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Research Article

Certain Results on Ricci Solitons in α-Sasakian Manifolds

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We study Ricci solitons in α-Sasakian manifolds and show that it is a shrinking or expanding soliton and the manifold is Einstein with Killing vector field. Further, we prove that if V is conformal Killilng vector field, then the Ricci soliton in 3-dimensional α-Sasakian manifolds is shrinking or expanding but cannot be steady.

1. Introduction

A Ricci soliton (g, V, λ) is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\mathscr{L}_V g + 2S + 2\lambda g = 0, \tag{1}$$

where V is a complete vector field on M and λ is a constant. The Ricci soliton is said to be shrinking, steady, or expanding according as λ is negative, zero, and positive, respectively. Long-existing solutions, that is, solutions which exist on an infinite time interval, are the self-similar solutions, which in Ricci flow are called Ricci soliton.

Compact Ricci solitons are the fixed points of the Ricci flow $\partial g/\partial t = -2\mathrm{Ric}(g)$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{2}$$

A Ricci soliton on a compact manifold is a gradient Ricci soliton. A Ricci soliton on a compact manifold has constant curvature in dimension 2 [1] and also in dimension 3 [2]. In [3], Perelman proved that a Ricci soliton on a compact n-manifold is a gradient Ricci soliton. In [4], Sharma studied Ricci solitons in K-contact manifolds, where the structure field ξ is Killing, and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. In [5], Tripathi studied Ricci solitons in $N(k)$-contact metric and (k, μ) manifolds. In [6], Ghosh and Sharma studied K-contact metrics as Ricci solitons. In [7], Nagaraja and Premalatha studied Ricci solitons in f-Kenmotsu manifolds and 3-dimensional trans-Sasakian manifolds. Recently, Bagewadi and Ingalahalli [8] studied Ricci solitons in Lorentzian α-Sasakian manifolds. Motivated by the previous studies on Ricci solitons, in this paper, we study Ricci solitons in an α-Sasakian manifolds, where α is some constant.

2. Preliminaries

Let M be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g, which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{4}$$

for all $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be α-Sasakian manifold if the following conditions hold:

$$\begin{gathered}(\nabla_X \phi) Y = \alpha (g(X, Y)\xi - \eta(Y) X), \\ \nabla_X \xi = -\alpha\phi X, \qquad (\nabla_X \eta) Y = \alpha g(X, \phi Y),\end{gathered} \tag{5}$$

for some nonzero constant α on M.

In an α-Sasakian manifold, we have the following relations:

$$R(X,Y)\xi = \alpha^2 [\eta(Y)X - \eta(X)Y], \quad (6)$$

$$R(\xi,X)Y = \alpha^2 [g(X,Y)\xi - \eta(Y)X], \quad (7)$$

$$\eta(R(X,Y)Z) = \alpha^2 [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \quad (8)$$

$$S(X,\xi) = \alpha^2 (n-1)\eta(X), \quad (9)$$

$$S(\xi,\xi) = \alpha^2 (n-1), \quad (10)$$

$$Q\xi = \alpha^2 (n-1)\xi, \quad (11)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor, and Q is the Ricci operator.

3. Ricci Solitons in α-Sasakian Manifold

In this section, we prove some theorems on Ricci solitons in α-Sasakian manifold.

Proposition 1. *A complete Einstein α-Sasakian manifold is compact.*

Proof. Let M be a complete Einstein α-Sasakian manifold, then the general form is given by

$$S(X,Y) = \frac{r}{n} g(X,Y) \Longrightarrow Q = \frac{r}{n} I. \quad (12)$$

Operating ξ in (12) and using (11) show $r = n(n-1)\alpha^2$. Hence we get $Q = \alpha^2(n-1)I$. So the Ricci curvatures are equal to $\alpha^2(n-1)$ which is a positive constant. By Myers's theorem [9], we conclude that M is compact. □

Theorem 2. *If the metric g of an α-Sasakian manifold (M, g) is a gradient Ricci soliton, then the Ricci soliton is a shrinking soliton and (M, g) is compact Einstein.*

Proof. Equation (2) can be written as

$$\nabla_Y Df = QY + \lambda Y, \quad (13)$$

where D denotes the gradient operator of g and Y denotes an arbitrary vector field on M. Using this we derive

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \quad (14)$$

Taking its inner product with ξ, substituting $X = \xi$, and using (7) and (11), we have

$$Df = (\xi f)\xi. \quad (15)$$

Substituting (15) in (13), we get

$$Y(\xi f)\eta(X) - \alpha(\xi f)g(\phi Y, X) = S(X,Y) + \lambda g(X,Y). \quad (16)$$

