m-1}(\lambda,l)\,f(z)\right)', \quad (z\in\mathcal{U}). \tag{4}$$

We see that for $f(z)\in\mathcal{A}_p(n)$, we have

$$J_p^m(\lambda,l)\,f(z) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m a_k z^k, \tag{5}$$

where $\lambda\ge 0, l>-p, p\in\mathbb{N}, m\in\mathbb{Z}=\{0,\pm1,\ldots\}$ and $z\in\mathcal{U}$. It is readily verified from (5) that

$$\lambda z\left(J_p^m(\lambda,l)\,f(z)\right)' = (l+p)\,J_p^{m+1}(\lambda,l)\,f(z) - (l+p(1-\lambda))\,J_p^m(\lambda,l)\,f(z), \quad (\lambda>0),$$

$$\lambda z\left(J_p^m(\lambda,l)\,f(z)\right)'' = (l+p)\,J_p^{m+1}(\lambda,l)\,f'(z) - (l+\lambda+p(1-\lambda))\,J_p^m(\lambda,l)\,f'(z), \quad (\lambda>0). \tag{6}$$

We observe that the operator $J_p^m(\lambda,l)$ generalize several previously studied familiar operators, and we will show some of the interesting particular cases as follows:

(i) $J_p^m(\lambda,l)f(z) = I_p^m(\lambda,l)f(z)$ $(l\ge 0, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ (see [2]);
(ii) $J_p^m(1,l)f(z) = I_p(m,l)f(z)$ $(l\ge 0, m\in\mathbb{N}_0)$ (see [3, 4]);
(iii) $J_p^m(1,0)f(z) = D_p^m f(z)$ $(m\in\mathbb{N}_0)$ (see [5–7]);
(iv) $J_1^m(1,l)f(z) = I_l^m f(z)$ $(l\ge 0, m\in\mathbb{N}_0)$ (see [8, 9]);
(v) $J_1^m(1,0)f(z) = D^m f(z)$ $(m\in\mathbb{N}_0)$ (see [10]);
(vi) $J_1^m(\lambda,0)f(z) = D_\lambda^m f(z)$ $(m\in\mathbb{N}_0)$ (see [11]);
(vii) $J_1^m(1,1)f(z) = \mathcal{D}^m f(z)$ $(m\in\mathbb{N}_0)$ (see [12]);
(viii) $J_p^{-n}(\lambda,l)f(z) = J_p^n(\lambda,l)f(z)$ $(n\in\mathbb{N}_0)$ (see [13]).

(For other generalizations of the operator $J_p^m(\lambda,l)$, see [1]).

Making use of the above operator $J_p^m(\lambda,l)$, we introduce the class $\mathbb{B}_p^m(\lambda,l,\mu,\alpha)$ of analytic and p-valent functions defined as follows.

Definition 1. A function $f(z)\in\mathcal{A}_p(n)$ is said to be a member of the class $\mathbb{B}_p^m(n,\lambda,l,\mu,\alpha)$ if and only if

$$\left|\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)\,f(z)\right)' - p\right| < p-\alpha, \quad (p\in\mathbb{N}), \tag{7}$$

for some α $(0\le\alpha<p)$, $\mu\ge 0, \lambda\ge 0$, $l>-p$, $p\in\mathbb{N}, m\in\mathbb{Z}$ and for all $z\in\mathcal{U}$.

Note that condition (7) implies that

$$\operatorname{Re}\left(\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)\,f(z)\right)'\right) > \alpha. \tag{8}$$

We note that $\mathbb{B}_p^0(n,\lambda,l,\mu,\alpha)\equiv\mathbb{B}_p^1(n,0,l,\mu,\alpha)\equiv\mathcal{B}(p,n,\mu,\alpha)$, the class which has been introduced and studied by the author in [14]. Also, we have $\mathbb{B}_p^0(n,\lambda,l,2,\alpha)\equiv\mathcal{S}_p^*(n,\alpha)$, $\mathbb{B}_p^0(\lambda,l,1,\alpha)\equiv\mathcal{C}_p(n,\alpha)$. The class $\mathbb{B}_1^0(1,\lambda,l,3,\alpha)\equiv\mathcal{B}(\alpha)$ is the class which has been introduced and studied by Frasin and Darus [15] (see also [16, 17]).

