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Research Article

New Subclasses of Analytic Functions with Respect to Symmetric and Conjugate Points

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We introduce new subclasses of close-to-convex and quasiconvex functions with respect to symmetric and conjugate points. The coefficient estimates for functions belonging to these classes are obtained.

1. Introduction

Let U be the class of functions which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$ given by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1}$$

and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| \le 1$, $z \in E$.

Let S denote the class of functions f which are analytic and univalent in E of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \tag{2}$$

Let S_s^* be the subclass of functions $f(z) \in S$ and satisfying the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E. \tag{3}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1].

Also, let S_c^* be the subclass of functions $f(z) \in S$ and satisfying the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, \quad z \in E. \tag{4}$$

These functions are called starlike with respect to conjugate points and were introduced by El-Ashwah and Thomas [2]. Further results on starlike functions with respect to symmetric points or conjugate points can be found in [3–5].

Then, Das and Singh [6] introduced another class C_s, namely, convex functions with respect to symmetric points and satisfying the condition

$$\operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'}\right) > 0, \quad z \in E. \tag{5}$$

Suppose that f and g are two analytic functions in E. Then, we say that the function g is subordinate to the function f, and we write $g(z) \prec f(z)$, $z \in E$, if there exists a Schwarz function $\varpi(z)$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that $g(z) = f(\varpi(z))$, $z \in E$.

In view of subordination definition, Goel and Mehrok [7] introduced a subclass of S_s^* denoted by $S_s^*(A, B)$.

Let $S_s^*(A, B)$ be the class of functions of the form (2) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \le B < A \le 1, \ z \in E. \tag{6}$$

Following them, many authors introduced the analogue definitions by extension as follows (see [8, 9]).

Definition 1. Let $S_c^*(A,B)$ be the subclass of S consisting of functions given by (2) satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1,\ z \in E. \tag{7}$$

Let $C_s(A,B)$ be the subclass of S consisting of functions given by (2) satisfying the condition

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1,\ z \in E. \tag{8}$$

Let $C_c(A,B)$ be the subclass of S consisting of functions given by (2) satisfying the condition

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1,\ z \in E. \tag{9}$$

Motivated by the pervious classes, Selvaraj and Vasanthi [10] defined the following classes of functions with respect to symmetric and conjugate points.

Definition 2. Let $M_s(\alpha,A,B)$ be the subclass of S consisting of functions given by (2) satisfying the condition

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z)-f(-z)\right)+\alpha z\left(f(z)-f(-z)\right)'} \prec \frac{1+Az}{1+Bz},$$
$$-1 \le B < A \le 1,\ 0 \le \alpha \le 1,\ z \in E. \tag{10}$$

Let $M_c(\alpha,A,B)$ be the subclass of S consisting of functions given by (2) satisfying the condition

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z)+\overline{f(\overline{z})}\right)+\alpha z\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+Az}{1+Bz},$$
$$-1 \le B < A \le 1,\ 0 \le \alpha \le 1,\ z \in E. \tag{11}$$

In this paper, we introduce the class $K_s^*(\alpha,A,B;C,D)$ consisting of analytic functions f of the form (2) and satisfying

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(g(z)-g(-z)\right)+\alpha z\left(g(z)-g(-z)\right)'} \prec \frac{1+Cz}{1+Dz},$$
$$-1 \le D \le B < A \le C \le 1,$$
$$0 \le \alpha \le 1,\ z \in E, \tag{12}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha,A,B)$.

In addition, we introduce the class $K_c^*(\alpha,A,B;C,D)$ consisting of analytic functions f of the form (2) and satisfying

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(g(z)+\overline{g(\overline{z})}\right)+\alpha z\left(g(z)+\overline{g(\overline{z})}\right)'} \prec \frac{1+Cz}{1+Dz},$$
$$-1 \le D \le B < A \le C \le 1,$$
$$0 \le \alpha \le 1,\ z \in E, \tag{13}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha,A,B)$.

