

Research Article

Bounded Nonlinear Functional Derived by the Generalized Srivastava-Owa Fractional Differential Operator

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By making use of the generalized Srivastava-Owa fractional differential operator, a class of analytical functions is imposed. The sharp bound for the nonlinear functional associated with the Hankel determinant is computed. We consider a new technique to prove our results. Important properties such as inclusion, subordination, and Hadamard product are studied. Some recent results are included.

1. Introduction

Fractional calculus (real and complex) is a rapidly growing subject of interest for physicists and mathematicians. The reason for this is that problems may be discussed in a much more stringent and elegant way than using traditional methods. Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. Several different derivatives were introduced: Riemann-Liouville, Hadamard, Grunwald-Letnikov, Riesz, Erdelyi-Kober operators, and Caputo [1–7].

Recently, the theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been employed for imposing, for example, the characterization properties, coefficient estimates [8], distortion inequalities [9], and convolution structures for various subclasses of analytic functions and the works in the research monographs. In [10], Srivastava and Owa defined the fractional operators (derivative and integral) in the complex z -plane \mathbb{C} as follows.

Definition 1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \leq \alpha < 1, \quad (1)$$

where the function $f(z)$ is analytical in simply-connected region of the complex z -plane \mathbb{C} containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta) (z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0, \quad (2)$$

where the function $f(z)$ is analytical in simply connected region of the complex z -plane (\mathbb{C}) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

In [11], the author generalized a formula for the fractional integral as follows: for natural $n \in \mathbb{N} = \{1, 2, \dots\}$ and real μ , the n -fold integral of the form

$$I_z^{\alpha, \mu} f(z) = \int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta_2^\mu d\zeta_2 \cdots \int_0^{\zeta_{n-1}} \zeta_n^\mu f(\zeta_n) d\zeta_n. \quad (3)$$

Employing the Dirichlet technique implies

$$\begin{aligned} \int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta^\mu f(\zeta) d\zeta &= \int_0^z \zeta^\mu f(\zeta) d\zeta \int_\zeta^z \zeta_1^\mu d\zeta_1 \\ &= \frac{1}{\mu + 1} \int_0^z (z^{\mu+1} - \zeta^{\mu+1}) \zeta^\mu f(\zeta) d\zeta. \end{aligned} \quad (4)$$

Repeating the above step $n - 1$ times yields

$$\begin{aligned} \int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta_2^\mu d\zeta_2 \cdots \int_0^{\zeta_{n-1}} \zeta_n^\mu f(\zeta_n) d\zeta_n \\ = \frac{(\mu + 1)^{1-n}}{(n - 1)!} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{n-1} \zeta^\mu f(\zeta) d\zeta, \end{aligned} \quad (5)$$

which imposes the fractional operator type

$$I_z^{\alpha, \mu} f(z) = \frac{(\mu + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{\alpha-1} \zeta^\mu f(\zeta) d\zeta, \quad (6)$$

where α and $\mu \neq -1$ are real numbers and the function $f(z)$ is analytic in simply connected region of the complex z -plane \mathbb{C} containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. When $\mu = 0$, we arrive at the standard Srivastava-Owa fractional integral. Further information can be found in [11].

Corresponding to the fractional integral operator, the fractional differential operator is

$$\begin{aligned} D_z^{\alpha, \mu} f(z) &:= \frac{(\mu + 1)^\alpha}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^\mu f(\zeta)}{(z^{\mu+1} - \zeta^{\mu+1})^\alpha} d\zeta; \\ &0 \leq \alpha < 1, \end{aligned} \quad (7)$$

where the function $f(z)$ is analytical in simply connected region of the complex z -plane \mathbb{C} containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. We have

$$D_z^{\alpha, \mu} z^\nu = \frac{(\mu + 1)^{\alpha-1} \Gamma((\nu/(\mu + 1)) + 1)}{\Gamma((\nu/(\mu + 1)) + 1 - \alpha)} z^{(1-\alpha)(\mu+1)+\nu-1}. \quad (8)$$

Let \mathcal{A} denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (9)$$

Also, let $\mathcal{S}, \mathcal{S}^*$ and \mathcal{C} denote the subclasses of \mathcal{A} consisting of functions which are, respectively, univalent, starlike $\Re(zf'(z)/f(z)) > 0$, and convex $\Re(1 + (z^2 f''(z)/f'(z))) > 0$ in U . It is well known that, if the function $f(z)$ given by (9) is in the class \mathcal{S} , then $|a_n| \leq n$, $n \in \mathbb{N} \setminus \{1\}$. Moreover, if the function $f(z)$ given by (9) is in the class \mathcal{C} , then $|a_n| \leq 1$, $n \in \mathbb{N}$.