In this paper, we obtain some sufficient conditions and some angular properties for functions belonging to the class $\mathbb{B}_p^m(n,\lambda,l,\mu,\alpha)$. Several corollaries and consequences of the main results are also considered.

In order to derive our main results, we have to recall the following lemmas.

Lemma 2 (see [18]). *Let $w(z)$ be analytic in $\mathcal{U}$ and such that $w(0)=0$. Then if $|w(z)|$ attains its maximum value on circle $|z|=r<1$ at a point $z_o\in\mathcal{U}$, one has*

$$z_o w'(z_o) = k w(z_o), \tag{9}$$

where $k\ge 1$ is a real number.

Lemma 3 (see [19]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2\times\mathcal{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z)\notin\Omega$ for $z\in\mathcal{U}$, and for all real x,y such that $y\le -n(1+x^2)/2$. If the function $q(z)=1+q_nz^n+q_{n+1}z^{n+1}+\cdots$ is analytic in $\mathcal{U}$ such that $\Phi(q(z),zq'(z);z)\in\Omega$ for all $z\in\mathcal{U}$, then $\operatorname{Re} q(z)>0$.*

Lemma 4 (see [20]). *Let $q(z)$ be analytic in $\mathcal{U}$ with $q(0)=1$ and $q(z)\ne 0$ for all $z\in\mathcal{U}$. If there exist two points z_1, $z_2\in\mathcal{U}$ such that*

$$-\frac{\pi\alpha_1}{2} = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi\alpha_2}{2} \tag{10}$$

for $\alpha_1>0$, $\alpha_2>0$, and for $|z|<|z_1|=|z_2|$, then we have

$$\frac{z_1q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)\beta, \qquad \frac{z_2q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)\beta, \tag{11}$$

where

$$\beta\ge\frac{1-|a|}{1+|a|}, \qquad a = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \tag{12}$$

2. Sufficient Conditions for Starlikeness and Close-to-Convexity

Unless otherwise mentioned, we shall assume in the remainder of this paper that

$$\mu, \gamma \geq 0; \quad \lambda > 0; \quad l > -p; \;\; p, n \in \mathbb{N}, \;\; m \in \mathbb{Z}. \tag{13}$$

Making use of Lemma 2, we first prove the following.

Theorem 5. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left| \frac{l+p}{\lambda} \left(\frac{J_p^{m+1}(\lambda,l) f'(z)}{J_p^m(\lambda,l) f'(z)} - (\mu-1) \frac{J_p^{m+1}(\lambda,l) f(z)}{J_p^m(\lambda,l) f(z)} + \mu - 2 \right) + \gamma \left(\left(\frac{z^p}{J_p^m(\lambda,l) f(z)} \right)^{\mu-1} z^{1-p} (J_p^m(\lambda,l) f(z))' - p \right) \right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \quad (z \in \mathcal{U}), \tag{14}$$

for some α $(0 \leq \alpha < p)$, then $f(z) \in \mathbb{B}_p^m(n,\lambda,l,\mu,\alpha)$.

Proof. Define the function $w(z)$ by

$$\left(\frac{z^p}{J_p^m(\lambda,l) f(z)} \right)^{\mu-1} z^{1-p} \left(J_p^m(\lambda,l) f(z) \right)' = p + (p-\alpha) w(z). \tag{15}$$