We note that

(i) for $\alpha = 0$, $K_s^*(0,A,B;C,D) = K_s(A,B;C,D)$ (see Mehrok et al. [11]) and $K_c^*(0,A,B;C,D) = K_c(A,B;C,D)$;

(ii) for $\alpha = 0$, $C = 1$ and $D = -1$, $K_s^*(0,A,B;1,-1) = K_s(A,B)$ (see Janteng and Halim [12]) and $K_c^*(0,A,B;1,-1) = K_c(A,B)$;

(iii) for $\alpha = 0$, $A = C = 1$ and $B = D = -1$, $K_s^*(0,1,-1;1,-1) \equiv K_s$ and $K_c^*(0,1,-1;1,-1) \equiv K_c$;

(iv) for $\alpha = 1$, $K_s^*(1,A,B;C,D) = K_s^*(A,B;C,D)$ and $K_c^*(1,A,B;C,D) = K_c^*(A,B;C,D)$;

(v) for $\alpha = 1, C = 1$ and $D = -1$, $K_s^*(1,A,B;1,-1) = K_s^*(A,B)$ (see Janteng and Halim [13]) and $K_c^*(1,A,B;1,-1) = K_c^*(A,B)$;

(vi) for $\alpha = 1$, $A = C = 1$ and $B = D = -1$, $K_s^*(1,1,-1;1,-1) \equiv K_s^*$ and $K_c^*(1,1,-1;1,-1) \equiv K_c^*$.

By the definition of subordination, it follows that $f \in K_s^*(\alpha,A,B;C,D)$ if and only if

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(g(z)-g(-z)\right)+\alpha z\left(g(z)-g(-z)\right)'} = \frac{1+C\omega(z)}{1+D\omega(z)} = P(z), \quad \omega(z) \in U, \tag{14}$$

and that $f \in K_c^*(\alpha,A,B;C,D)$ if and only if

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)\left(g(z)+\overline{g(\overline{z})}\right)+\alpha z\left(g(z)+\overline{g(\overline{z})}\right)'} = \frac{1+C\omega(z)}{1+D\omega(z)} = P(z), \quad \omega(z) \in U, \tag{15}$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{16}$$

In the next section, we discuss the coefficient estimates for functions belonging to the classes $K_s^*(\alpha,A,B;C,D)$ and $K_c^*(\alpha,A,B;C,D)$.

2. Some Preliminary Lemmas

We will require the following lemmas for proving our main results.

Lemma 3 (see [7]). *If $P(z)$ is given by* (14), (15) *and* (16), *then for* $-1 \le D < C \le 1$,

$$|p_n| \le (C - D), \quad n = 1, 2, \ldots. \tag{17}$$

Lemma 4 (see [10]). *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha, A, B)$. Then, for $n \ge 1$, $0 \le \alpha \le 1$,*

$$\begin{aligned} |b_{2n}| &\le \frac{(A-B)}{2^n \cdot n!\,(1+(2n-1)\alpha)} \prod_{j=1}^{n-1} (A - B + 2j), \\ |b_{2n+1}| &\le \frac{(A-B)}{2^n \cdot n!\,(1+2n\alpha)} \prod_{j=1}^{n-1} (A - B + 2j). \end{aligned} \tag{18}$$

Lemma 5 (see [10]). *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha, A, B)$. Then, for $n \ge 1$, $0 \le \alpha \le 1$,*

$$\begin{aligned} |b_{2n}| &\le \frac{(A-B)}{(2n-1)!\,(1+(2n-1)\alpha)} \prod_{j=1}^{2n-2} (A - B + j), \\ |b_{2n+1}| &\le \frac{(A-B)}{(2n)!\,(1+2n\alpha)} \prod_{j=1}^{2n-1} (A - B + j). \end{aligned} \tag{19}$$

3. Main Results

Unless otherwise mentioned, we will assume in the reminder of this paper that $-1 \le D \le B < A \le C \le 1$, $0 \le \alpha \le 1$, and $z \in E$.

Theorem 6. *Let $f \in K_s^*(\alpha, A, B; C, D)$, then for $n \ge 1$,*

$$|a_{2n}| \le \frac{(C-D)}{2^n \cdot n!\,(1+(2n-1)\alpha)} \prod_{j=1}^{n-1} (A - B + 2j), \tag{20}$$

$$\begin{aligned} |a_{2n+1}| \le{}& \frac{1}{(2n+1)(1+2n\alpha)} \\ &\times \left\{ \left[(C-D) + \frac{(A-B)}{2n} \right] \right. \\ &\left. \times \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right] \right\}. \end{aligned} \tag{21}$$

Proof. Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha, A, B)$, it follows that

$$\begin{aligned} &2zg'(z) + 2\alpha z^2 g''(z) \\ &\quad = \left[(1-\alpha)(g(z) - g(-z)) + \alpha z (g(z) - g(-z))' \right] K(z), \end{aligned} \tag{22}$$

for $z \in E$, with $\mathrm{Re}(K(z)) > 0$, where $K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$.