In our present investigation, we will also make use of the Fox-Wright generalization ${}_q\Psi_p[z]$ of the hypergeometric ${}_qF_p$ function defined by [12]

$$\begin{aligned} {}_q\Psi_p \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_q, A_q); \\ (\beta_1, B_1), \dots, (\beta_p, B_p); \end{matrix} z \right] \\ = {}_q\Psi_p \left[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z \right] \\ := \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \cdots \Gamma(\alpha_q + nA_q)}{\Gamma(\beta_1 + nB_1) \cdots \Gamma(\beta_p + nB_p)} \frac{z^n}{n!} \\ = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^q \Gamma(\alpha_j + nA_j)}{\prod_{j=1}^p \Gamma(\beta_j + nB_j)} \frac{z^n}{n!}, \end{aligned} \quad (10)$$

where $A_j > 0$ for all $j = 1, \dots, q$, $B_j > 0$ for all $j = 1, \dots, p$, and $1 + \sum_{j=1}^p B_j - \sum_{j=1}^q A_j \geq 0$ for suitable values $|z| < 1$, and α_i, β_j are complex parameters.

It is well known that

$$\begin{aligned} {}_q\Psi_p \left[\begin{matrix} (\alpha_1, 1), \dots, (\alpha_q, 1); \\ (\beta_1, 1), \dots, (\beta_p, 1); \end{matrix} z \right] \\ = \Lambda^{-1} {}_qF_p(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p; z), \end{aligned} \quad (11)$$

where

$$\Lambda := \frac{\prod_{j=1}^p \Gamma(\beta_j)}{\prod_{i=1}^q \Gamma(\alpha_i)}, \quad (12)$$

and ${}_qF_p$ is the generalized hypergeometric function.

Now by making use of the operator (7), we introduce the following extension operator $\Phi^{\alpha, \mu} : \mathcal{A} \rightarrow \mathcal{A}$:

$$\begin{aligned} \Phi^{\alpha, \mu} f(z) &:= \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{(\mu + 1)^{\alpha-1} \Gamma((1/(\mu + 1)) + 1)} \\ &\times z^{\alpha-\mu+\mu\alpha} D_z^{\alpha, \mu} f(z) \\ &= \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{(\mu + 1)^{\alpha-1} \Gamma((1/(\mu + 1)) + 1)} z^{\alpha-\mu+\mu\alpha} D_z^{\alpha, \mu} \\ &\times \left(z + \sum_{n=2}^{\infty} a_n z^n \right) \\ &= \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{(\mu + 1)^{\alpha-1} \Gamma((1/(\mu + 1)) + 1)} z^{\alpha-\mu+\mu\alpha} \end{aligned}$$

$$\begin{aligned}
 & \times \left[\frac{(\mu + 1)^{\alpha-1} \Gamma((1/(\mu + 1)) + 1)}{\Gamma((1/(\mu + 1)) + 1 - \alpha)} z^{-\alpha+\mu-\mu\alpha+1} \right. \\
 & \quad \left. + \sum_{n=2}^{\infty} \frac{\Gamma((n/(\mu + 1)) + 1)}{\Gamma((n/(\mu + 1)) + 1 - \alpha)} a_n z^{n-\alpha+\mu-\mu\alpha} \right] \\
 & = z + \sum_{n=2}^{\infty} \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{\Gamma((1/(\mu + 1)) + 1)} \\
 & \quad \times \frac{\Gamma((n/(\mu + 1)) + 1)}{\Gamma((n/(\mu + 1)) + 1 - \alpha)} a_n z^n \\
 & := z + \sum_{n=2}^{\infty} \phi_n^{\alpha,\mu} a_n z^n.
 \end{aligned} \tag{13}$$