Interchanging X and Y in (16), we have

$$X(\xi f)\eta(Y) - \alpha(\xi f)g(\phi X, Y) = S(Y,X) + \lambda g(Y,X). \quad (17)$$

Adding (16) and (17), we have

$$X(\xi f)\eta(Y) + Y(\xi f)\eta(X) = 2S(Y,X) + 2\lambda g(Y,X). \quad (18)$$

Putting $Y = \xi$ in (18), we have

$$X(\xi f) = [\alpha^2(n-1) + \lambda]\eta(X). \quad (19)$$

The use of the previous two equations provides

$$S(X,Y) = [\alpha^2(n-1) + \lambda]\eta(X)\eta(Y) - \lambda g(Y,X). \quad (20)$$

Consequently, (13) assumes the form

$$\nabla_Y Df = [\alpha^2(n-1) + \lambda]\eta(Y)\xi. \quad (21)$$

Using this, we compute $R(X,Y)Df$, and taking inner product with ξ (bearing in mind that $Df = (\xi f)\xi$), we obtain $[\alpha^2(n-1) + \lambda] = 0$. Therefore, from (19), we have $X(\xi f) = 0$; that is, ξf is constant or $\xi f = c$. Hence (15) can be written as $df = c\eta$. Its exterior derivative implies $cd\eta = 0$. Hence $c = 0$. Thus f is constant.

Consequently, (13) reduces to $S = \alpha^2(n-1)g$; that is, an α-Sasakian manifold is an Einstein. Also, as $\lambda = -\alpha^2(n-1)$ is negative for $\alpha > 0$ or $\alpha < 0$; that is, Ricci soliton in α-Sasakian manifolds is shrinking. □

From above-mentioned theorem, we state the following corollary.

Corollary 3. *If a metric g of a compact α-Sasakian manifold (M, g) is a Ricci soliton, then g is a shrinking soliton and the manifold is Einstein.*

Theorem 4. *If a metric g in an α-Sasakian manifold is a Ricci soliton with $V = \xi$, then it is Einstein.*

Proof. Putting $V = \xi$ in (1), then we have

$$(\mathscr{L}_\xi g + 2S + 2\lambda g)(X,Y) = 0, \quad (22)$$

where

$$(\mathscr{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \quad (23)$$

Substituting (23) in (22), then we get the result. □

Proposition 5. *If an α-Sasakian manifold is a Ricci soliton with V point-wise collinear with ξ, then V is a constant multiple of ξ and the manifold is Einstein.*

Proof. From (1), we have

$$(\mathscr{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \quad (24)$$

where

$$(\mathscr{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V). \quad (25)$$

Substituting (25) in (24), then we obtain

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \quad (26)$$

Putting $V = a\xi$ in (26), we get

$$(Xa)\,\eta(Y) + (Ya)\,\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \quad (27)$$

Putting $X = Y = \xi$ in (27), we have

$$(\xi a) + \alpha^2(n-1) + \lambda = 0. \quad (28)$$

Again putting $X = \xi$ in (27), we obtain

$$(Ya) = \left[-\alpha^2(n-1) - \lambda\right]\eta(Y). \quad (29)$$

Equation (30) implies that

$$da = \left[-\alpha^2(n-1) - \lambda\right]\eta. \quad (30)$$

Applying d on both sides,

$$d^2a = \left[-\alpha^2(n-1) - \lambda\right]d\eta. \quad (31)$$

Equation (31) implies that $d^2a = 0$, but $d\eta$ is nowhere vanishing. Therefore, $-\lambda-\alpha^2(n-1) = 0$ which implies $da = 0$; that is, a is constant. As ξ is Killing, we conclude that the manifold is Einstein which completes the proof. □

Definition 6. A vector field V is said to be conformal Killing vector field if it satisfies

$$\mathscr{L}_V g = 2\rho g \quad (32)$$

for some scalar function ρ.

Theorem 7. *Let (g, V, λ) be a Ricci soliton in an α-Sasakian manifolds (M, g). Then (M, g) is Ricci-semisymmetric if and only if V is conformal Killing.*

Proof. Suppose that V is a conformal Killing vector field, and from (1), we have

$$2\rho g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \quad (33)$$

The previous equation implies that

$$S(X,Y) = [-\rho - \lambda]\,g(X,Y). \quad (34)$$

This shows that the Ricci soliton is Einstein as follows:

$$QX = [-\rho - \lambda]\,X. \quad (35)$$

Let M be an α-Sasakian manifolds; then we have [10]

(1) Einstein,

(2) locally Ricci symmetric,

(3) Ricci semisymmetric; that is, $R \cdot S = 0$.