On equating the coefficients of like powers of z in (22), we get

$$\begin{aligned} 2(1+\alpha) b_2 &= d_1, \\ 2(1+2\alpha) b_3 &= d_2, \\ 4(1+3\alpha) b_4 &= d_3 + (1+2\alpha) b_3 d_1, \\ 4(1+4\alpha) b_5 &= d_4 + (1+2\alpha) b_3 d_2, \end{aligned} \tag{23}$$

and continuing in this way, we obtain

$$\begin{aligned} 2n(1+(2n-1)\alpha) b_{2n} &= d_{2n-1} + (1+2\alpha) b_3 d_{2n-3} \\ &\quad + \cdots + (1+(2n-2)\alpha) b_{2n-1} d_1, \end{aligned} \tag{24}$$

$$\begin{aligned} 2n(1+2n\alpha) b_{2n+1} &= d_{2n} + (1+2\alpha) b_3 d_{2n-2} \\ &\quad + \cdots + (1+(2n-2)\alpha) b_{2n-1} d_2. \end{aligned} \tag{25}$$

From (14) and (16), we have

$$\begin{aligned} &\left(z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots + 2na_{2n} z^{2n} + \cdots\right) \\ &\quad + \alpha \left(2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 + 20a_5 z^5 + \cdots \right. \\ &\qquad \left. + (2n-1)\, 2na_{2n} z^{2n} + \cdots\right) \\ &= \left[(1-\alpha)\left(z + b_3 z^3 + b_5 z^5 + \cdots + b_{2n-1} z^{2n-1} \right.\right. \\ &\qquad \left. + b_{2n+1} z^{2n+1} + \cdots\right) \\ &\quad + \alpha \left(z + 3b_3 z^3 + 5b_5 z^5 + \cdots + (2n-1) b_{2n-1} z^{2n-1} \right. \\ &\qquad \left.\left. + (2n+1) b_{2n+1} z^{2n+1} + \cdots\right)\right] \\ &\times \left(1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 \right. \\ &\quad \left. + p_5 z^5 + \cdots + p_{2n-1} z^{2n-1} + p_{2n} z^{2n} + \cdots\right). \end{aligned} \tag{26}$$

On equating the coefficients, we obtain

$$\begin{aligned} 2(1+\alpha) a_2 &= p_1, \\ 3(1+2\alpha) a_3 &= p_2 + (1+2\alpha) b_3, \end{aligned} \tag{27}$$

$$\begin{aligned} 4(1+3\alpha) a_4 &= p_3 + (1+2\alpha) b_3 p_1, \\ 5(1+4\alpha) a_5 &= p_4 + (1+2\alpha) b_3 p_2 + (1+4\alpha) b_5, \end{aligned} \tag{28}$$

and so

$$\begin{aligned}
&2n(1+(2n-1)\alpha)a_{2n}\\
&\quad= p_{2n-1}+(1+2\alpha)b_3p_{2n-3}\\
&\qquad+\cdots+(1+(2n-2)\alpha)b_{2n-1}p_1,\\
&(2n+1)(1+2n\alpha)a_{2n+1}\\
&\quad= p_{2n}+(1+2\alpha)b_3p_{2n-2}\\
&\qquad+\cdots+(1+(2n-2)\alpha)b_{2n-1}p_2+(1+2n\alpha)b_{2n+1}.
\end{aligned} \tag{29}$$

By using Lemma 3 and (27), we have

$$|a_2|\le\frac{(C-D)}{2\cdot1\cdot(1+\alpha)},\qquad |a_3|\le\frac{(A-B)+2(C-D)}{3\cdot2\cdot(1+2\alpha)}. \tag{30}$$

Again, by applying Lemma 3 and using (23), we obtain from (28)

$$\begin{aligned}
|a_4|&\le\frac{(C-D)(A-B+2)}{4\cdot2\cdot(1+3\alpha)},\\
|a_5|&\le\frac{(A-B+2)[(A-B)+4(C-D)]}{5\cdot8\cdot(1+4\alpha)}.
\end{aligned} \tag{31}$$

It follows that (20) and (21) hold for $n=1,2$. We now prove (20) and (21) by induction.

