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Research Article

A Study on Fuzzy Ideals of N-Groups

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Using the idea of the new sort of fuzzy subnear-ring of a near-ring, fuzzy subgroups, and their generalizations defined by various researchers, we try to introduce the notion of $(\epsilon, \epsilon \vee q)$-fuzzy ideals of N-groups. These fuzzy ideals are characterized by their level ideals, and some other related properties are investigated.

1. Introduction and Basic Definitions

The concept of a fuzzy set was introduced by Zadeh [1] in 1965, utilizing what Rosenfeld [2] defined as fuzzy subgroups. This was studied further in detail by different researchers in various algebraic systems. The concept of a fuzzy ideal of a ring was introduced by Liu [3]. The notion of fuzzy subnear-ring and fuzzy ideals was introduced by Abou-Zaid [4]. Then in many papers, fuzzy ideals of near-rings were discussed for example, see [5–11]. In [12], the idea of fuzzy point and its belongingness to and quasi coincidence with a fuzzy set were used to define (α, β)-fuzzy subgroup, where α, β take one of the values from $\{\epsilon, q, \epsilon \wedge q, \epsilon \vee q\}$, $\alpha \neq \epsilon \wedge q$. A fuzzy subgroup in the sense of Rosenfeld is in fact an (ϵ, ϵ)-fuzzy subgroup. Thus, the concept of $(\epsilon, \epsilon \vee q)$-fuzzy subgroup was introduced and discussed thoroughly in [7]. Bhakat and Das [13] introduced the concept of $(\epsilon, \epsilon \vee q)$-fuzzy subrings and ideals of a ring. Davvaz [14, 15], Narayanan and Manikantan [16], and Zhan and Davvaz [17] studied a new sort of fuzzy subnear-ring (ideal and prime ideal) called $(\epsilon, \epsilon \vee q)$-fuzzy subnear-ring (ideal and prime ideal) and gave characterizations in terms of the level ideals. In [18, 19], the idea of fuzzy ideals of N-groups was defined, and various properties such as fundamental theorem of fuzzy ideals and fuzzy congruence were studied, respectively. In the present paper, we extend the idea of $(\epsilon, \epsilon \vee q)$-fuzzy ideals of near-rings to the case of N-groups and introduce the idea of fuzzy cosets with some results.

We first recall some basic concepts for the sake of completeness.

By a *near-ring* we mean a nonempty set N with two binary operations "+" and "·" satisfying the following axioms:

(i) $(N, +)$ is a group,

(ii) $(N, \cdot)$ is a semigroup,

(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in N$.

It is in fact a right near-ring because it satisfies the right distributive law. We will use the word "near-ring" to mean "right near-ring." N is said to be *zero symmetric* if $0 \cdot x = x \cdot 0 = 0$ for all $x \in N$. We denote $x \cdot y$ by xy.

Note that the missing left distributive law, $x \cdot (y + z) = x \cdot y + x \cdot z$, has to do with linearity if x is considered as a function.

Example 1. Let $\mathcal{G}$ be a group, and let $M(\mathcal{G})$ be the set of all mappings from $\mathcal{G}$ into $\mathcal{G}$. We define + and · on $M(\mathcal{G})$ by

$$\begin{aligned} (f + g)(x) &:= f(x) + g(x), \\ (f \cdot g)(x) &:= f(g(x)). \end{aligned} \tag{1}$$

Then, $(M(\mathcal{G}), +, \cdot)$ is a near-ring.

Just in the same way as R-modules or vector spaces are used in ring theory, N-groups are used in near-ring theory.

By an N-*group* we mean a nonempty set G together with a map $\Phi : N \times G \to G$ written as $\Phi(n, g) = ng$ satisfying the following conditions:

(i) $(G, +)$ is a group (not necessarily abelian),

(ii) $(n_1 + n_2)g = n_1 g + n_2 g$,

(iii) $(n_1 n_2)g = n_1(n_2 g)$ for all $n_1, n_2 \in N$, $g \in G$.

Example 2. Let N be a subnear-ring of $M(\mathcal{G})$. Then, $\mathcal{G}$ is an N-group via function application as operation.

Example 3. The additive group $(N, +)$ of a near-ring $(N, +, \cdot)$ is an N-group via the near-ring multiplication.

An *ideal* I of N-group G is an additive normal subgroup of G such that $NI \subseteq I$ and $n(g + h) - ng \in I$ for all $h \in I$, $g \in G$, $n \in N$. A mapping between two N-groups G and G' is called an N-*homomorphism* if $f(g + h) = f(g) + f(h)$ and $f(ng) = nf(g)$ for all $g, h \in G$, $n \in N$.

Throughout this study, we use N to denote a zero-symmetric near-ring and G to denote an N-group.

For any fuzzy subset A of G, $ImA = \{A(x) \mid x \in G\}$ denotes the image of A. For any subset I of G, χ_I denotes the characteristic function of I.

Definition 4 (see [2]). A fuzzy subset A of a group G is called a *fuzzy subgroup* of G if it satisfies the following conditions:

(i) $A(x + y) \geq \min\{A(x), A(y)\}$,

(ii) $A(-x) \geq A(x)$,

for all $x, y \in G$.