Then $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$. It follows from (15) and the identities (6) and (1.6) that

$$\frac{l+p}{\lambda} \left(\frac{J_p^{m+1}(\lambda,l) f'(z)}{J_p^m(\lambda,l) f'(z)} - (\mu-1) \frac{J_p^{m+1}(\lambda,l) f(z)}{J_p^m(\lambda,l) f(z)} + \mu - 2 \right) + \gamma \left(\left(\frac{z^p}{J_p^m(\lambda,l) f(z)} \right)^{\mu-1} z^{1-p} \left(J_p^m(\lambda,l) f(z) \right)' - p \right) = \gamma (p-\alpha) w(z) + \frac{(p-\alpha) z w'(z)}{p + (p-\alpha) w(z)}. \tag{16}$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that

$$\max_{|z|<z_0} |w(z)| = |w(z_0)| = 1. \tag{17}$$

Then from Lemma 2, we have (9). Therefore, letting $w(z_0) = e^{i\theta}$, we obtain that

$$\begin{aligned} &\left| \frac{l+p}{\lambda} \left(\frac{J_p^{m+1}(\lambda,l) f'(z_0)}{J_p^m(\lambda,l) f'(z_0)} - (\mu-1) \frac{J_p^{m+1}(\lambda,l) f(z_0)}{J_p^m(\lambda,l) f(z_0)} + \mu - 2 \right) + \gamma \left(\left(\frac{z_0^p}{J_p^m(\lambda,l) f(z_0)} \right)^{\mu-1} z^{1-p} \left(J_p^m(\lambda,l) f(z_0) \right)' - p \right) \right| \\ &= \left| \gamma (p-\alpha) w(z_0) + \frac{(p-\alpha) z w'(z_0)}{p + (p-\alpha) w(z_0)} \right| \\ &\geq \operatorname{Re} \left\{ \gamma (p-\alpha) + \frac{(p-\alpha) k}{p + (p-\alpha) w(z_0)} \right\} \\ &> \gamma (p-\alpha) + \frac{p-\alpha}{2p-\alpha} \\ &= \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}. \end{aligned} \tag{18}$$

Which contradicts our assumption (14). Therefore we have $|w(z)| < 1$ in $\mathcal{U}$. Finally, we have

$$\left| \left(\frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right| = (p-\alpha) |w(z)| < p - \alpha \quad (z \in \mathcal{U}), \tag{19}$$

that is, $f(z) \in \mathbb{B}_p^m(n,\lambda,l,\mu,\alpha)$. This completes the proof of the theorem. □

Putting $m = l = 0$ and $\lambda = 1$ in Theorem 5, we obtain the following.

Corollary 6. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left| 1 + \frac{z f''(z)}{f'(z)} - p + (\mu-1) \times \left(p - \frac{z f'(z)}{f(z)} \right) + \gamma \left(\left(\frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right) \right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \quad (z \in \mathcal{U}), \tag{20}$$

for some α $(0 \leq \alpha < p)$, then $f(z) \in \mathscr{B}(p,n,\mu,\alpha)$.

Putting $\mu = \lambda = 1$ and m = l = 0 in Theorem 5, one obtains the following.

Corollary 7. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} - p + \gamma\left(z^{1-p}f'(z) - p\right)\right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha} \quad (z \in \mathscr{U}), \tag{21}$$

for some α $(0 \le \alpha < p)$, *then* $f(z) \in \mathscr{C}_p(n,\alpha)$.

Letting $\gamma = p = n = 1$ in Corollary 7, one has

Corollary 8. *If $f(z) \in \mathscr{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} + f'(z) - 1\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \quad (z \in \mathscr{U}), \tag{22}$$

for some α $(0 \le \alpha < 1)$, *then* $f(z) \in \mathscr{C}(\alpha)$. *In particular, if* $f(z) \in \mathscr{A}$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)} + f'(z) - 1\right| < \frac{3}{2} \quad (z \in \mathscr{U}), \tag{23}$$

then $f(z)$ is close-to-convex in $\mathscr{U}$.

Putting $\mu = 2$, $\lambda = 1$, and $l = 0$ in Theorem 5, one obtains the following.

Corollary 9. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \gamma\left(\left(\frac{z^p}{f(z)}\right)z^{1-p}\left(f'(z) - p\right)\right)\right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \tag{24}$$

for some α $(0 \le \alpha < p)$, *then* $f(z) \in \mathscr{S}_p^*(n,\alpha)$.

