

Research Article

Geometric Mesh Three-Point Discretization for Fourth-Order Nonlinear Singular Differential Equations in Polar System

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Numerical method based on three geometric stencils has been proposed for the numerical solution of nonlinear singular fourth-order ordinary differential equations. The method can be easily extended to the sixth-order differential equations. Convergence analysis proves the third-order convergence of the proposed scheme. The resulting difference equations lead to block tridiagonal matrices and can be easily solved using block Gauss-Seidel algorithm. The computational results are provided to justify the usefulness and reliability of the proposed method.

1. Introduction

Consider the fourth-order boundary value problem:

$$u^{(4)}(r) = f(r, u(r), u^{(1)}(r), u^{(2)}(r), u^{(3)}(r)), \quad (1)$$

$$\alpha < r < \beta$$

subject to the necessary boundary conditions:

$$\begin{aligned} u(\alpha) &= \alpha_0, & u^{(2)}(\alpha) &= \alpha_1, \\ u(\beta) &= \beta_0, & u^{(2)}(\beta) &= \beta_0, \end{aligned} \quad (2)$$

where $f \in C^4[\alpha, \beta]$ and $\alpha_0, \alpha_1, \beta_0$, and β_1 are real constants and $-\infty < \alpha < r < \beta < \infty$.

Or equivalently

$$u^{(2)}(r) = v(r),$$

$$v^{(2)}(r) = f(r, u(r), u^{(1)}(r), v(r), v^{(1)}(r)), \quad (3)$$

$$\alpha < r < \beta$$

subject to the natural boundary conditions:

$$\begin{aligned} u(\alpha) &= \alpha_0, & v(\alpha) &= \alpha_1, \\ u(\beta) &= \beta_0, & v(\beta) &= \beta_1. \end{aligned} \quad (4)$$

Fourth-order differential equations occur in various areas of mathematics such as viscoelastic and inelastic flows, beam theory, Lifshitz point in phase transition physics (e.g., nematic liquid crystal, crystals, and ferroelectric crystals) [1], the rolls in a Rayleigh-Benard convection cell (two parallel plates of different temperature with a liquid in between) [2], spontaneous pattern formation in second-order materials (e.g., polymeric fibres) [3], the waves on a suspension bridge [4, 5], geological folding of rock layers [6], buckling of a strut on a nonlinear elastic foundation [7], traveling water waves in a shallow channel [8], pulse propagation in optical fibers [9], system of two reaction diffusion equation [10], and so forth.

The existence and uniqueness of the solution for the fourth and higher-order boundary value problems have been discussed in [11–14]. In the recent past, the numerical solution of fourth-order differential equations has been developed using multiderivative, finite element method, Ritz method, spline collocation, and finite difference method [15–18]. The determination of eigen values of self adjoint fourth-order differential equations was developed in [19] using finite difference scheme. The motivation of variable mesh technique for differential equations arises from the theory of electrochemical reaction-convection-diffusion problems in one-dimensional space geometry [20]. The geometric mesh method for self-adjoint singular perturbation problems using

finite difference approximations was discussed in [21]. The use of geometric mesh in the context of boundary value problems was studied extensively in [22–24]. In this paper, we derive a geometric mesh finite difference method for the solution of fourth- and sixth-order differential boundary value problems with order of accuracy being three. The simplicity of the proposed method lies in its three-point discretization without any use of fictitious nodes. The scheme is compact and applicable to both singular and nonsingular problems. The resulting difference equations are solved using block Gauss-Seidel algorithm for linear case, and corresponding Newton's method has been applied to nonlinear problems.

The paper is outlined in the following manner: In Section 2, the derivation of the method is discussed in detail. In Section 3, we define the procedure for numerical solution to singular problems in such a way that the method retains the order and accuracy even in the vicinity of singularity. In Section 4, algorithmic details are provided for the numerical solution of sixth-order differential equations. The convergence property has been discussed briefly in Section 4. The numerical illustrations based on geometric mesh as well as uniform mesh were provided in Section 5. The paper is concluded in the last section with future development and remarks.

2. Derivation of the Numerical Scheme

We discretize the solution region $[\alpha, \beta]$ such that $\alpha = r_0 < r_1 < \dots < r_n < r_{n+1} = \beta$. Let $h_k = r_k - r_{k-1}$ and $k = 1(1)n + 1$ be the nonuniform step size, and let $\sigma_k = h_{k+1}/h_k > 0$ be the geometric mesh ratio. Let $U_k = u(r_k)$ and $V_k = v(r_k)$ denote the exact solution values of u and v at the mesh r_k and u_k, v_k be their approximate solutions, respectively.