Definition 5. For a fuzzy subset A of G, $t \in (0, 1]$, the subset $A_t = \{x \in G \mid A(x) \geq t\}$ is called a *level subset* of G determined by A and t.

The set $\{x \in G \mid A(x) > 0\}$ is called the *support* of A and is denoted by Supp$A$. A fuzzy subset A of G of the form

$$A(y) = \begin{cases} t\,(\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases} \tag{2}$$

is said to be a *fuzzy point* denoted by x_t. Here x is called the *support point*, and t is called its *value*. A fuzzy point x_t is said to belong to (resp., quasi coincident with) a fuzzy set A written as $x_t \epsilon A$ (resp., $x_t q A$) if $A(x) \geq t$ (resp., $A(x) + t > 1$). If $x_t \in A$ or $x_t q A$, then we write $x_t \in \vee q A$. The symbols $x_t \overline{\epsilon} A$, $x_t \overline{q} A$, $x_t \overline{\in \vee q A}$ mean that $x_t \in A$, $x_t q A$, $x_t \in \vee q A$ do not hold, respectively.

Definition 6 (see [7, 12]). A fuzzy subset of a group G is said to be an $(\epsilon, \epsilon \vee q)$-*fuzzy subgroup* of G if for all $x, y \in G$ and $t, r \in (0, 1]$,

(i) $x_t, y_r \in A \Rightarrow (x + y)_{\min\{t,r\}} \in \vee q A$,

(ii) $x_t \in A \Rightarrow (-x)_t \in \vee q A$.

Remark 7 (see [7]). The conditions (i) and (ii) of Definition 6 are respectively equivalent to

(i) $A(x + y) \geq \min\{A(x), A(y), 0.5\}$,

(ii) $A(-x) \geq \min\{A(x), 0.5\}$,

for all $x, y \in G$.

Remark 8. For any $(\epsilon, \epsilon \vee q)$-fuzzy subgroup A of G such that $A(x) \geq 0.5$ for some $x \in G$, then $A(0) \geq 0.5$ and if $A(0) < 0.5$, then $A(x) < 0.5$ for all $x \in G$. So, A is just the usual fuzzy subgroup in the sense of Rosenfeld.

Remark 9. It is noted that if A is a fuzzy subgroup then it is an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup of G. However the converse may not be true.

Here onwards we assume that A is an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup in the nontrivial sense for which case we have $A(0) \geq 0.5$.

Definition 10 (see [7]). An $(\epsilon, \epsilon \vee q)$-fuzzy subgroup of a group G is said to be $(\epsilon, \epsilon \vee q)$-*fuzzy normal subgroup* if for any $x, y \in G$ and $t \in (0, 1]$,

$$x_t \in A \Longrightarrow (y + x - y)_t \in \vee q A. \tag{3}$$

Remark 11 (see [7]). The condition of $(\epsilon, \epsilon \vee q)$-fuzzy normal subgroup is given in the equivalent forms as

(i) $A(y + x - y) \geq \min\{A(x), 0.5\}$,

(ii) $A(x + y) \geq \min\{A(y + x), 0.5\}$,

(iii) $A([x, y]) \geq \min\{A(x), 0.5\}$, for all $x, y \in G$.

Here $[x, y]$ denotes the commutator of x, y in G.

In the light of this fact, the condition of Definition 10 can be replaced by any one of the above conditions in Remark 8.

Definition 12 (see [18]). Let A be a fuzzy subset of an N-group G. It is called a *fuzzy N-subgroup of G* if it satisfies the following conditions:

(i) $A(x + y) \geq \min\{A(x), A(y)\}$,

(ii) $A(nx) \geq A(x)$,

for all $x, y \in G$, $n \in N$.

Remark 13. If G is a unitary N-group, the above conditions are equivalent to conditions $A(x - y) \geq \min\{A(x), A(y)\}$ and $A(nx) \geq A(x)$ for all $x, y \in G$, $n \in N$.

Definition 14 (see [18, 19]). A nonempty fuzzy subset A of an N-group G is called a *fuzzy ideal* if it satisfies the following conditions:

(i) $A(x - y) \geq \min\{A(x), A(y)\}$,

(ii) $A(nx) \geq A(x)$,

(iii) $A(y + x - y) \geq A(x)$,

(iv) $A(n(x + y) - nx) \geq A(y)$,

for all $x, y \in G$, $n \in N$.

Definition 15 (see [14]). A fuzzy set A of a near-ring N is called an $(\epsilon, \epsilon\vee q)$-*fuzzy subnear-ring* of N if for all $t, r \in (0,1]$, and $x, y \in N$

(i) (a) $x_t, y_r \in A \Rightarrow (x+y)_{\min\{t,r\}} \in \vee qA$,

(b) $x_t \in A \Rightarrow (-x)_t \in \vee qA$,

(ii) $x_t, y_r \in A \Rightarrow (xy)_{\min\{t,r\}} \in \vee qA$.

A is called an $(\epsilon, \epsilon\vee q)$-fuzzy ideal of N if it is $(\epsilon, \epsilon\vee q)$-fuzzy subnear-ring of N and

(iii) $x_t \in A \Rightarrow (y+x-y) \in \vee qA$,

(iv) $y_r \in A, x \in N \Rightarrow (yx)_r \in \vee qA$,

(v) $a_t \in A \Rightarrow (y(x+a)-yx)_t \in \vee qA$,

for all $x, y, a \in N$.