Obviously, when $\mu = 0$, we have the extension fractional differential operator defined in [13] ([14] for recent work), which contains the Carlson-Shaffer operator. In term of the Fox-Wright generalized function,

$$\begin{aligned}
 \Phi^{\alpha,\mu} f(z) & = z + \sum_{n=2}^{\infty} \frac{\Gamma(n + 1) \Gamma((1/(\mu + 1)) + 1 - \alpha)}{\Gamma((1/(\mu + 1)) + 1)} \\
 & \quad \times \frac{\Gamma((n/(\mu + 1)) + 1)}{\Gamma((n/(\mu + 1)) + 1 - \alpha)} \frac{a_n}{n!} z^n \\
 & = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1) \Gamma((1/(\mu + 1)) + 1 - \alpha)}{\Gamma((1/(\mu + 1)) + 1)} \\
 & \quad \times \frac{\Gamma((n/(\mu + 1)) + 1)}{\Gamma((n/(\mu + 1)) + 1 - \alpha)} \frac{a_n}{n!} z^n \\
 & = \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{\Gamma((1/(\mu + 1)) + 1)} \\
 & \quad \times {}_2\Psi_1 \left[\begin{matrix} (1, 1), \left(1, \frac{1}{\mu + 1}\right); \\ \left(1 - \alpha, \frac{1}{\mu + 1}\right); \end{matrix} z \right] * f(z) \\
 & = \frac{\Gamma((1/(\mu + 1)) + 1 - \alpha)}{\Gamma((1/(\mu + 1)) + 1)} {}_2\Psi_1 [z] * f(z) \\
 & := \Psi(\alpha, \mu; z) * f(z),
 \end{aligned} \tag{14}$$

where $a_0 = 0, a_1 = 1$, and $*$ is the Hadamard product. Note that

$$\begin{aligned}
 \Phi^{0,0} f(z) & = f(z), & \Phi^{1,0} f(z) & = z f'(z), \\
 (\Phi^{\alpha,0} f)(z) & = (L(2, 2 - \alpha) f)(z),
 \end{aligned} \tag{15}$$

where $L(a, c)f$ is the Carlson-Shaffer operator. Moreover, operator (14) can be viewed as a linear operator which is essentially analogous to the Dziok-Srivastava operator whenever used instead of the Fox-Wright generalization of the hypergeometric function.

Recently, various results, such as convolution and inclusion properties, distortion theorem, extreme points, and coefficient estimates, are proposed by many authors for the operators due to Srivastava involving the Wright function, generalized hypergeometric function, and Meijer's G -functions. These operators are Dziok-Srivastava, Srivastava-Wright, Cho-Kwon-Srivastava operator, Cho-Saigo-Srivastava operator, Jung-Kim-Srivastava, and Srivastava-Owa operators (see [15–24]). Going on in this generalization, we have finally the Erdelyi-Kober operator of fractional integration with three parameters used in [25].

Definition 3 (subordination principal). For two functions f and g analytical in U , we say that the function $f(z)$ is subordinated to $g(z)$ in U and write $f(z) \prec g(z) (z \in U)$, if there exists a Schwartz function $w(z)$ analytical in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in U$. In particular, if the function $g(z)$ is univalent in U , the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 4. For the function f defined by (9), the Hankel determinant of f is defined by

$$\begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}. \tag{16}$$

Now we proceed to define a new class of analytic function involving the operator (13).

Definition 5. The function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_{\alpha,\mu}(\theta, \rho)$, where $0 \leq \alpha < 1, \mu \geq 0, |\theta| < \pi/2, 0 \leq \rho \leq 1$, if it satisfies the inequality

$$\Re \left\{ e^{i\theta} \frac{\Phi^{\alpha,\mu} f(z)}{z} \right\} > \rho \cos \theta, \quad (z \in U). \tag{17}$$

Consequently, from Definition 4, we have

$$\begin{aligned}
 f \in \mathcal{R}_{\alpha,\mu}(\theta, \rho) & \iff e^{i\theta} \frac{\Phi^{\alpha,\mu} f(z)}{z} \\
 & = [(1 - \rho) p(z) + \rho] \cos \theta + i \sin \theta,
 \end{aligned} \tag{18}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \dots, z \in U$ satisfies the following properties [26]:

- (i) $|c_n| \leq 2$ and $\Re(p(z)) > 0$,
- (ii) $2c_2 = c_1^2 + w(4 - c_1^2)$,
- (iii) $4c_3 = c_1^3 + 2(4 - c_1^2)c_1 w - c_1(4 - c_1^2)w^2 + 2(4 - c_1^2)(1 - |w|^2)z, (|w| \leq 1, |z| \leq 1)$.

We denote this class by \mathcal{P} .