Putting $p = \gamma = n = 1$ in Corollary 9, one easily obtains the following result due to Owa [21].

Corollary 10. *If $f(z) \in \mathscr{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha}, \quad (z \in \mathscr{U}), \tag{25}$$

for some $\alpha (0 \le \alpha < 1)$, *then* $f(z) \in \mathscr{S}^*(\alpha)$. *In particular, if* $f(z) \in \mathscr{A}$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2} \quad (z \in \mathscr{U}), \tag{26}$$

then $f(z)$ is starlike in $\mathscr{U}$.

Remark 11. We note that the results obtained by the author [14, Theorem 2.1, Corollaries 2.2–2.5] are not corrects. The correct results are given by Corollaries 6, 7, and 9.

Next we prove the following.

Theorem 12. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\begin{aligned}\operatorname{Re}\Bigg\{&\left[\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)f(z)\right)'\right] \\ &\left(\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)f(z)\right)' + \frac{l+p}{\lambda}\right. \\ &\left.\times\left(\frac{J_p^{m+1}(\lambda,l)f'(z)}{J_p^m(\lambda,l)f'(z)} - (\mu-1)\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right)\right)\Bigg\} \\ &> \delta\left(\frac{l(2-\mu)}{\lambda} + \delta + \frac{n}{2}\right) + p\left(\frac{\delta(2-\mu)}{\lambda} - \frac{n}{2}\right).\end{aligned} \tag{27}$$

then $f(z) \in \mathbb{B}_p^m(n,\lambda,l,\mu,\alpha)$, *where* $0 \le \delta < p$.

Proof. Define the function $q(z)$ by

$$\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)f(z)\right)' = \delta + (p-\delta)q(z). \tag{28}$$

Then, we see that $q(z) = 1 + q_nz^n + q_{n+1}z^{n+1} + \cdots$ is analytic in $\mathscr{U}$. A computation shows that

$$\begin{aligned}&\left[\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)f(z)\right)'\right] \\ &\Bigg\{\left(\frac{z^p}{J_p^m(\lambda,l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda,l)f(z)\right)' \\ &\quad+\frac{l+p}{\lambda}\left(\frac{J_p^{m+1}(\lambda,l)f'(z)}{J_p^m(\lambda,l)f'(z)} - (\mu-1)\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right)\Bigg\} \\ &= (p-\delta)zq'(z) + (p-\delta)^2q^2(z) + (p-\delta)q(z) \\ &\quad\times\left(\frac{(l+p)(2-\mu)}{\lambda} + 2\delta\right) + \frac{\delta(l+p)(2-\mu)}{\lambda} + \delta^2 \\ &= \Phi\left(q(z), zq'(z); z\right),\end{aligned} \tag{29}$$

where

$$\begin{aligned}\Phi(r,s;t) &= (p-\delta)s + (p-\delta)^2r^2 + (p-\delta)r\left(\frac{(l+p)(2-\mu)}{\lambda} + 2\delta\right) \\ &\quad+ \frac{\delta(l+p)(2-\mu)}{\lambda} + \delta^2.\end{aligned} \tag{30}$$

For all real x, y satisfying $y \le -n(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\Phi(ix, y; z) &= (p-\delta)y - (p-\delta)^2 x^2 \\ &\quad + \frac{\delta(l+p)(2-\mu)}{\lambda} + \delta^2 \\ &\le -\frac{n}{2}(p-\delta) - (p-\delta)\left[\frac{n}{2} + p - \delta\right]x^2 \\ &\quad + \frac{\delta(l+p)(2-\mu)}{\lambda} + \delta^2 \\ &\le \frac{\delta(l+p)(2-\mu)}{\lambda} + \delta^2 - \frac{n}{2}(p-\delta) \\ &= \delta\left(\frac{l(2-\mu)}{\lambda} + \delta + \frac{n}{2}\right) + p\left(\frac{\delta(2-\mu)}{\lambda} - \frac{n}{2}\right).\end{aligned} \tag{31}$$