Equation (29) together with Lemma 3 yield

$$|a_{2n}|\le\frac{(C-D)}{2n(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right], \tag{32}$$

$$\begin{aligned}
|a_{2n+1}|&\le\frac{1}{(2n+1)(1+2n\alpha)}\\
&\quad\times\left\{(C-D)\left[1+\sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right]\right.\\
&\qquad\left.+(1+2n\alpha)|b_{2n+1}|\right\}.
\end{aligned} \tag{33}$$

Again, using Lemma 3 in (25), we have

$$|b_{2n+1}|\le\frac{(A-B)}{2n(1+2n\alpha)}\left[1+\sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right]. \tag{34}$$

Using (34) in (33), we obtain

$$\begin{aligned}
&|a_{2n+1}|\\
&\quad\le\frac{1}{(2n+1)(1+2n\alpha)}\\
&\qquad\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[1+\sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right]\right\}.
\end{aligned} \tag{35}$$

We suppose that (20) and (21) hold for $k=3,4,\dots,(n-1)$.

Using Lemma 4 in (32) and (35), we get

$$\begin{aligned}
|a_{2n}|&\le\frac{(C-D)}{2n(1+(2n-1)\alpha)}\\
&\quad\times\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right],
\end{aligned} \tag{36}$$

$$\begin{aligned}
|a_{2n+1}|&\le\frac{1}{(2n+1)(1+2n\alpha)}\\
&\quad\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\right.\\
&\qquad\left.\times\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}.
\end{aligned} \tag{37}$$

In order to prove (20), it is sufficient to show that

$$\begin{aligned}
&\frac{(C-D)}{2m(1+(2m-1)\alpha)}\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\\
&\quad=\frac{(C-D)}{2^m\cdot m!(1+(2m-1)\alpha)}\prod_{j=1}^{m-1}(A-B+2j)\\
&\qquad(m=3,4,\dots,n).
\end{aligned} \tag{38}$$

Thus, (38) is valid for $m=3$.

Let us assume that (38) is true for all m, $3<m\le(n-1)$. Then, from (36), we have

$$\begin{aligned}
&\frac{(C-D)}{2n(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\\
&\quad=\frac{(n-1)}{n}\left\{\frac{(C-D)}{2(n-1)(1+(2n-1)\alpha)}\right.\\
&\qquad\left.\times\left[1+\sum_{k=1}^{n-2}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}\\
&\qquad+\frac{(C-D)}{2n(1+(2n-1)\alpha)}\cdot\frac{(A-B)}{2^{n-1}\cdot(n-1)!}\\
&\qquad\times\prod_{j=1}^{n-2}(A-B+2j)
\end{aligned}$$

$$= \frac{(n-1)}{n} \cdot \frac{(C-D)}{2^{n-1} \cdot (n-1)!\,(1+(2n-1)\alpha)} \times \prod_{j=1}^{n-2} (A-B+2j) + \frac{(C-D)}{2n\,(1+(2n-1)\alpha)} \cdot \frac{(A-B)}{2^{n-1} \cdot (n-1)!} \times \prod_{j=1}^{n-2} (A-B+2j)$$
$$= \frac{(C-D)}{2^n \cdot n!\,(1+(2n-1)\alpha)} \prod_{j=1}^{n-2} (A-B+2j) \times (A-B+2(n-1))$$
$$= \frac{(C-D)}{2^n \cdot n!\,(1+(2n-1)\alpha)} \prod_{j=1}^{n-1} (A-B+2j). \tag{39}$$

Thus, (38) holds for $m = n$, and, hence, (20) follows. Next, we prove (21).

From (38), we have

$$1 + \sum_{k=1}^{n-1} \frac{(A-B)}{2^k \cdot k!} \prod_{j=1}^{k-1} (A-B+2j) = \frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A-B+2j). \tag{40}$$

By using (40) in (37), we obtain

$$|a_{2n+1}| \le \frac{1}{(2n+1)(1+2n\alpha)} \times \left\{ \left[(C-D) + \frac{(A-B)}{2n} \right] \times \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A-B+2j) \right] \right\} \tag{41}$$

which proves (21). □

By taking $\alpha = 0$ and $\alpha = 1$ in Theorem 6, respectively, we can readily deduce the following corollaries.