Note that

$$\mathcal{R}_{\alpha,0}(0, \rho) = \Re \left\{ \frac{\Phi^{\alpha,0} f(z)}{z} \right\} > \rho \tag{20}$$

(see [27]),

$$\mathcal{R}_{\alpha,0}(\theta, \rho) = \mathbb{R} \left\{ e^{i\theta} \frac{\Phi^{\alpha,0} f(z)}{z} \right\} > \rho \cos \theta \quad (21)$$

(see [28]).

It is well known that, for the univalent function f of the form (9), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. In the recent paper, we assume the Hankel determinant for $n = 2, q = 2$ and calculate the sharp bound for the functional $|a_2 a_4 - a_3^2|$ for $f \in \mathcal{R}_{\alpha,\mu}(\theta, \rho)$. Properties of this class are illustrated, and some well-known results are generalized. For this purpose, we need the following preliminary in the sequel, which can be found in [29].

Lemma 6. Let ϕ and ψ be univalent convex in U . Then, the Hadamard product $\phi * \psi$ is also univalent convex function in U .

Lemma 7. Let Φ and Ψ be univalent convex in U , and $\phi < \Phi$ and $\psi < \Psi$. Then, $\phi * \psi < \Phi * \Psi$.

Lemma 8. Let ϕ and ψ be starlike of order $1/2$ then, for function Φ satisfying $\Re(\Phi(z)) > \sigma$ ($\sigma \in [0, 1)$),

$$\mathbb{R} \left(\frac{\phi(z) * \Phi(z) \psi(z)}{\phi(z) * \psi(z)} \right) > \sigma, \quad (z \in U). \quad (22)$$

2. Main Results

We have the following result.

Theorem 9. Let the function f be in the class $\mathcal{R}_{\alpha,\mu}(\theta, \rho)$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} (80\beta + 16\mu_3^2), \quad (23)$$

where $\beta := |\mu_3^2 - \mu_2 \mu_4|$ and

$$\begin{aligned} \mu_1 &= \frac{\Gamma((1/(\mu+1)) + 1 - \alpha)}{\Gamma((1/(\mu+1)) + 1)}, \\ \mu_2 &= \frac{\Gamma((2/(\mu+1)) + 1)}{\Gamma((2/(\mu+1)) + 1 - \alpha)}, \\ \mu_3 &= \frac{\Gamma((3/(\mu+1)) + 1)}{\Gamma((3/(\mu+1)) + 1 - \alpha)}, \\ \mu_4 &= \frac{\Gamma((4/(\mu+1)) + 1)}{\Gamma((4/(\mu+1)) + 1 - \alpha)}. \end{aligned} \quad (24)$$

The estimate (23) is sharp.

Proof. Since $f \in \mathcal{R}_{\alpha,\mu}(\theta, \rho)$, then

$$\begin{aligned} e^{i\theta} \frac{\Phi^{\alpha,\mu} f(z)}{z} \\ = [(1-\rho)(1 + c_1 z + c_2 z^2 + \dots) + \rho] \cos \theta + i \sin \theta. \end{aligned} \quad (25)$$

Comparing the coefficients of (13) and (25), we receive

$$\begin{aligned} a_2 &= \frac{(1-\rho)c_1 \cos \theta}{e^{i\theta} \mu_1 \mu_2}, \quad a_3 = \frac{(1-\rho)c_2 \cos \theta}{e^{i\theta} \mu_1 \mu_3}, \\ a_4 &= \frac{(1-\rho)c_3 \cos \theta}{e^{i\theta} \mu_1 \mu_4}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \mu_1 &= \frac{\Gamma((1/(\mu+1)) + 1 - \alpha)}{\Gamma((1/(\mu+1)) + 1)}, \\ \mu_2 &= \frac{\Gamma((2/(\mu+1)) + 1)}{\Gamma((2/(\mu+1)) + 1 - \alpha)}, \\ \mu_3 &= \frac{\Gamma((3/(\mu+1)) + 1)}{\Gamma((3/(\mu+1)) + 1 - \alpha)}, \\ \mu_4 &= \frac{\Gamma((4/(\mu+1)) + 1)}{\Gamma((4/(\mu+1)) + 1 - \alpha)}. \end{aligned} \quad (27)$$

Therefore, (26) implies

$$|a_2 a_4 - a_3^2| = \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2} \left| \frac{c_1 c_3}{\mu_2 \mu_4} - \frac{c_2^2}{\mu_3^2} \right|. \quad (28)$$

