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Research Article

On $S_\lambda^L(I)$-Asymptotically Statistical Equivalence of Sequences of Sets

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This paper presents the notion of $S_\lambda^L(I)$-asymptotically statistical equivalence, which is a natural combination of asymptotic I-equivalence, and λ-statistical equivalence for sequences of sets. We find its relations to I-asymptotically statistical convergence, strong λ_I-asymptotically equivalence, and strong Cesaro I-asymptotically equivalence for sequences of sets.

1. Introduction and Background

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞, such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \tag{1}$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if

$$\lim_{n \to \infty} t_n(x) = L. \tag{2}$$

If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability.

We write

$$[C, 1] = \left\{ x = (x_n) : \exists L \in \mathbb{R}, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\},$$

$$[V, \lambda] = \left\{ x = (x_n) : \exists L \in \mathbb{R}, \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \right\}, \tag{3}$$

for the sets of sequences $x = (x_k)$, which are strongly Cesaro summable and strongly (V, λ)-summable to L, that is, $x_k \to L[C, 1]$ and $x_k \to L[V, \lambda]$, respectively. Let Λ denote the set of all nondecreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞, such that $\lambda_{n+1} \leq \lambda_n$ and $\lambda_1 = 1$.

Statistical convergence of sequences of points was introduced by Fast (see [1]), and under different names, it has been discussed in number theory, trigonometric series, and summability. In 1993, Marouf presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Mursalen defined λ-statistical convergence by using the λ sequence. He denoted this new method by S_λ, and found its relation to statistical convergence, $[C, 1]$-summability, and $[V, \lambda]$-summability (see [2]). Savaş introduced and studied the concepts of strongly λ-summability and λ-statistical convergence for fuzzy numbers (see [3]). He also presented asymptotically λ-statistical equivalent sequences of fuzzy numbers (see [4]). Kostyrko et al. (see [5, 6]) introduced the concept of I-convergence of sequences in a metric space and studied some properties of this convergence. In addition to these definitions, natural inclusion theorems are also presented. The concept of convergence of sequences of points has been extended by several authors to convergence

of sequences of sets. One of these extensions that we will consider in this paper is Wijsman convergence. The concept of Wijsman statistical convergence is an implementation of the concept of statistical convergence presented by Nuray and Rhoades (see [7]).

Definition 1. The sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \left|x_n - L\right| \ge \varepsilon\right\}\right| = 0. \tag{4}$$

In this case one writes $st - \lim x_k = L$ (see [8]).

Definition 2. A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in I$,

(ii) for each $A, B \in I$ one has $A \cup B \in I$,

(iii) for each $A \in I$ and each $B \subseteq A$ one has $B \in I$ (see [5]).

An ideal is called nontrivial if $\mathbb{N} \notin I$, and nontrivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Definition 3. A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin I$,

(ii) for each $A, B \in F$ one has $A \cap B \in F$,

(iii) for each $A \in F$ and each $B \supseteq A$ one has $B \in F$ (see [5]).

Proposition 4. *I is a nontrivial ideal in $\mathbb{N}$ if and only if*

$$F = F(I) = \{M = \mathbb{N} \setminus A : A \in I\}, \tag{5}$$

is a filter in $\mathbb{N}$ (see [5]).

Definition 5. Let I be a nontrivial ideal of subsets of $\mathbb{N}$, and let (X, d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be I-convergent to L. Therefore $L = I - \lim_{n\to\infty} x_n$ if and only if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{n \in \mathbb{N} : \left|x_n - L\right| \ge \varepsilon\right\}, \tag{6}$$

belongs to I. The number L is called the I limit of the sequence $x = (x_n)_{n\in\mathbb{N}} \in X$ (see [5]).

Definition 6. Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, one says that the sequence $\{A_k\}$ is Wijsman convergent to A:

$$\lim_{k\to\infty} d(x, A_k) = d(x, A), \tag{7}$$

for each $x \in X$. In this case one writes $W - \lim_{k\to\infty} A_k = A$ (see [9, 10]).