Consider the following three-point geometric mesh discretizations for $u^{(2)}(r) = v(r)$ and $v^{(2)}(r) = f(r)$:

$$\begin{aligned} U_{k+1} - (1 + \sigma_k)U_k + \sigma_k U_{k-1} + \frac{h^2}{12} (p_k V_{k+1} + q_k V_k + r_k V_{k-1}) \\ = O(h_k^5), \quad k = 1(1)n, \end{aligned} \quad (5)$$

$$\begin{aligned} V_{k+1} - (1 + \sigma_k)V_k + \sigma_k V_{k-1} + \frac{h^2}{12} (p_k f_{k+1} + q_k f_k + r_k f_{k-1}) \\ = E_k, \quad k = 1(1)n, \end{aligned} \quad (6)$$

where

$$\begin{aligned} E_k &= \frac{h_k^5}{360} \sigma_k (2\sigma_k + 1) (\sigma_k + 2) (1 - \sigma_k^2) f_k^{(5)}(\xi) \\ &\quad + O(h_k^6), \quad r_{k-1} < \xi < r_{k+1}, \\ p_k &= 1 - \sigma_k - \sigma_k^2, \\ q_k &= -(\sigma_k + 1)(\sigma_k^2 + 3\sigma_k + 1), \\ r_k &= \sigma_k(\sigma_k^2 - \sigma_k - 1). \end{aligned} \quad (7)$$

Define

$$\widetilde{U}'_k = \frac{U_{k+1} - (1 - \sigma_k^2)U_k - \sigma_k^2 U_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \quad (8)$$

$$\widetilde{V}'_k = \frac{V_{k+1} - (1 - \sigma_k^2)V_k - \sigma_k^2 V_{k-1}}{h_k \sigma_k (1 + \sigma_k)},$$

$$\widetilde{U}'_{k+1} = \frac{(1 + 2\sigma_k)U_{k+1} - (1 + \sigma_k)^2 U_k + \sigma_k^2 U_{k-1}}{h_k \sigma_k (1 + \sigma_k)},$$

$$\widetilde{V}'_{k+1} = \frac{(1 + 2\sigma_k)V_{k+1} - (1 + \sigma_k)^2 V_k + \sigma_k^2 V_{k-1}}{h_k \sigma_k (1 + \sigma_k)},$$

$$\widetilde{U}'_{k-1} = \frac{-U_{k+1} + (1 + \sigma_k)^2 U_k - \sigma_k (2 + \sigma_k) U_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \quad (9)$$

$$\widetilde{V}'_{k-1} = \frac{-V_{k+1} + (1 + \sigma_k)^2 V_k - \sigma_k (2 + \sigma_k) V_{k-1}}{h_k \sigma_k (1 + \sigma_k)},$$

$$\widetilde{f}_{k\pm 1} = f(r_{k\pm 1}, U_{k\pm 1}, \widetilde{U}'_{k\pm 1}, V_{k\pm 1}, \widetilde{V}'_{k\pm 1}).$$

It is easy to verify that

$$\begin{aligned} \widetilde{f}_{k+1} &= f_{k+1} - \frac{h_k^2}{6} \sigma_k (\sigma_k + 1) (G_k U_k^{(3)} + H_k V_k^{(3)}) + O(h_k^3), \\ \widetilde{f}_{k-1} &= f_{k-1} - \frac{h_k^2}{6} (\sigma_k + 1) (G_k U_k^{(3)} + H_k V_k^{(3)}) + O(h_k^3), \end{aligned} \quad (10)$$

where $G_k = (\partial f / \partial U^{(1)})_{r_k}$, $H_k = (\partial f / \partial V^{(1)})_{r_k}$, and so forth.

Now, let

$$\begin{aligned} \widehat{U}'_k &= \widetilde{U}'_k + \gamma h_k (V_{k+1} - V_{k-1}), \\ \widehat{V}'_k &= \widetilde{V}'_k + \delta h_k (\widetilde{f}_{k+1} - \widetilde{f}_{k-1}), \end{aligned} \quad (11)$$

where γ and δ are free parameters to be determined.

With the help of (8) and (11), it follows that

$$\begin{aligned} \widehat{U}'_k &= U_k^{(1)} + \frac{h_k^2}{6} (\sigma_k + 6\gamma(1 + \sigma_k)) U_k^{(3)} + O(h_k^3), \\ \widehat{V}'_k &= V_k^{(1)} + \frac{h_k^2}{6} (\sigma_k + 6\delta(1 + \sigma_k)) V_k^{(3)} + O(h_k^3). \end{aligned} \quad (12)$$

Further, we define

$$\widehat{f}_k = f(r_k, U_k, \widehat{U}'_k, V_k, \widehat{V}'_k). \quad (13)$$

With the help of (12)-(13), it follows that

$$\begin{aligned} \widehat{f}_k &= f_k + \frac{h_k^2}{6} (\sigma_k + 6\gamma(1 + \sigma_k)) G_k U_k^{(3)} \\ &\quad + \frac{h_k^2}{6} (\sigma_k + 6\delta(1 + \sigma_k)) H_k V_k^{(3)} + O(h_k^3). \end{aligned} \quad (14)$$