2. Generalized Fuzzy Ideals

In this section, we give the definition of $(\epsilon, \epsilon \vee q)$-fuzzy subgroup and ideal of an N-group G based on Definitions 14 and 15.

Definition 16. A fuzzy subset A of an N-group G is said to be an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup of G if $x, y \in G$, $n \in N$, $t, r \in (0,1]$,

(i) $x_t, y_r \in A \Rightarrow (x+y)_{\min\{t,r\}} \in \vee qA$,

(ii) $x_t \in A \Rightarrow (-x)_t \in \vee qA$,

(iii) $x_t \in A, n \in N \Rightarrow (nx)_t \in \vee qA$.

Lemma 17. *Let A be a fuzzy subset of G and $t, r \in (0,1]$. Then,*

(i) $x_t, y_r \in A \Rightarrow (x+y)_{\min\{t,r\}} \epsilon \vee qA \Leftrightarrow A(x+y) \geq \min\{A(x), A(y), 0.5\}$,

(ii) $x_t \in A \Rightarrow (-x)_t \in \vee qA \Leftrightarrow A(-x) \geq \min\{A(x), 0.5\}$, *for all $x \in G$,*

(iii) $x_t \in A \Rightarrow (nx)_t \epsilon \vee qA \Leftrightarrow A(nx) \geq \min\{A(x), 0.5\}$, *for all $x \in G$, $n \in N$.*

Proof. (i) Let $x, y \in G$. Consider the case (a): $\min\{A(x), A(y)\} < 0.5$.

Assume that $A(x+y) < \min\{A(x), A(y), 0.5\} = \min\{A(x), A(y)\}$. Choose t such that $A(x+y) < t < \min\{A(x), A(y)\}$ which implies that $x_t \in A$, $y_t \in A$ but $(x+y)_t \overline{\in \vee qA}$ [as $A(x+y)+t < 1$ and $A(x+y) < t$]. Consider the case (b): $\min\{A(x), A(y)\} \geq 0.5$. Assume that $A(x+y) < \min\{A(x), A(y), 0.5\} = 0.5$. Choose t such that $A(x+y) < t < 0.5$ so that $x_t, y_t \in A$ but $(x+y)_t \overline{\in \vee qA}$.

Conversely, let $x_t, y_r \in A \Rightarrow A(x) \geq t, A(y) \geq r$. Then, $A(x+y) \geq \min\{A(x), A(y), 0.5\} \geq \min\{t, r, 0.5\}$. Thus $A(x+y) \geq \min\{t, r\}$ if either t or $r \leq 0.5$ and $A(x+y) \geq 0.5$ if both t and $r > 0.5$ which means $(x+y)_{\min\{t,r\}} \epsilon \vee qA$.

(ii) Let $x \in G$, $\min\{A(x), 0.5\} \leq 0.5$. Suppose $A(-x) < \min\{A(x), 0.5\} \leq 0.5$. Choose r such that $A(-x) < r < \min\{A(x), 0.5\} \leq 0.5$. Then, $x_r \in A$ but $(-x)_r \overline{\in \vee qA}$ which contradicts the hypothesis. So, $A(-x) \geq \min\{A(x), 0.5\}$ for all $x \in G$.

Conversely, let $x_t \in A$. Then, $A(x) \geq t$. But we have $A(-x) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} \Rightarrow A(-x) \geq t$ or $A(-x) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5 \Rightarrow (-x)_t \in \vee qA$.

(iii) Let $x \in G$ and $\min\{A(x), 0.5\} \leq 0.5$. Suppose $A(nx) < \min\{A(x), 0.5\} \leq 0.5$. Choose r such that $A(nx) < r < \min\{A(x), 0.5\} \leq 0.5$. Then, $A(x) > r$ that is, $x_r \in A$, but $(nx)_r \overline{\in \vee qA}$ as $A(nx) < r$ and $A(nx) + r \leq 1$.

Conversely let $x_t \in A, n \in N$; then $A(x) \geq t$. But $A(nx) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} \Rightarrow A(nx) \geq t$ or $A(nx) \geq 0.5$ according as $t \leq 0.5$ or $r > 0.5 \Rightarrow A(nx) \geq t$ or $A(nx) + t > 1 \Rightarrow (nx)_t \in \vee qA$. □

Theorem 18. *Let A be a fuzzy subset of G. Then, A is an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup of G if and only if the following conditions are satisfied:*

(i) $A(x+y) \geq \min\{A(x), A(y), 0.5\}$,

(ii) $A(-x) \geq \min\{A(x), 0.5\}$,

(iii) $A(nx) \geq \min\{A(x), 0.5\}$,

for all $x, y \in G$, $n \in N$.

Proof. It follows from the previous lemma. □

Definition 19. A fuzzy subset A of an N-group G is said to be $(\epsilon, \epsilon \vee q)$-*fuzzy ideal* of G if it is an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup and satisfies the following conditions:

(i) $x_t \in A \Rightarrow (y+x-y)_t \in vqA$,

(ii) $a_t \in A \Rightarrow (n(x+a)-nx)_t \in vqA$,

for any $n \in N$, $x, a \in G$.