Corollary 7 (see [11]). *Let $f \in K_s(A,B;C,D)$, then, for $n \ge 1$,*

$$|a_{2n}| \le \frac{(C-D)}{2^n \cdot n!} \prod_{j=1}^{n-1} (A-B+2j),$$
$$|a_{2n+1}| \le \frac{1}{2n+1} \left\{ \left[(C-D) + \frac{(A-B)}{2n} \right] \times \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A-B+2j) \right] \right\}. \tag{42}$$

Corollary 8. *Let $f \in K_s^*(A,B;C,D)$, then, for $n \ge 1$,*

$$|a_{2n}| \le \frac{(C-D)}{2^{n+1} \cdot n \cdot n!} \prod_{j=1}^{n-1} (A-B+2j),$$
$$|a_{2n+1}| \le \frac{1}{(2n+1)^2} \left\{ \left[(C-D) + \frac{(A-B)}{2n} \right] \times \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A-B+2j) \right] \right\}. \tag{43}$$

Further, putting $C = 1$ and $D = -1$ in Corollary 8,one obtains the following.

Corollary 9. *Let $f \in K_s^*(A,B)$, then, for $n \ge 1$,*

$$|a_{2n}| \le \frac{1}{2^n \cdot n \cdot n!} \prod_{j=1}^{n-1} (A-B+2j),$$
$$|a_{2n+1}| \le \frac{1}{(2n+1)^2} \times \left\{ \left[2 + \frac{(A-B)}{2n} \right] \times \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A-B+2j) \right] \right\}. \tag{44}$$

Remark 10. Corollary 9 improves the result obtained by Janteng and Halim [13, Theorem 3.1].

Theorem 11. *Let $f \in K_c^*(\alpha, A, B; C, D)$, then, for $n \geq 1$,*

$$|a_{2n}| \leq \frac{1}{2n(1+(2n-1)\alpha)} \times \left\{\left[(C-D)+\frac{(A-B)}{2n-1}\right]\left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right]\right\}, \tag{45}$$

$$|a_{2n+1}| \leq \frac{1}{(2n+1)(1+2n\alpha)} \times \left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[\frac{1}{(2n-1)!}\prod_{j=1}^{2n-1}(A-B+j)\right]\right\}. \tag{46}$$

Proof. Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha, A, B)$, it follows that

$$2zg'(z) + 2\alpha z^2 g''(z) = \left[(1-\alpha)\left(g(z)+\overline{g(\overline{z})}\right)+\alpha z\left(g(z)+\overline{g(\overline{z})}\right)'\right]K(z), \tag{47}$$

where $K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$.

On equating the coefficients of like powers of z in (47), we get

$$(1+\alpha)b_2 = d_1, \tag{48}$$

$$2(1+2\alpha)b_3 = d_2 + (1+\alpha)b_2 d_1,$$

$$3(1+3\alpha)b_4 = d_3 + (1+\alpha)b_2 d_2 + (1+2\alpha)b_3 d_1, \tag{49}$$

$$4(1+4\alpha)b_5 = d_4 + (1+\alpha)b_2 d_3 + (1+2\alpha)b_3 d_2 + (1+3\alpha)b_4 d_1, \tag{50}$$

and continuing in this way, we obtain

$$(2n-1)(1+(2n-1)\alpha)b_{2n} = d_{2n-1} + (1+\alpha)b_2 d_{2n-2} + \cdots + (1+(2n-2)\alpha)b_{2n-1}d_1, \tag{51}$$

$$2n(1+2n\alpha)b_{2n+1} = d_{2n} + (1+\alpha)b_2 d_{2n-1} + \cdots + (1+(2n-1)\alpha)b_{2n}d_1. \tag{52}$$

From (15) and (16), we have

$$\begin{aligned}
&\left(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots + 2na_{2n}z^{2n} + \cdots\right) \\
&\quad + \alpha\left(2a_2z^2 + 6a_3z^3 + 12a_4z^4 + 20a_5z^5 + \cdots + (2n-1)2na_{2n}z^{2n} + \cdots\right) \\
&= \left[(1-\alpha)\left(z + b_2z^2 + b_3z^3 + b_4z^4 + b_5z^5 + \cdots + b_{2n}z^{2n} + \cdots\right)\right. \\
&\quad \left. + \alpha\left(z + 2b_2z^2 + 3b_3z^3 + 4b_4z^4 + 5b_5z^5 + \cdots + 2nb_{2n}z^{2n} + \cdots\right)\right] \\
&\quad \times \left(1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + p_5z^5 + \cdots + p_{2n}z^{2n} + \cdots\right).
\end{aligned} \tag{53}$$