As an example, consider the following sequence of circles in the (x, y)-plane:

$$A_k = \left\{(x, y) : x^2 + y^2 + 2kx = 0\right\}. \tag{8}$$

As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 7. Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, one says that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \left|d(x, A_k) - d(x, A)\right| \ge \varepsilon\right\}\right| = 0. \tag{9}$$

In this case one writes $st - \lim_W A_k = A$ or $A_k \to A(WS)$ (see [7]).

$$WS := \left\{\{A_k\} : st - \lim_W A_k = A\right\}, \tag{10}$$

where WS denotes the set of Wijsman statistical convergence sequences.

Also, the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades (see [7]). Let (X, ρ) be a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

Definition 8. Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman Cesaro summable to A if $\{d(x, A_k)\}$ is Cesaro summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A), \tag{11}$$

and one says that the sequence $\{A_k\}$ is Wijsman strongly Cesaro summable to A if $\{d(x, A_k)\}$ is strongly summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left|d(x, A_k) - d(x, A)\right| = 0, \tag{12}$$

(see [7]).

Definition 9. Let (X, d) be a metric space. For any non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_{k\to\infty} \frac{d(x, A_k)}{d(x, B_k)} = 1, \tag{13}$$

(denoted by $A_k \sim B_k$) (see [11]).

As an example, consider the following sequences of circles in the (x, y)-plane.

$$\begin{aligned} A_k &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 2ky = 0\right\}, \\ B_k &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\right\}. \end{aligned} \tag{14}$$

Since

$$\lim_{k\to\infty} \frac{d(x, A_k)}{d(x, B_k)} = 1, \tag{15}$$

the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense); that is, $A_k \sim B_k$.

Definition 10. Let (X, d) be a metric space. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistically equivalent (Wijsman sense) of multiple L provided that for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| = 0, \tag{16}$$

(denoted by $A_k \overset{WS_L}{\sim} B_k$) and simply asymptotically statistically equivalent (Wijsman sense) if $L = 1$ (see [11]).

2. Main Results

Definition 11 (see [12]). Let (X, d) be a metric space and let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman I-convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$, the set

$$A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \ge \varepsilon\}, \tag{17}$$

belongs to I. In this case one writes $I_W - \lim A_k = A$ or $A_k \to A(I_W)$, and the set of Wijsman I-convergent sequences of sets will be denoted by

$$I_W = \{\{A_k\} : \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \in I\}. \tag{18}$$

As an example, consider the following sequence. Let $X = \mathbb{R}^2$, and let $\{A_k\}$ be the following sequence:

$$A_k = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\} & \text{if } k \ne n^2 \\ \{(x, y) \in \mathbb{R}^2 : y = -1\} & \text{if } k = n^2, \end{cases} \tag{19}$$

and $A = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. The sequence $\{A_k\}$ is not Wijsman convergent to the set A. But, if we take $I = I_d$, then $\{A_k\}$ is Wijsman I-convergent to set A, where I_d is the ideal of sets that have zero density.

Definition 12. Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is said to be Wijsman λ-statistically convergent or WS_λ-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \right| = 0. \tag{20}$$

In this case one writes $S_\lambda - \lim_W A_k = A$, or $\{A_k\} \to A(WS_\lambda)$ and,

$$WS_\lambda := \{\{A_k\} : A \subseteq X, WS_\lambda - \lim A_k = A\}. \tag{21}$$

If $\lambda_n = n$, then Wijsman λ-statistical convergence is the same as Wijsman statistical convergence for the sequences of sets.

Definition 13. Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly (V, λ) summable to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |d(x, A_k) - d(x, A)| = 0. \tag{22}$$

In this case one writes $\{A_k\} \to A[V, \lambda]$.

If $\lambda_n = n$, then $[V, \lambda]$-summability reduces to $[C, 1]$-summability for sequences of sets.

Theorem 14. *Let $\lambda \in \Lambda$, (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, then*

(i) *$\{A_k\} \to A[V, \lambda] \Rightarrow \{A_k\} \to A(WS_\lambda)$ and the inclusion $[V, \lambda] \subsetneqq (WS_\lambda)$ is proper for sequences of sets,*

(ii) *if $\{A_k\}$ is bounded (i.e., $\{A_k\} \in L_\infty$) and $\{A_k\} \to A(WS_\lambda)$, then $\{A_k\} \to A[V, \lambda]$,*

(iii) *$WS_\lambda \cap L_\infty = [V, \lambda] \cap L_\infty$,*

where L_∞ denotes the set of bounded sequences of sets.