Let $\Omega = \{w : \operatorname{Re} w > \delta(l(2-\mu)/\lambda + \delta + n/2) + p(\delta(2-\mu)/\lambda - n/2)\}$. Then $\Phi(q(z), zq'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \le -n(1+x_2^2)/2$, $z \in \mathcal{U}$. By using Lemma 3, we have $\operatorname{Re} q(z) > 0$, that is, $f(z) \in \mathbb{B}_p^m(n, \lambda, l, \mu, \alpha)$. □

Putting $\mu = \lambda = 1$ and $m = l = 0$ in Theorem 12, we have the following.

Corollary 13 (see [14]). *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\begin{aligned}&\operatorname{Re}\left\{\left(z^{1-p}f'(z)\right)^2 + z^{1-p}f'(z) + z^{2-p}f''(z)\right\} \\ &\quad > \delta\left(\delta + \frac{n}{2}\right) + p\left(\delta - \frac{n}{2}\right).\end{aligned} \tag{32}$$

then $f(z) \in \mathcal{C}_p(n, \delta)$, where $0 \le \delta < p$. In particular, if $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left\{\left(f'(z)\right)^2 + f'(z) + zf''(z)\right\} > -\frac{1}{2} \tag{33}$$

then $f(z)$ is close-to-convex in $\mathcal{U}$.

Putting $m = l = 0$, $\lambda = 1$, and $\mu = 2$ in Theorem 12, one has the following.

Corollary 14 (see [14]). *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) > \delta\left(\delta + \frac{n}{2}\right) - \frac{n}{2}p, \tag{34}$$

then $f(z) \in \mathcal{S}_p^*(n, \delta)$, where $0 \le \delta < p$. In particular, if $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) > -\frac{1}{2}. \tag{35}$$

then $f(z)$ is starlike in $\mathcal{U}$.

3. Argument Properties

Finally, we prove the following.

Theorem 15. *Suppose that*

$$\left(\frac{z^p}{J_p^m(\lambda, l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda, l)f(z)\right)' \ne \delta \tag{36}$$

for $z \in \mathcal{U}$ and $0 \le \delta < p$. If $f(z) \in \mathcal{A}_p(n)$ satisfies

$$\begin{aligned}&-\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right) \\ &\quad < \arg\left\{\left(\left(\frac{z^p}{J_p^m(\lambda, l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda, l)f(z)\right)'\right)\right. \\ &\qquad \cdot\left(\left[\frac{l+p}{\lambda}\left(\frac{J_p^{m+1}(\lambda, l)f'(z)}{J_p^m(\lambda, l)f'(z)}\right.\right.\right. \\ &\qquad \left.\left.\left. -(\mu-1)\frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)} + \mu - 2\right)\right]\right. \\ &\qquad \left.\left. + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\} \\ &\quad < \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right).\end{aligned} \tag{37}$$

for $\alpha_1, \alpha_2, \gamma > 0$, then

$$\begin{aligned}&-\frac{\pi}{2}\alpha_1 < \arg\left(\left(\frac{z^p}{J_p^m(\lambda, l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda, l)f(z)\right)' - \delta\right) \\ &\quad < \frac{\pi}{2}\alpha_2.\end{aligned} \tag{38}$$

Proof. Define the function $q(z)$ by

$$q(z) = \frac{1}{p-\delta}\left(\left(\frac{z^p}{J_p^m(\lambda, l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda, l)f(z)\right)' - \delta\right). \tag{39}$$