Lemma 20. *Let A be a fuzzy subset of G and $t, r \in (0,1]$. Then,*

(i) $x_t \in A \Rightarrow (y+x-y)_t \in vqA \Leftrightarrow A(y+x-y) \geq \min\{A(x), 0.5\}$,

(ii) $a_t \in A \Rightarrow (n(x+a)-nx)_t \in vqA \Leftrightarrow A(n(x+a)-nx) \geq \min\{A(x), 0.5\}$.

Proof. (i) Assume that $A(y+x-y) < \min\{A(x), 0.5\}$. Choose t such that $A(y+x-y) < t < \min\{A(x), 0.5\}$. But $\min\{A(x), 0.5\} \leq A(x)$ or 0.5 according as $A(x) < 0.5$ or $A(x) \geq 0.5$. So, $A(x) > t$ or $A(x) \geq 0.5 \Rightarrow x_t \in A$ or $x_{0.5} \in A$. But $(y+x-y)_t \overline{\in vqA}$ or $(y+x-y)_{0.5} \overline{\in vqA}$, respectively, which contradicts the hypothesis.

Conversely, assume that $x_t \in A$, then $A(x) \geq t$. For any $y \in G$, we have $A(y+x-y) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} \Rightarrow A(y+x-y) \geq t$ or 0.5 according as $t < 0.5$ or $t \geq 0.5 \Rightarrow (y+x-y)_t \in A$ or $A(y+x-y)+t > 1$. So, $\Rightarrow (y+x-y)_t \in vqA$.

(ii) Assume that $A(n(x+a)-nx) < \min\{A(a), 0.5\} = A(a)$ or 0.5 for some $n \in N$, $x, a \in G$. According as $A(a) < 0.5$ or $A(a) \geq 0.5$. Choose $t \in (0,1]$ such that $A(n(x+a)-nx) < t < \min\{A(a), 0.5\} \leq 0.5$. In either case, $A(n(x+a)-nx) < t$ and $A(n(x+a)-nx)+t < 1$. So, $(n(x+a)-nx)_t \overline{\in vqA}$, a contradiction.

Conversely, assume that $A(n(x+a)-nx) \geq \min\{A(a), 0.5\}$ for all $a, x \in G$, $n \in N$. Let $a_t \in A$. Then, $A(a) \geq t$. So, $A(n(x+a)-nx) \geq \min\{t, 0.5\} \leq 0.5 = t$ or 0.5 according as $t \leq 0.5$ or $t > 0.5$. So, $A(n(x+a)-nx)_t \in vqA$. □

Theorem 21. *Let A be an $(\epsilon, \epsilon \vee q)$ fuzzy subgroup of G. Then, A is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G if and only if*

(i) $A(y + x - y) \geq \min\{A(x), 0.5\}$, *for all* $x, y \in G$,

(ii) $A(n(x + a) - nx) \geq \min\{A(a), 0.5\}$, *for all* $n \in N$, $x, a \in G$.

Proof. It is immediate from Lemma 20. □

By definition, a fuzzy ideal of G is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. But the converse is not true in general as shown by the following example.

Example 22. Consider $G = \mathbb{S}_3 = \{i, \rho_1, \rho_2, \tau_1, \tau_2, \tau_3\}$ (written additively) to be a $\mathbb{Z}$-group. Define a fuzzy subset A of G as $A(i) = 1$, $A(\rho_1) = A(\rho_2) = A(\tau_2) = A(\tau_3) = 0.6$, $A(\tau_1) = 0.8$ which is not fuzzy ideal as $A[2(\tau_1 + \tau_2) - 2\tau_2] = A(\rho_1) = A(\rho_2) = 0.6 < A(\tau_1)$; it contradicts the condition (iv) of Definition 14. As $A(x - y)$, $A(nx)$, $A(y + x - y)$ and $A(n(x+a)-nx) = 0.6$ or $0.8 \geq \min\{0.5, 0.6$ or $0.8\} = 0.5$, thus, the notion of $(\epsilon, \epsilon \vee q)$-fuzzy ideal is a successful generalization of fuzzy ideals of G as introduced in [18].

Theorem 23. *Let $\{A_i, i \in J\}$ be any family of $(\epsilon, \epsilon \vee q)$-fuzzy ideals of G. Then, $A = \bigcap A_i$ is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G.*

Proof. It is straightforward. □

Theorem 24. *A nonempty subset I of G is an ideal of G if and only if χ_I is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G.*

Proof. If I is an ideal of G, it is clear from [18, Proposition 2.11] that χ_I is fuzzy ideal of G. Since every fuzzy ideal is $(\epsilon, \epsilon \vee q)$-fuzzy ideal, χ_I is $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G.