Proof. (i) Let $\varepsilon > 0$ and $\{A_k\} \to A[V, \lambda]$. One has

$$\begin{aligned} &\sum_{k\in I_n} |d(x, A_k) - d(x, A)| \\ &\quad \ge \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\ge\varepsilon}} |d(x, A_k) - d(x, A)| \\ &\quad \ge \varepsilon \cdot \left| \{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \right|. \end{aligned} \tag{23}$$

Therefore $\{A_k\} \to A[V, \lambda] \Rightarrow \{A_k\} \to A(WS_\lambda)$.

The following example shows that $(WS_\lambda) \subsetneqq [V, \lambda]$ for sequences of sets:

$$A_k = \begin{cases} \{k\}, & \text{for } n - \left[\sqrt{\lambda_n}\right] + 1 \le k \le n, \\ \{0\}, & \text{otherwise.} \end{cases} \tag{24}$$

Then $\{A_k\} \notin L_\infty$ and for every ε $(0 < \varepsilon \le 1)$,

$$\begin{aligned} &\frac{1}{\lambda_n} \left| \{k \in I_n : |d(x, A_k) - d(x, \{0\})| \ge \varepsilon\} \right| \\ &\quad = \frac{\left[\sqrt{\lambda_n}\right]}{\lambda_n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{25}$$

that is, $\{A_k\} \to \{0\}\,(WS_\lambda)$. On the other hand,

$$\frac{1}{\lambda_n} \sum_{k\in I_n} |d(x, A_k) - d(x, \{0\})| \nrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{26}$$

that is, $\{A_k\} \nrightarrow \{0\}\,[V, \lambda]$.

(ii) Suppose that $\{A_k\}$ is bounded and $\{A_k\} \to A(WS_\lambda)$. Then there is a M such that $|d(x, A_k) - d(x, A)| \leq M$ for all k. Given $\varepsilon > 0$, one has

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left|d(x,A_k)-d(x,A)\right| \\
&\quad = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left|d(x,A_k)-d(x,A)\right| \\
&\qquad + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|<\varepsilon}}\left|d(x,A_k)-d(x,A)\right| \\
&\quad \leq \frac{M}{\lambda_n}\left|\left\{k\in I_n : \left|d(x,A_k)-d(x,A)\right|\geq\varepsilon\right\}\right| + \varepsilon,
\end{aligned} \tag{27}$$

which implies that $\{A_k\} \to A[V, \lambda]$.

(iii) This immediately follows from (i) and (ii). □

Definition 15. Let (X, d) be a metric space, and let I be an admissible ideal. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be asymptotically Wijsman I-equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{k\in\mathbb{N} : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\} \in I. \tag{28}$$

This will be denoted by $A_k \overset{I_W}{\sim} B_k$.

Definition 16. Let (X, d) be a metric space, and let I be an admissible ideal. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strong Cesaro I-asymptotically equivalent (Wijsman sense) of multiple L if every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{n\in\mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\} \in I. \tag{29}$$

This will be denoted by $A_k \overset{C_1^L(I_W)}{\sim} B_k$.

Definition 17. Let (X, d) be a metric space. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman I-asymptotically statistically equivalent of multiple L if for every $\varepsilon, \delta > 0$ and for each $x \in X$,

$$\left\{n\in\mathbb{N} : \frac{1}{n}\left|\left\{k\leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I. \tag{30}$$

This will be denoted by $A_k \overset{S^L(I_W)}{\sim} B_k$.

Example 18. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$, and let (X, d) be a metric space, then $A, A_k \subset X$ are non-empty closed subsets. Let $X = \mathbb{R}^2$, $\{A_k\}, \{B_k\}$ be the following sequences:

$$\begin{aligned}
A_k &= \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq n,\ 0\leq y\leq \frac{1}{n}\cdot x\right\}, & \text{if, } k\neq n^2, \\ \{0,0\}, & \text{otherwise,} \end{cases} \\
B_k &= \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq n,\ 0\leq y\leq -\frac{1}{n}\cdot x\right\}, & \text{if, } k\neq n^2, \\ \{0,0\}, & \text{otherwise.} \end{cases}
\end{aligned} \tag{31}$$

If we take $I = I_d$ we have

$$\left\{k\in\mathbb{N} : \left|\frac{d(x,A_k)}{d(x,B_k)} - 1\right| \geq \varepsilon\right\} \in I. \tag{32}$$

Thus, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically I-equivalent (Wijsman sense); that is, $A_k \overset{I_W^1}{\sim} B_k$, where I_d is the ideal of sets that have zero density.