The implication (1) $\rightarrow$ (2) $\rightarrow$ (3) is trivial. Now, we prove the implication (3) $\rightarrow$ (1).

Now,

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \quad (36)$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in (36), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \quad (37)$$

By using (7) in (37), we obtain

$$\begin{aligned}\alpha^2&\left[g(Y,U)S(\xi,V) - \eta(U)S(Y,V)\right]\\ &+ \alpha^2\left[g(Y,V)S(U,\xi) - \eta(V)S(U,Y)\right] = 0.\end{aligned} \quad (38)$$

Putting $U = \xi$ in (38) and by using (3), (9), and (10) on simplification, we obtain

$$S(Y,V) = (n-1)\,\alpha^2 g(Y,V). \quad (39)$$

Substituting (39) in (1), we have

$$(\mathscr{L}_V g)(X,Y) = \rho g(X,Y), \quad (40)$$

where $\rho = 2[-\alpha^2(n-1) - \lambda]$; that is, V is conformal Killing. □

Now we study Ricci solitons in 3-dimensional α-Sasakian manifolds.

Theorem 8. *In a 3-dimensional α-Sasakian manifolds, a Ricci soliton (g, V, λ), where V is conformal Killing, is*

(i) *shrinking for $\rho = 2\alpha^2$,*

(ii) *expanding for $\rho < 2\alpha^2$ and $\rho > 2\alpha^2$.*

Proof. Suppose that (M, g) is a 3-dimensional α-Sasakian manifolds and (g, V, λ) is a Ricci soliton in (M, g). If V is a conformal Killing vector field, then

$$\mathscr{L}_V g = 2\rho g \quad (41)$$

for some scalar function ρ.

In a 3-dimensional α-Sasakian manifolds and from (1), we have

$$S(X,Y) = [-\rho - \lambda]\,g(X,Y), \quad (42)$$

$$QX = [-\rho - \lambda]\,X, \quad (43)$$

$$r = 3[-\rho - \lambda]. \quad (44)$$

In a 3-dimensional α-Sasakian manifolds, the curvature tensor R is given by

$$\begin{aligned}R(X,Y)Z = {}& g(Y,Z)QX - g(X,Z)QY\\ &+ S(Y,Z)X - S(X,Z)Y\\ &- \frac{r}{2}\left[g(Y,Z)X - g(X,Z)Y\right].\end{aligned} \quad (45)$$

Using (42), (43), and (44) in (45), we get

$$R(X,Y)Z = \left\{2[-\rho - \lambda] - \frac{r}{2}\right\}\left[g(Y,Z)X - g(X,Z)Y\right]. \quad (46)$$

Putting $X = Z = \xi$ in (46), we get

$$R(\xi, Y)\xi = \left\{2[-\rho - \lambda] - \frac{r}{2}\right\}[\eta(Y)\xi - Y]. \tag{47}$$

In an α-Sasakian manifolds $R(\xi, Y)\xi$ is given by

$$R(\xi, Y)\xi = \alpha^2 [\eta(Y)\xi - Y]. \tag{48}$$

From (47) and (48), we have

$$\left\{2[-\rho - \lambda] - \frac{r}{2} - \alpha^2\right\}[\eta(Y)\eta(W) - g(Y, W)] = 0. \tag{49}$$

The previous equation implies that

$$\left\{2[-\rho - \lambda] - \frac{r}{2} - \alpha^2\right\} = 0. \tag{50}$$

From (44) and (50), we have

$$\lambda = -\left[\rho + 2\alpha^2\right]. \tag{51}$$

(i) If $\rho = 2\alpha^2$ implies $\lambda = -4\alpha^2$, that is, $\lambda < 0$. Hence Ricci soliton is shrinking.

(ii) Let $\rho < 2\alpha^2$, suppose $\rho = (-2\alpha^2 - 1) < 2\alpha^2$ implies $\rho + 2\alpha^2 = -1$, that is, $-\lambda = 1 > 0$. Hence Ricci soliton is expanding.

(iii) Let $\rho > 2\alpha^2$ implies $-2\alpha^2 < -\rho$. If $-2\alpha^2 - 2\alpha^2 < -\rho - 2\alpha^2$ then $-4\alpha^2 < -\lambda$ implies $\lambda > 4\alpha^2$. Hence Ricci soliton is expanding.

□

If $\rho = 0$ in (41), then $\mathscr{L}_V g = 0$; that is, conformal vector field does not exist, and then the Ricci soliton is generalization of Einstein metric, then V is a Killing vector field. On base of this condition, we state the following.

Remark 9. A Ricci soliton in α-Sasakian manifolds is shrinking, if V is a Killing vector field.

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