On equating the coefficients, we obtain

$$2(1+\alpha)a_2 = p_1 + (1+\alpha)b_2, \tag{54}$$

$$3(1+2\alpha)a_3 = p_2 + (1+\alpha)b_2p_1 + (1+2\alpha)b_3,$$

$$4(1+3\alpha)a_4 = p_3 + (1+\alpha)b_2p_2 + (1+2\alpha)b_3p_1 + (1+3\alpha)b_4, \tag{55}$$

$$5(1+4\alpha)a_5 = p_4 + (1+\alpha)b_2p_3 + (1+2\alpha)b_3p_2 + (1+3\alpha)b_4p_1 + (1+4\alpha)b_5, \tag{56}$$

and so

$$\begin{aligned}
2n(1+(2n-1)\alpha)a_{2n} = {} & p_{2n-1} + (1+\alpha)b_2p_{2n-2} \\
& + (1+2\alpha)b_3p_{2n-3} + \cdots \\
& + (1+(2n-2)\alpha)b_{2n-1}p_1 \\
& + (1+(2n-1)\alpha)b_{2n},
\end{aligned} \tag{57}$$

$$\begin{aligned}
(2n+1)(1+2n\alpha)a_{2n+1} = {} & p_{2n} + (1+\alpha)b_2p_{2n-1} \\
& + (1+2\alpha)b_3p_{2n-2} + \cdots \\
& + (1+(2n-1)\alpha)b_{2n}p_1 \\
& + (1+2n\alpha)b_{2n+1}.
\end{aligned} \tag{58}$$

By using Lemma 3, (48), and (54), we have

$$|a_2| \leq \frac{(C-D)+(A-B)}{2\cdot 1\cdot(1+\alpha)}, \qquad |a_3| \leq \frac{(A-B+1)\left[(A-B)+2(C-D)\right]}{3\cdot 2\cdot(1+2\alpha)}. \tag{59}$$

Again, by applying Lemma 3 and using (48)–(50), we obtain from (55) and (56)

$$\begin{aligned} |a_4| &\le \frac{(A-B+1)(A-B+2)[(A-B)+3(C-D)]}{4\cdot 6\cdot(1+3\alpha)}, \\ |a_5| &\le (A-B+1)(A-B+2)(A-B+3) \\ &\quad \times [(A-B)+4(C-D)](5\cdot 24\cdot(1+4\alpha))^{-1}. \end{aligned} \tag{60}$$

It follows that (45) and (46) hold for $n = 1, 2$. We now prove (45) by induction.

Equation (57) together with Lemma 3 yield

$$\begin{aligned} |a_{2n}| &\le \frac{1}{2n(1+(2n-1)\alpha)} \\ &\quad \times \left\{ (C-D)\left[1+\sum_{k=1}^{n-1}(1+(2k-1)\alpha)|b_{2k}| \right.\right. \\ &\qquad \left. + \sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right] \\ &\qquad \left. + (1+(2n-1)\alpha)|b_{2n}|\right\}. \end{aligned} \tag{61}$$

Again, by using Lemma 3 in (51), we have

$$\begin{aligned} |b_{2n}| &\le \frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)} \\ &\quad \times \left[1+\sum_{k=1}^{n-1}(1+(2k-1)\alpha)|b_{2k}| \right. \\ &\qquad \left. + \sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right]. \end{aligned} \tag{62}$$

Using (62) in (61), we obtain

$$\begin{aligned} |a_{2n}| &\le \frac{1}{2n(1+(2n-1)\alpha)} \\ &\quad \times \left\{ \left[(C-D)+\frac{(A-B)}{2n-1}\right] \right. \\ &\quad \times \left[1+\sum_{k=1}^{n-1}(1+(2k-1)\alpha)|b_{2k}| \right. \\ &\qquad \left.\left. + \sum_{k=1}^{n-1}(1+2k\alpha)|b_{2k+1}|\right]\right\}. \end{aligned} \tag{63}$$

We suppose that (45) holds for $k = 3, 4, \ldots, (n-1)$.