Example 19. Let$I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let, (X, d) be a metric space, then $A, A_k \subset X$ are non-empty closed subsets. Let $X = \mathbb{R}^2$, $\{A_k\}, \{B_k\}$ be the following sequences:

$$\begin{aligned}
A_k &= \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : x^2+(y-1)^2 = \frac{1}{k}\right\}, & \text{if } k\neq n^2 \\ \{0,0\}, & \text{otherwise,} \end{cases} \\
B_k &= \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : x^2+(y+1)^2 = \frac{1}{k}\right\}, & \text{if } k\neq n^2 \\ \{0,0\}, & \text{otherwise.} \end{cases}
\end{aligned} \tag{33}$$

If we take $I = I_d$ we have

$$\left\{k\in\mathbb{N} : \left|\frac{d(x,A_k)}{d(x,B_k)} - 1\right| \geq \varepsilon\right\} \in I. \tag{34}$$

Thus, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically I-equivalent (Wijsman sense); that is, $A_k \overset{I_W^1}{\sim} B_k$, where I_d is the ideal of sets which have zero density.

Definition 20. Let (X, d) be a metric space. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly λ_I-asymptotically equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\} \in I. \tag{35}$$

This will be denoted by $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$.

Definition 21. Let (X, d) be a metric space. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, one says that the sequences $\{A_k\}$ and $\{B_k\}$ are I-asymptotically λ-statistically equivalent (Wijsman sense) of multiple L, provided that for every $\varepsilon, \delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I. \tag{36}$$

This will be denoted by $A_k \overset{S_\lambda^L(I_W)}{\sim} B_k$.

Theorem 22. *Let $\lambda \in \Lambda$ and let I be an admissible ideal in $\mathbb{N}$. If $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$, then $A_k \overset{S_\lambda^L(I_W)}{\sim} B_k$.*

Proof. Assume that $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$ and $\varepsilon > 0$. Then,

$$\begin{aligned}\sum_{k\in I_n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| &\geq \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \\ &\geq \varepsilon\left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right|,\end{aligned} \tag{37}$$

and so,

$$\begin{aligned}&\frac{1}{\varepsilon\lambda_n}\sum_{k\in I_n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \\ &\quad\geq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right|.\end{aligned} \tag{38}$$

Then for any $\delta > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\quad\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\delta\right\}.\end{aligned} \tag{39}$$

Since right hand belongs to I, then left hand also belongs to I, and this completes the proof. □

Theorem 23. *Let $\lambda \in \Lambda$ and let I be an admissible ideal in $\mathbb{N}$. If $\{A_k\}$ and $\{B_k\}$ are bounded and $A_k \overset{S_\lambda^L(I_W)}{\sim} B_k$, then $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$.*

Proof. Let $\{A_k\}, \{B_k\}$ be bounded sequences and let $A_k \overset{S_\lambda^L(I_W)}{\sim} B_k$. Then there is an M such that

$$\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \leq M, \tag{40}$$

for all k. For each $\varepsilon > 0$,

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \\ &\quad= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \\ &\qquad+ \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|<\varepsilon}}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \\ &\quad\leq M\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.\end{aligned} \tag{41}$$

Then,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\} \\ &\quad\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \times \left|\left\{k \in I_n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in I.\end{aligned} \tag{42}$$

Therefore, $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$. □

The following example shows that if $\{A_k\}$ and $\{B_k\}$ are not bounded, then Theorem 23 cannot be true.

Example 24. Take $L = 1$ and define $\{A_k\}$ to be,

$$A_k = \begin{cases}\{k\}, & k = k_{r-1} + 1,\ k_{r-1} + 2, \ldots, k_{r-1} + \left[\sqrt{\lambda_n}\right] \\ \{1\}, & \text{otherwise,}\end{cases} \tag{43}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function and $B_k = \{1\}$ for all k. Note that $\{A_k\}$ is not bounded. Then $A_k \overset{S_\lambda^L(I)}{\sim} B_k$, but $A_k \overset{V_\lambda^L(I)}{\sim} B_k$ is not true.