By letting $c_1 := c$ and using (i)–(iii), we have

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} \\ &\times |c^4 (\mu_3^2 - \mu_2 \mu_4) + 2wc^2 (4 - c^2) \\ &\times (\mu_3^2 - \mu_2 \mu_4) - w^2 (4 - c^2) \\ &\times (\mu_3^2 c^2 + \mu_2 \mu_4 (4 - c^2)) \\ &+ (1 - |w|^2) z (2c (4 - c^2) \mu_3^2)|. \end{aligned} \quad (29)$$

By employing the triangle inequality and assuming $|w| := x, \beta := |\mu_3^2 - \mu_2 \mu_4|, c > 0$, and $|z| \leq 1$, we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} \\ &\times \{c^4 \beta + 2xc^2 (4 - c^2) \beta \\ &+ x^2 (4 - c^2) \beta (c^2 + 2c) \\ &+ 2c (4 - c^2) \mu_3^2\} := F(x, c). \end{aligned} \quad (30)$$

Our aim is to maximize F in the interior of the domain $\mathcal{D} = [0, 1] \times [0, 2]$. Since

$$\begin{aligned} \frac{\partial F(x, c)}{\partial x} &= \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} \{2c^2 (4 - c^2) \beta \\ &+ 2x (4 - c^2) \beta (c^2 + 2c)\} > 0, \end{aligned} \quad (31)$$

thus F cannot have a maximum in the interior of \mathcal{D} . Furthermore,

$$\max_{x \in [0,1]} F(x, c) = F(1, c) := H(c), \tag{32}$$

where

$$H(c) \leq \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} [16\beta + 16\beta(4-c^2) + 4(4-c^2)\mu_3^2] := G(c). \tag{33}$$

But

$$\max_{c \in [0,2]} G(c) = G(0); \tag{34}$$

hence the upper bound of (28) is

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\rho)^2 \cos^2 \theta}{\mu_1^2 \mu_2 \mu_4 \mu_3^2} (80\beta + 16\mu_3^2). \tag{35}$$

The equality holds for the functions

$$f(z) = \Psi(\alpha, \mu; z) * e^{i\theta} \left[z \left(\frac{1+(1-2\rho)z^2}{1-z^2} \cos \theta + i \sin \theta \right) \right]. \tag{36}$$

□

Remark 10. Letting $\mu = 0$, we receive a recent result due to Mishra and Gochhayat [28]; putting $\alpha \rightarrow 1, \mu = 0, \rho = 0$, we obtain a result given by Janteng et al. [30].

Theorem 11. Assume that $\theta \in (-\pi/2, \pi/2), \rho \in [0, 1)$ and $\alpha_1, \alpha_2 \in [0, 1)$, with $\alpha_1 < \alpha_2$. If the subordination

$$z\mathcal{G}'' \left(\frac{1}{\mu+1} + 1 - \alpha_2, \frac{1}{\mu+1} + 1 - \alpha_1; z \right) + \mathcal{G}' \left(\frac{1}{\mu+1} + 1 - \alpha_2, \frac{1}{\mu+1} + 1 - \alpha_1; z \right) < 1 + z, \tag{37}$$

where

$$\mathcal{G}(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad (c \neq 0, -1, -2, \dots), \tag{38}$$

holds, then

$$\mathcal{R}_{\alpha_2, \mu}(\theta, \rho) \subset \mathcal{R}_{\alpha_1, \mu}(\theta, \rho). \tag{39}$$

Proof. Let $f \in \mathcal{R}_{\alpha_2, \mu}(\theta, \rho)$. We rewrite

$$\begin{aligned} \Phi^{\alpha_1, \mu} f(z) &= \Psi(\alpha_1, \mu; z) * f(z) \\ &= (\Psi^{(-1)}(\alpha_2, \mu; z) * \Psi(\alpha_2, \mu; z)) \\ &\quad * \Psi(\alpha_1, \mu; z) * f(z) \end{aligned}$$

$$\begin{aligned} &= (\Psi^{(-1)}(\alpha_2, \mu; z) * \Psi(\alpha_1, \mu; z)) \\ &\quad * \Psi(\alpha_2, \mu; z) * f(z) \\ &= \mathcal{G} \left(\frac{1}{\mu+1} + 1 - \alpha_2, \frac{1}{\mu+1} + 1 - \alpha_1; z \right) \\ &\quad * \Phi^{\alpha_2, \mu} f(z), \end{aligned} \tag{40}$$