Conversely, let χ_I be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. Let $x, y \in I$, $\chi_I(x - y) \geq \min\{\chi_I(x), \chi_I(y), 0.5\} = 0.5$. So, $\chi_I(x - y) = 1 \Rightarrow x - y \in I$. Let $n \in N$, $x \in I$, $\chi_I(nx) \geq \min\{\chi_I(x), 0.5\} = 0.5 \Rightarrow \chi_I(nx) = 1 \Rightarrow nx \in Iy \in G$, $x \in I$, $\chi_I(y + x - y) \geq \min\{\chi_I(x), 0.5\} = 0.5 \Rightarrow \chi_I(y + x - y) = 1 \Rightarrow y + x - y \in In \in N$, $x \in I$, $y \in G$, $\chi_I(n(y + x) - ny) \geq \min\{\chi_I(x), 0.5\} = 0.5 \Rightarrow \chi_I(n(y + x) - ny) = 1 \Rightarrow n(y + x) - ny \in I$. Then, I is an ideal of G. □

Theorem 25. *A fuzzy subset A of G is an $(\epsilon, \epsilon \vee q)$-fuzzy (subgroup) ideal of G if and only if the level subset A_t is a (subgroup) ideal for $0 < t \leq 0.5$.*

Proof. We prove the result for $(\epsilon, \epsilon \vee q)$-fuzzy ideal S. Let A be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. Let $t \leq 0.5$, $x, y, i \in A_t$, $n \in N$.

(i) $A(x - y) \geq \min\{A(x), A(y), 0.5\} \geq \min\{t, 0.5\} = t \Rightarrow x - y \in A_t$,

(ii) $A(nx) \geq \min\{A(a), 0.5\} \geq \min\{t, 0.5\} = t \Rightarrow nx \in A_t$,

(iii) $A(y + x - y) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} = t \Rightarrow y + x - y \in A_t$, $y \in G$

(iv) $A(n(y + x) - ny) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} = t \Rightarrow n(y + x) - ny \in A_t$, $y \in G$, $n \in N$.

Hence, A_t is an ideal of G. Again, let A_t be an ideal of G for all $t \leq 0.5$. If possible, let there exist $x, y \in G$ such that $A(x - y) < t < \min\{A(x), A(y), 0.5\}$. Let t be such that $A(x - y) < t < \min\{A(x), A(y), 0.5\} \Rightarrow x, y \in A_t$ and $x - y \notin A_t$, a contradiction. So, $A(x - y) \geq \min\{A(x), A(y), 0.5\}$, for all $x, y \in G$. For $n \in N$, $x \in G$ let $A(nx) < \min\{A(x), 0.5\}$. If possible let t be such that $A(nx) < t < \min\{A(x), 0.5\}$. This implies $x \in A_t$, but $nx \notin A_t$, a contradiction. Similarly, we can prove that $A(y + x - y) \geq \min\{A(x), 0.5\}$, $A[n(y + x) - ny] \geq \min\{A(x), 0.5\}$, $x, y \in G$, $n \in N$. □

Remark 26. For $t \in (0.5, 1)$, A may be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G, but A_t may not be an ideal of G. Let $t = 0.8$ in Example 22. Then, $A_t = \{i, \tau_1\}$. A_t is not an ideal of $\mathbb{S}_3$ as it is not a normal subgroup of $\mathbb{S}_3$.

We are looking for a corresponding result when A_t is an ideal of G for all $t \in (0.5, 1]$.

Theorem 27. *Let A be a fuzzy subset of an N-group G. Then, $A_t \neq \phi$ is an ideal of G for all $t \in (0.5, 1]$ if and only if A satisfies the following conditions:*

(i) $\max\{A(x - y), 0.5\} \geq \min\{A(x), A(y)\}$,

(ii) $\max\{A(nx), 0.5\} \geq A(x)$,

(iii) $\max\{A(y + x - y), 0.5\} \geq A(x)$,

(iv) $\max\{A(n(y + x) - ny), 0.5\} \geq A(x)$,

for all $x, y \in G$, $n \in N$.

Proof. Suppose that $A_t \neq \phi$ is an ideal of G for all $t \in (0.5, 1]$. In order to prove (i), suppose that for some $x, y \in G$, $\max\{A(x - y), 0.5\} < \min\{A(x), A(y)\}$. Let $t = \min\{A(x), A(y)\}$. So, $x, y \in A_t$ and $t \in (0.5, 1]$. Since A_t is an ideal, $x - y \in A_t$. So, $A(x - y) \geq t > \max\{A(x - y), 0.5\}$, a contradiction. In order to prove (ii), suppose that $x \in G$, $n \in N$ and $\max\{A(nx), 0.5\} < A(x) = t$ (say). Then, $x \in A_t \Rightarrow nx \in A_t \Rightarrow A(nx) \geq t > \max\{A(nx), 0.5\}$, a contradiction. Similarly, we can prove (iii) and (iv).

Conversely, suppose that conditions (i) to (iv) hold. We show that A_t is an ideal of G for all $t \in (0.5, 1]$. Let $x, y \in A_t$. Then, $0.5 < t \leq \min\{A(x), A(y)\} \leq \max\{A(x - y), 0.5\} = A(x - y)$. So, $x - y \in A_t$. Let $n \in N$, $x \in A_t$. Then, $0.5 < t \leq A(x) \leq \max\{A(x), 0.5\} = A(x)$ so $nx \in A_t$. For $x \in A_t$, $y \in G$, $0.5 < t \leq A(x) \leq \max\{A(y + x - y), 0.5\} = A(y + x - y) \Rightarrow y + x - y \in A_t$. Also, if $n \in N$, $x \in A_t$, $y \in G$, $0.5 < t \leq A(x) \leq \max\{A(n(y + x) - ny), 0.5\} = A(n(y + x) - ny)$. Hence, $n(y + x) - ny \in A_t$. Then, A_t is an ideal of G. □

A definition for the previous kind of fuzzy subset was given for the case of near-rings in [17]. Now, we give the definition for N-groups.