Theorem 25. *Let $\lambda \in \Lambda$ and let I be an admissible ideal in $\mathbb{N}$. If $A_k \overset{V_\lambda^L(I_W)}{\sim} B_k$, then $A_k \overset{C_1^L(I_W)}{\sim} B_k$.*

Proof. Assume that $A_k \overset{V_\lambda^L(I)}{\sim} B_k$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| \\
&\quad= \frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| + \frac{1}{n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| \\
&\quad\le \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| + \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| \\
&\quad\le \frac{2}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|,
\end{aligned} \tag{44}$$

and so,

$$\begin{aligned}
&\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\} \\
&\quad\subseteq \left\{n\in\mathbb{N}: \frac{1}{\lambda_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\frac{\varepsilon}{2}\right\}\in I.
\end{aligned} \tag{45}$$

Hence $A_k \overset{C_1^L(I_W)}{\sim} B_k$. □

Theorem 26. *If* $\liminf \lambda_n/n > 0$, then $A_k \overset{S^L(I_W)}{\sim} B_k$ implies $A_k \overset{S_\lambda^L(I_W)}{\sim} B_k$.

Proof. Assume that $\liminf(\lambda_n/n) > 0$ and there exists a $\delta > 0$ such that $\lambda_n/n \ge \delta$ for sufficiently large n. For given $\varepsilon > 0$ one has,

$$\begin{aligned}
&\frac{1}{n}\left\{k\le n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\} \\
&\quad\supseteq \frac{1}{n}\left\{k\in I_n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}.
\end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k\le n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right| \\
&\quad\ge \frac{1}{n}\left|\left\{k\in I_n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right| \\
&\quad\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k\in I_n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right| \\
&\quad\ge \delta\frac{1}{\lambda_n}\left|\left\{k\in I_n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right|;
\end{aligned} \tag{47}$$

then for any $\eta > 0$ we get

$$\begin{aligned}
&\left\{n\in\mathbb{N}: \frac{1}{\lambda_n}\left|\left\{k\in I_n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right|\ge\eta\right\} \\
&\quad\subseteq \left\{n\in\mathbb{N}: \frac{1}{n}\left|\left\{k\le n: \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right|\ge\eta\delta\right\}\in I,
\end{aligned} \tag{48}$$

and this completes the proof. □

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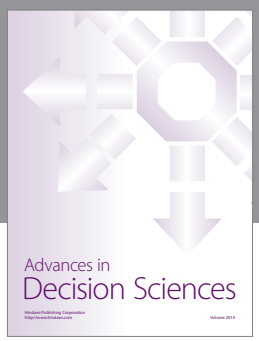

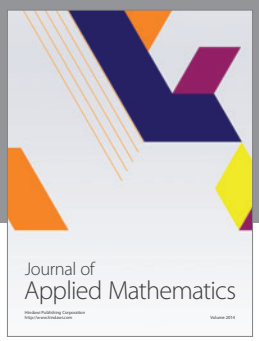

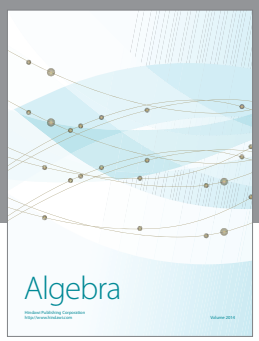

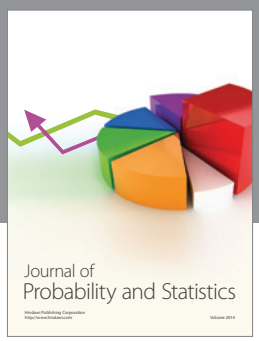

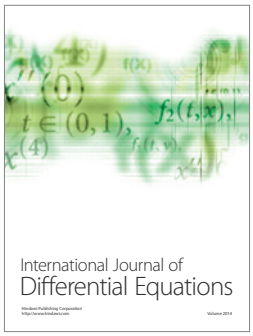

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