Using Lemma 5 in (63), we get

$$\begin{aligned} |a_{2n}| &\le \frac{1}{2n(1+(2n-1)\alpha)} \\ &\quad \times \left\{ \left[(C-D)+\frac{(A-B)}{2n-1}\right] \right. \\ &\quad \times \left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j) \right. \\ &\qquad \left.\left. + \sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\}. \end{aligned} \tag{64}$$

In order to prove (45), it is sufficient to show that

$$\begin{aligned} &\frac{1}{2m(1+(2m-1)\alpha)} \\ &\quad \times \left\{ \left[(C-D)+\frac{(A-B)}{2m-1}\right] \right. \\ &\quad \times \left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j) \right. \\ &\qquad \left.\left. + \sum_{k=1}^{m-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\} \\ &= \frac{1}{2m(1+(2m-1)\alpha)} \\ &\quad \times \left\{ \left[(C-D)+\frac{(A-B)}{2m-1}\right] \right. \\ &\quad \times \left.\left[\frac{1}{(2m-2)!}\prod_{j=1}^{2m-2}(A-B+j)\right]\right\} \\ &\qquad (m = 3, 4, \ldots, n). \end{aligned} \tag{65}$$

Thus, (65) is valid for $m = 3$.

Let us assume that (65) is true for all m, $3 < m \le (n-1)$. Then, from (64), we have

$$\begin{aligned} &\frac{1}{2n(1+(2n-1)\alpha)} \\ &\quad \times \left\{ \sum_{k=1}^{n-1}\left[(C-D)+\frac{(A-B)}{2n-1}\right] \right. \\ &\quad \times \left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j) \right. \\ &\qquad \left.\left. + \sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n-1}{n}\right)\left\{\sum_{k=1}^{n-2}\frac{1}{2(n-1)(1+(2n-1)\alpha)}\right.\\
&\qquad\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\qquad\times\left[1+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)\right.\\
&\qquad\left.\left.+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\}\\
&\quad+\frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\right.\\
&\qquad\left.+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]\\
&= \left(\frac{n-1}{n}\right)\cdot\frac{1}{2(n-1)(1+(2n-1)\alpha)}\\
&\quad\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{1}{(2(n-1)-2)!}\prod_{j=1}^{2n-4}(A-B+j)\right]\\
&\quad+\frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\right.\\
&\qquad\left.+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]\\
&= \frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{1}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\right.\\
&\qquad\times(A-B+(2n-3))\\
&\qquad\left.+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]\\
&= \frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{1}{(2(n-1)-1)!}\prod_{j=1}^{2n-3}(A-B+j)\right.\\
&\qquad\left.+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]\\
&= \frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{1}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right.\\
&\qquad\left.\times(A-B+(2n-2))\right]\\
&= \frac{1}{2n(1+(2n-1)\alpha)}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\\
&\quad\times\left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right].
\end{aligned}
\tag{66}
$$

Thus, (65) holds for $m = n$, and, hence, (45) follows. Similarly, we can prove (46). □

By taking $\alpha = 0$ and $\alpha = 1$ in Theorem 11, respectively, we can easily derive the following corollaries.

Corollary 12. *Let $f \in K_c(A,B;C,D)$, then, for $n \geq 1$,*

$$
\begin{aligned}
|a_{2n}| &\leq \frac{1}{2n}\left\{\left[(C-D)+\frac{(A-B)}{2n-1}\right]\right.\\
&\qquad\left.\times\left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right]\right\},\\
|a_{2n+1}| &\leq \frac{1}{2n+1}\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\right.\\
&\qquad\left.\times\left[\frac{1}{(2n-1)!}\prod_{j=1}^{2n-1}(A-B+j)\right]\right\}.
\end{aligned}
\tag{67}
$$

Corollary 13. *Let $f \in K_c^*(A,B;C,D)$, then, for $n \geq 1$,*

$$|a_{2n}| \leq \frac{1}{(2n)^2}\left\{\left[(C-D)+\frac{(A-B)}{2n-1}\right]\times\left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right]\right\},$$

$$|a_{2n+1}| \leq \frac{1}{(2n+1)^2}\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\times\left[\frac{1}{(2n-1)!}\prod_{j=1}^{2n-1}(A-B+j)\right]\right\}. \tag{68}$$

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