Definition 28. A fuzzy subset of G is called an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$-*fuzzy subgroup* of G if for all $t, r \in (0, 1]$ and for all $x, y \in G$, $n \in N$,

(i) (a) $(x + y)_{\min(t,r)} \overline{\epsilon} A$ implies $x_t \overline{\epsilon} \vee \overline{q} A$ or $y_r \overline{\epsilon} \vee \overline{q} A$,

(b) $(-x)_t \overline{\epsilon} A$ implies $x_t \overline{\epsilon} \vee \overline{q} A$,

(ii) $(nx)_t \overline{\in} A$ implies $x_t \overline{\in} \vee \overline{q} A$.

Moreover, A is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of G if A is $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G and

(iii) $(y + x - y)_t \overline{\in} A$ implies $x_t \overline{\in} \vee \overline{q} A$,

(iv) $(n(x + y) - ny)_t \overline{\in} A$ implies $x_t \overline{\in} \vee \overline{q} A$.

Theorem 29. *A fuzzy subset A of G is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of G if and only if*

(1) (a) $\max\{A(x + y), 0.5\} \geq \min\{A(x), A(y)\}$,

(b) $\max\{A(-x), 0.5\} \geq A(x)$,

(2) $\max\{A(y + x - y), 0.5\} \geq A(x)$,

(3) $\max\{A(nx), 0.5\} \geq A(x)$,

(4) $\max\{A(n(y + x) - ny), 0.5\} \geq A(x)$.

Proof. (i)a $\Leftrightarrow$ (1)a. Let $x, y \in G$ be such that $\max\{A(x + y), 0.5\} < \min\{A(x), A(y)\}$. Let $t = \min\{A(x), A(y)\}$; then $0.5 < t \leq 1$, $(x + y)_t \overline{\in} A$. So we must have $x_t \overline{\in} \vee \overline{q} A$ or $y_t \overline{\in} \vee \overline{q} A$. But $x_t \in A$ and $y_t \in A$. Here $x_t \overline{q} A$ or $y_t \overline{q} A$ then $t \leq A(x)$ and $A(x) + t \leq 1$ or $t \leq A(y)$ and $A(y) + t \leq 1$ then $t \leq 0.5$, a contradiction.

Conversely, let $(x + y)_{\min(t,r)} \overline{\in} A$. Then, $A(x + y) < \min(t, r)$. If $A(x + y) \geq \min\{A(x), A(y)\}$, then $\min\{A(x), A(y)\} < \min(t, r)$. Hence, either $A(x) < t$ or $A(y) < r$ which implies $x_t \overline{\in} A$ or $y_r \overline{\in} A$. Thus, $x_t \overline{\in} \vee \overline{q} A$ or $y_r \overline{\in} \vee \overline{q} A$.

Again if $A(x + y) < \min\{A(x), A(y)\}$, then by (1)a

$$0.5 \geq \min\{A(x), A(y)\} > A(x + y). \tag{4}$$

Suppose that $x_t \in A$ or $y_r \in A$ then $t \leq A(x) \leq 0.5$ or $r \leq A(y) \leq 0.5$. It follows that either $x_t \overline{q} A$ or $y_t \overline{q} A$, and thus $x_t \overline{\in} \vee \overline{q} A$ or $y_r \overline{\in} \vee \overline{q} A$.

(i)b $\Leftrightarrow$ (1)b: Suppose that there exists $x \in G$ such that $\max\{A(-x), 0.5\} < A(x)$. If $A(x) = t$ then $0.5 < t \leq 1$ and $A(-x) < t$ so that $(-x)_t \overline{\in} A$. But then we must have either $(x)_t \overline{\in} A$ or $x_t \overline{q} A$. Also we have $(x)_t \in A$. So, $A(x) + 1 \leq 1$ which means that $t \leq 0.5$, a contradiction.

Conversely, suppose that $(x)_t \overline{\in} A$ then $A(-x) < t$. If $A(-x) \geq A(x)$, then $A(x) < t$ which gives $x_t \overline{\in} \vee \overline{q} A$. Again if $A(-x) < A(x)$ by (1)b we have $0.5 \geq A(x)$. Putting $(x)_t \in A$, then $t \leq A(x) \leq 0.5$ so that $x_t \overline{q} A$ which means that $x_t \overline{\in} \vee \overline{q} A$. Similarly, we can prove the remaining parts. □

Theorem 30. *A fuzzy subset A of G is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal if and only if $A_t (\neq \phi)$ is an ideal of G for all $t \in (0.5, 1]$.*

3. Fuzzy Cosets and Isomorphism Theorem

In this section, we first study the properties of $(\epsilon, \epsilon \vee q)$-fuzzy ideals under a homomorphism. Then, we introduce the fuzzy cosets and prove the fundamental isomorphism theorem on N-groups with respect to the structure induced by these fuzzy cosets.