where

$$\begin{aligned} \Psi(\alpha_2, \mu; z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \Gamma((1/(\mu+1)) + 1 - \alpha)}{\Gamma((1/(\mu+1)) + 1)} \\ &\quad \times \frac{\Gamma((n/(\mu+1)) + 1)}{\Gamma((n/(\mu+1)) + 1 - \alpha)} \frac{1}{n!} z^n, \\ \Psi^{(-1)}(\alpha_2, \mu; z) &= \sum_{n=0}^{\infty} \frac{\Gamma((1/(\mu+1)) + 1)}{\Gamma(n+1) \Gamma((1/(\mu+1)) + 1 - \alpha)} \\ &\quad \times \frac{\Gamma((n/(\mu+1)) + 1 - \alpha)}{\Gamma((n/(\mu+1)) + 1)} n! z^n, \\ \mathcal{G}(a, c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0, -1, -2, \dots), \end{aligned} \tag{41}$$

and $(x)_k = \Gamma(x+k)/\Gamma(x)$ is a Pochhammer symbol. Therefore,

$$\begin{aligned} &\frac{e^{i\theta} \Phi^{\alpha_1, \mu} f(z)}{z} \\ &= \frac{\mathcal{G}((1/(\mu+1)) + 1 - \alpha_2, (1/(\mu+1)) + 1 - \alpha_1; z) * (e^{i\theta} \Phi^{\alpha_2, \mu} f(z)/z) z}{\mathcal{G}((1/(\mu+1)) + 1 - \alpha_2, (1/(\mu+1)) + 1 - \alpha_1; z) * z}. \end{aligned} \tag{42}$$

Assumption (37) implies that $\mathcal{G}((1/(\mu+1)) + 1 - \alpha_2, (1/(\mu+1)) + 1 - \alpha_1; z)$ is convex (see [31, Theorem 1.9]) and consequently $\mathcal{G}((1/(\mu+1)) + 1 - \alpha_2, (1/(\mu+1)) + 1 - \alpha_1; z) \in \mathcal{S}^*(1/2)$ (Marx-Strohhäcker Theorem [32]). Moreover, the function $\psi(z) = z$ is starlike of order $1/2$, then in view of Lemma 8, we obtain that

$$\Re \left(\frac{e^{i\theta} \Phi^{\alpha_1, \mu} f(z)}{z} \right) > \rho \cos \theta, \tag{43}$$

and consequently $\mathcal{R}_{\alpha_2, \mu}(\theta, \rho) \subset \mathcal{R}_{\alpha_1, \mu}(\theta, \rho)$. □

Remark 12. Condition (37) can be replaced by another condition to obtain the convexity of the function \mathcal{G} , such that

$$\left| z\mathcal{G}'' \left(\frac{1}{\mu+1} + 1 - \alpha_2, \frac{1}{\mu+1} + 1 - \alpha_1; z \right) \right| < \frac{1}{2} \tag{44}$$

yields that $\mathcal{G}((1/(\mu+1)) + 1 - \alpha_2, (1/(\mu+1)) + 1 - \alpha_1; z)$ is convex (see [31]).

Theorem 13. Let $f \in \mathcal{S}^*$ and $h \in \mathcal{R}_{\alpha, \mu}(\theta, \rho)$ ($\rho \in [0, 1], \theta \in (-\pi/2, \pi/2), \alpha \in [0, 1), \mu \geq 0$). Then $f * h \in \mathcal{R}_{\alpha, \mu}(\theta, \rho)$.

Proof. By employing the properties of the Hadamard product, we receive

$$\Phi^{\alpha,\mu}(f * h)(z) = f(z) * \Phi^{\alpha,\mu}h(z). \tag{45}$$

Therefore,

$$\frac{e^{i\theta}\Phi^{\alpha,\mu}(f * h)(z)}{z} = \frac{f(z) * ((e^{i\theta}\Phi^{\alpha,\mu}h(z))/z)z}{f(z) * z}. \tag{46}$$

In virtue of Lemma 8, we have

$$\Re\left(\frac{e^{i\theta}\Phi^{\alpha,\mu}(f * h)(z)}{z}\right) > \rho \cos \theta. \tag{47}$$

Hence $f * h \in \mathcal{R}_{\alpha,\mu}(\theta, \rho)$. □

Theorem 14. Let $f \in \mathcal{R}_{\alpha,\mu}(\theta, \rho)$ ($\rho \in [0, 1]$, $\theta \in (-\pi/2, \pi/2)$, $\alpha \in [0, 1]$, $\mu \geq 0$). Then the integral

$$(\mathcal{J}f)(z) = \frac{\tau + 1}{z^\tau} \int_0^z \xi^{\tau-1} f(\xi) d\xi \quad (z \in U, \tau > -1) \tag{48}$$

is also in $\mathcal{R}_{\alpha,\mu}(\theta, \rho)$.