Then we see that $q(z)$ analytic in $\mathcal{U}$, $q(0) = 1$, and $q(z) \ne 0$ for all $z \in \mathcal{U}$. It follows from (39) that

$$\begin{aligned}&\left(\left(\frac{z^p}{J_p^m(\lambda, l)f(z)}\right)^{\mu-1} z^{1-p}\left(J_p^m(\lambda, l)(f(z))'\right)\right) \\ &\quad \cdot\left(\left[\frac{l+p}{\lambda}\left(\frac{J_p^{m+1}(\lambda, l)f'(z)}{J_p^m(\lambda, l)f'(z)} - (\mu-1)\frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)}\right.\right.\right. \\ &\qquad \left.\left.\left. + \mu - 2\right)\right] + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta} \\ &= (p-\delta)zq'(z) + \gamma q(z).\end{aligned} \tag{40}$$

Suppose that there exists two points $z_1, z_2 \in \mathcal{U}$ such that the condition (10) is satisfied, then by Lemma 4, we obtain (11) under the constraint (12). Therefore, we have

$$\begin{aligned}
&\arg\left(\gamma q(z_1) + (p-\delta) z q'(z_1)\right) \\
&\quad = \arg q(z_1) + \arg\left(\gamma + (p-\delta)\frac{z_1 q'(z_1)}{q(z_1)}\right) \\
&\quad = -\frac{\pi}{2}\alpha_1 + \arg\left(\gamma - i\frac{(\alpha_1+\alpha_2)(p-\delta)}{2}\beta\right) \\
&\quad = -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\beta\right) \\
&\quad \le \frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right), \\
&\arg\left(\gamma q(z_2) + (p-\delta) z q'(z_2)\right) \ge \frac{\pi}{2}\alpha_2 \\
&\quad + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),
\end{aligned} \tag{41}$$

which contradict the assumption of the theorem. This completes the proof. □

Putting $\mu = \lambda = 1$ and $m = l = 0$ in Theorem 15, one has the following.

Corollary 16 (see [14]). *Suppose that $z^{1-p} f'(z) \neq \delta$ for $z \in \mathcal{U}$ and $0 \le \delta < p$. If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\begin{aligned}
&-\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right) \\
&\quad < \arg\left\{z^{1-p} f'(z)\left(1 + \frac{z f''(z)}{f'(z)} - p + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\} \\
&\quad < \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),
\end{aligned} \tag{42}$$

for $\alpha_1, \alpha_2, \gamma > 0$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(z^{1-p} f'(z) - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{43}$$

In particular, if $f(z) \in \mathcal{A}$ satisfies

$$\begin{aligned}
&\left|\arg\left(z f''(z) + f'(z)(\gamma+1) - \frac{z\left(f'(z)\right)^2}{f(z)}\right)\right| \\
&\quad < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\gamma},
\end{aligned} \tag{44}$$

for $\gamma > 0$, then

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha, \quad (0 < \alpha \le 1). \tag{45}$$

Putting $\mu = 2$, $\lambda = 1$, and $l = 0$ in Theorem 15, one has the following.

Corollary 17 (see [14]). *Suppose that $z f'(z)/f(z) \neq \delta$ for $z \in \mathcal{U}$ and $0 \le \delta < p$. If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\begin{aligned}
&-\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right) \\
&\quad < \arg\left\{\frac{z f'(z)}{f(z)}\left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\} \\
&\quad < \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),
\end{aligned} \tag{46}$$

for $\alpha_1, \alpha_2, \gamma > 0$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z f'(z)}{f(z)} - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{47}$$

In particular, if $f(z) \in \mathcal{A}$ satisfies

$$\begin{aligned}
&\left|\arg\left(\frac{z^2 f''(z)}{f(z)} - \left(\frac{z f'(z)}{f(z)}\right)^2 + \frac{z f'(z)}{f(z)}(\gamma+1)\right)\right| \\
&\quad < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\gamma},
\end{aligned} \tag{48}$$

for $\gamma > 0$, then

$$\left|\arg\frac{z f'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha, \quad (0 < \alpha \le 1). \tag{49}$$

Remark 18. Taking different choices of m, p, λ, and l in the above theorems, we obtain some sufficient conditions for starlikeness and close-to-convexity and some angular properties for functions belonging to new classes defined by the previously operators mentioned in Section 1.

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