Theorem 31. *Let G and G' be two N-groups, and let $f : G \to G'$ be an N-homomorphism. If f is surjective and A is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G, then so is $f(A)$. If B is a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G', then $f^{-1}(B)$ is a fuzzy ideal of G.*

Proof. We assume that A is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. For any $x, y \in G'$; it follows that

$$\begin{aligned} f(A)(x + y) &= \sup_{x+y=f(z)} \{A(z)\} \\ &\geq \sup_{f(u)=x, f(v)=y} \{A(u + v)\} \\ &\geq \sup_{f(u)=x, f(v)=y} \{\min\{A(u), A(v), 0.5\}\} \\ &= \min\left\{\sup_{f(u)=x} \{A(u)\}, \sup_{f(v)=y} \{A(v)\}, 0.5\right\} \\ &= \min\{f(A)(x), f(A)(y), 0.5\}. \end{aligned} \tag{5}$$

Also,

$$\begin{aligned} f(A)(-x) &= \sup_{f(z)=-x} \{A(z)\} = \sup_{f(-z)=x} \{A(z)\} \\ &= \sup_{f(z)=x} \{A(-z)\} \\ &\geq \sup_{f(z)=x} \{\min\{A(u), 0.5\}\} \\ &= \min\left\{\sup_{f(z)=x} \{A(z)\}, 0.5\right\} \\ &= \min\{f(A)(x), 0.5\}. \end{aligned} \tag{6}$$

Again,

$$\begin{aligned} f(A)(nx) &= \sup_{f(z)=nx} \{A(z)\} \geq \sup_{f(u)=x} \{A(nu)\} \\ &\geq \sup_{f(u)=x} \{\min\{A(u), 0.5\}\} \\ &= \min\left\{\sup_{f(u)=x} \{A(u)\}, 0.5\right\} \\ &= \min\{f(A)(x), 0.5\}, \end{aligned}$$

$$\begin{aligned} f(A)(y + x - y) &= \sup_{f(z)=y+x-y} \{A(z)\} \\ &\geq \sup_{f(v)=x, f(u)=y} \{A(u + v - u)\} \end{aligned}$$

$$\begin{aligned}
&\geq \sup_{f(v)=x} \{\min\{A(v), 0.5\}\} \\
&= \min\left\{\sup_{f(v)=x} \{A(v)\}, 0.5\right\} \\
&= \min\{f(A)(x), 0.5\}, \\
f(A)&(n(y+x)-ny) \\
&= \sup_{f(z)=n(y+x)-ny} \{A(z)\} \\
&\geq \sup_{f(v)=x, f(u)=y} \{A(n(u+v)-nu)\} \\
&\geq \sup_{f(v)=x, f(u)=y} \{\min\{A(u), 0.5\}\} \\
&= \min\{f(A)(x), 0.5\}.
\end{aligned} \tag{7}$$

Therefore, $f(A)$ is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. Similarly, we can show that $f^{-1}(A)$ is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. □

Definition 32. Let A be $(\in, \in \vee q)$-fuzzy subgroup of G. For any $x \in G$, let A_x be defined by $A_x(g) = \min\{A(g-x), 0.5\}$ for all $g \in G$. This fuzzy subset A_x is called the $(\in, \in \vee q)$-fuzzy left coset of G determined by A and x.

Remark 33. Let A be an $(\epsilon, \epsilon \vee q)$-fuzzy subgroup of G. Then, A is an $(\epsilon, \epsilon \vee q)$-fuzzy normal if and only if $\min\{A(g-x), 0.5\} = \min\{A(-x+g), 0.5\}$ for all $x, g \in G$. If A is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal, we simply denote fuzzy coset by A_x.

Lemma 34. *Let A be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G. Then, $A_x = A_y$ if an only if $A(x-y) \geq 0.5$.*

Proof. Assume that $A(x-y) \geq 0.5$ and $g \in G$. $A_x(g) = \min\{A(g-x), 0.5\} \geq \min\{A(g-y), A(y-x), 0.5\} \geq \min\{A(g-y), A(x-y), 0.5\} = \min\{A(g-y), 0.5\} = A_y(g)$ which implies that $A_x \geq A_y$. Similarly, we can verify that $A_x \leq A_y$. Conversely we assume that $A_x = A_y$. Then, $A_y(x) = A_x(x) \Rightarrow \min\{A(x-y), 0.5\} = 0.5 \Rightarrow A(x-y) \geq 0.5$. □

Proposition 35. *Every fuzzy coset A_x is constant on every coset of $G_0 = \{x \in G \mid A(x) = A(0)\}$.*

Proof. Let $y + y_0 \in y + G_0$. Now, we have $A_x(y+y_0) = \min\{A(y+y_0-x), 0.5\} \geq \min\{A(y-x), A(x+y_0-x), 0.5\} \geq \min\{A(y-x), A(y_0), 0.5\} = \min\{A(y-x), 0.5\} = A_x(y)$. Also $A_x(y) = \min\{A(y-x), 0.5\} \geq \min\{A(-x+y+y_0-y_0), 0.5\} \geq \min\{A(-x+y+y_0), A(-y_0), 0.5\} = \min\{A(-x+y+y_0), 0.5\} \geq \min\{A(y+y_0-x), 0.5\} = A_x(y+y_0)$. Thus $A_x(y+y_0) = A_x(y)$ for all $y_0 \in G_0$. □