Proof. It is easy to show that

$$\Phi^{\alpha,\mu}(\mathcal{J}f)(z) = \mathcal{G}(\tau + 1, \tau + 2) * \Phi^{\alpha,\mu}f(z). \tag{49}$$

Therefore,

$$\begin{aligned} \frac{e^{i\theta}\Phi^{\alpha,\mu}(\mathcal{J}f)(z)}{z} &= \frac{\mathcal{G}(\tau + 1, \tau + 2) * ((e^{i\theta}\Phi^{\alpha,\mu}f(z))/z)z}{\mathcal{G}(\tau + 1, \tau + 2) * z}. \end{aligned} \tag{50}$$

But $\mathcal{G}(\tau + 1, \tau + 2) \in \mathcal{S}^*(1/2)$, thus in view of Lemma 8, the proof is complete. □

Remark 15. When $\mu = 0$ in Theorems 13 and 14, we have the results given in [28].

Theorem 16. Let $\rho \in [0, 1]$, $\theta \in (-\pi/2, \pi/2)$, $\alpha \in [0, 1]$, $\mu \geq 0$. If the subordination

$$\begin{aligned} &\mathcal{G}''\left(\frac{1}{\mu + 1} + 1 - \alpha, \frac{1}{\mu + 1} + 1; z\right) \\ &- \frac{2}{z}\mathcal{G}'\left(\frac{1}{\mu + 1} + 1 - \alpha, \frac{1}{\mu + 1} + 1; z\right) \\ &+ \frac{2}{z^2}\mathcal{G}\left(\frac{1}{\mu + 1} + 1 - \alpha, \frac{1}{\mu + 1} + 1; z\right) < 1 + z \end{aligned} \tag{51}$$

holds, then $g(z) := (e^{-i\theta}/z)(\mathcal{G}((1/(\mu+1))+1-\alpha, (1/(\mu+1))+1; z) * z(((1+(1-2\rho)z^2)/(1-z^2)) \cos \theta + i \sin \theta))$ is univalent convex function.

Proof. Condition (51) yields that $\mathcal{G}(1/(\mu+1))+1-\alpha, (1/(\mu+1))+1; z/z$ is univalent convex (see [31, Theorem 1.9]) and consequently, in view of Lemma 6, $g(z)$ is univalent convex function. □

Theorem 17. Let $f \in \mathcal{R}_{\alpha_2,\mu}(\theta, \rho)$. Then

$$\frac{f(z)}{z} < g(z), \tag{52}$$

where g is defined in Theorem 16.

Proof. Since $f \in \mathcal{R}_{\alpha_2,\mu}(\theta, \rho)$, then we have

$$\frac{\Phi^{\alpha,\mu}f(z)}{z} < e^{-i\theta}\left(\frac{1+(1-2\rho)z^2}{1-z^2}\cos\theta + i\sin\theta\right). \tag{53}$$

But g is univalent convex function (Theorem 16); thus by an application of Lemma 7, we obtain the desired assertion. □

3. Conclusion

We defined a new fractional differential operator which generalized well-known linear and nonlinear operators such as Carlson-Shaffer operator and the Dziok-Srivastava (linear operators) and Srivastava-Owa fractional differential operators (nonlinear operator). By making uses this operator a generalized class of analytic functions is defined and studied. The sharp bound for nonlinear functional based on the second-order Hankel determinant $|a_2a_4 - a_3^2|$, involving the generalized fractional differential operator, is computed. Several properties, depending on the Hadamard product, are imposed. We have shown that some results are generalized by recent works due to Mishra-Gochhayat, Ling-Ding, and Janteng et al. Furthermore, a new approach is introduced in the proof of Theorems 11 and 16 based on the subordination concept and employing the result due to Ponnusamy and Singh.

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