Theorem 36. *For any $(\epsilon, \epsilon \vee q)$-fuzzy ideal A of G, G_A the set of all fuzzy cosets of A in G is an N-group under the addition and scalar multiplication defined by $A_x + A_y = A_{x+y}$, $n(A_x) = A_{nx}$ for all $x, y \in G$, $n \in N$. The function $\overline{A}(A_x) : G_A \to [0,1]$ defined by $\overline{A}(A_x) = A(x)$ for all $A_x \in G_A$ is $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G_A.*

Proof. First, we show that the compositions are well defined. Let $x, y, c, d \in G$, $n \in N$ such that $A_x = A_y$ and $A_c = A_d$. Then, $A(x-y) \geq 0.5$, $A(c-d) \geq 0.5$. Now, we have $\min\{A(x+c-d-y), 0.5\} \geq \min\{A(-y+x+c-d), 0.5\} = 0.5$ which implies that $A(x+c-d-y) \geq 0.5$. So, by Lemma 34, we have $A_{x+c} = A_{y+d}$. Again $A_{nx}(g) = \min\{A(g-nx), 0.5\} \geq \min\{A(g-ny), A(nx-ny), 0.5\} = \min\{A(g-ny), A[n(y-y+x)-ny], 0.5\} \geq \min\{A(g-ny), A(-y+x), 0.5\} = A_{ny}(g)$. Similarly, we show that $A_{ny}(g) \geq A_{nx}(g)$. Thus, for $A_{ny}(g) = A_{nx}(g)$ for all $g \in G$. Hence the compositions are well defined. It is now easy to verify that G_A is an N-group with null element A_0 and negative element A_{-x}. Next, we check that $\overline{A}$ is $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G_A. Let $A_x, A_y \in G_A$. We have

(i) $\overline{A}(A_x - A_y) = \overline{A}(A_{x-y}) = A(x-y) \geq \min\{A(x), A(y), 0.5\} = \min\{\overline{A}(A_x), \overline{A}(A_y), 0.5\}$,

(ii) $\overline{A}(A_{nx}) = A(nx) \geq \min\{A(x), 0.5\} = \min\{\overline{A}(A_x), 0.5\}$,

(iii) $\overline{A}(A_y + A_x - Ay) = \overline{A}(A_{y+x-y}) = A(y+x-y) \geq \min\{A(x), 0.5\} = \min\{\overline{A}(A_x), 0.5\}$,

(iv) $\overline{A}[n(A_y + A_x) - nA_y] = \overline{A}[A_{n(y+x)-ny}] = A[n(y+x)-ny] \geq \min\{A(x), 0.5\} = \min\{\overline{A}(A_x), 0.5\}$.

Hence, the proof is completed. □

Lemma 37. *Let $H = \{x \in G \mid A_x \geq A_0\}$, $K = \{x \in G \mid A_x = A_0\}$. Then, $K = H = A_{0.5}$.*

Proof. Let $x \in H$. Then, for all $g \in G$, $A_x(g) \geq A_0(g) \Rightarrow \min\{A(g-x), 0.5\} \geq \min\{A(g), 0.5\} \Rightarrow \min\{A(-x), 0.5\} \geq \min\{A(0), 0.5\} = 0.5 \Rightarrow A(x) \geq 0.5$. Also, for $g \in G$, $A_0(g) = \min\{A(g), 0.5\} = \min\{A(g-x+x), 0.5\} \geq \min\{A(g-x), A(x), 0.5\} = \min\{A(g-x), 0.5\} = A_x(g) \Rightarrow A_0 \geq A_x$. So, $H = K$. As we have seen, if $x \in H$ then $A(x) \geq 0.5 \Rightarrow x \in A_{0.5}$. Conversely, if $x \in A_{0.5}$, for any $g \in G$, $A(g-x) \geq \min\{A(g), A(x), 0.5\} = \min\{A(g), 0.5\} \Rightarrow \min\{A(g-x), 0.5\} \geq \min\{A(g), 0.5\} \Rightarrow A_x \geq A_0$. Thus, $H = A_{0.5}$, which means that H is an ideal of G. □

Theorem 38. *If A is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G, then the map $f : G \to G_A$ given by $f(x) = A_x$ is an N-homomorphism with kernel $f = A_{0.5}$ and so $G/A_{0.5}$ is isomorphic to G_A.*

Proof. It is clear that f is an onto N-homomorphism from G to G_A with kernel $f = \{x \in G \mid f(x) = f(0)\} = \{x \in G \mid A_x = A_0\} = A_{0.5}$. □

Corollary 39. *Let A be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G, and let $\overline{B}$ be an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G_A. Then, $B : G \to [0,1]$ defined by $B(x) = \overline{B}(A_x)$ is an $(\epsilon, \epsilon \vee q)$-fuzzy ideal of G containing A.*

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