With the help of approximations (10), (14), the difference equation (6) at each internal mesh r_k , is approximated as follows:

$$V_{k+1} - (1 + \sigma_k) V_k + \sigma_k V_{k-1} + \frac{h_k^2}{12} (p_k \tilde{f}_k + q_k \hat{f}_k + r_k \tilde{f}_{k-1}) = \tilde{E}_k, \quad k = 1(1)n, \quad (15)$$

where

$$\begin{aligned} \tilde{E}_k = & -\frac{h_k^4}{72} U_k^{(3)} G_k (1 + \sigma_k) (6\gamma + (1 + 24\gamma) \sigma_k (1 + \sigma_k) \\ & + (1 + 6\gamma) \sigma_k^3) - \frac{h_k^4}{72} V_k^{(3)} H_k \\ & \times (1 + \sigma_k) (6\delta + (1 + 24\delta) \sigma_k + (1 + 6\delta) \sigma_k^3) \\ & + O(h_k^5), \quad \sigma_k \neq 1. \end{aligned} \quad (16)$$

The difference scheme (15) to be of $O(h_k^3)$, the coefficients of h_k^4 in (16) must be zero, and hence we obtain

$$\begin{aligned} 6\gamma + (1 + 24\gamma) \sigma_k (1 + \sigma_k) + (1 + 6\gamma) \sigma_k^3 &= 0, \\ 6\delta + (1 + 24\delta) \sigma_k (1 + \sigma_k) + (1 + 6\delta) \sigma_k^3 &= 0. \end{aligned} \quad (17)$$

Thus the values associated with (17) are given by $\gamma = \delta = -\sigma_k(\sigma_k^2 + \sigma_k + 1)/(6(1 + \sigma_k)(\sigma_k^2 + 3\sigma_k + 1))$, and the local truncation errors given by (16) becomes $\tilde{E}_k = O(h_k^5)$, $\sigma_k \neq 1$. However, for $\sigma_k \neq 1$, the error reduces to $\tilde{E}_k = O(h_k^6)$.

3. Application to Singular Problem

Consider the singular fourth-order linear differential equation in cylindrical polar coordinates:

$$\Delta^4 u(r) \equiv \left(\frac{d^2}{dr^2} + \frac{\lambda}{r} \frac{d}{dr} \right)^2 u(r) = g(r), \quad 0 < r < 1 \quad (18)$$

or equivalently

$$u^{(2)}(r) = v(r), \quad (19)$$

$$v^{(2)}(r) = a(r) v^{(1)}(r) + b(r) v(r) + c(r) u^{(1)}(r) + g(r), \quad (20)$$

where $a(r) = -2\lambda/r$, $b(r) = -\lambda(\lambda - 2)/r^2$, and $c(r) = \lambda(\lambda - 2)/r^3$ are the singular coefficients. The associated boundary conditions are given by (4). For $\lambda = 0, 1$, or 2 , the differential equation (18) shows planar, cylindrical, or spherical geometries (see, [25, 26]).

Applying the difference schemes (5) and (15) to (19) and (20), respectively, we obtain a system of coupled difference equations for $k = 1(1)n$:

$$\begin{aligned} & \begin{bmatrix} sb_k^{11} & sb_k^{12} \\ sb_k^{21} & sb_k^{22} \end{bmatrix} \begin{bmatrix} U_{k-1} \\ V_{k-1} \end{bmatrix} + \begin{bmatrix} dg_k^{11} & dg_k^{12} \\ dg_k^{21} & dg_k^{22} \end{bmatrix} \begin{bmatrix} U_k \\ V_k \end{bmatrix} \\ & + \begin{bmatrix} sp_k^{11} & sp_k^{12} \\ sp_k^{21} & sp_k^{22} \end{bmatrix} \begin{bmatrix} U_{k+1} \\ V_{k+1} \end{bmatrix} = \begin{bmatrix} R_k^{11} \\ R_k^{21} \end{bmatrix}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} sb_k^{11} &= \sigma_k, & sp_k^{11} &= 1, \\ dg_k^{11} &= -1 - \sigma_k, & sb_k^{12} &= \frac{h_k^2}{12} r_k, \\ sp_k^{12} &= \frac{h_k^2}{12} p_k, & dg_k^{12} &= \frac{h_k^2}{12} q_k, & R_k^{11} &= 0, \end{aligned}$$

$$\begin{aligned} sb_k^{21} &= \frac{h_k}{12(1 + \sigma_k)} \\ & \times (\gamma h_k q_k a_k (\sigma_k c_{k+1} + (2 + \sigma_k) (a_{k-1} + c_{k-1})) \\ & - \sigma_k (q_k c_k - p_k c_{k+1}) + (2 + \sigma_k) r_k (a_{k-1} + c_{k-1})), \end{aligned}$$

$$\begin{aligned} sp_k^{21} &= \frac{h_k}{12\sigma_k(1 + \sigma_k)} \\ & \times (\gamma h_k q_k a_k ((2\sigma_k + 1) c_{k+1} + c_{k-1}) \\ & + (2\sigma_k + 1) c_{k+1} p_k + c_k q_k - r_k c_{k-1}), \end{aligned}$$

$$\begin{aligned} dg_k^{21} &= \frac{h_k}{12\sigma_k} (\sigma_k (q_k c_k - p_k c_{k+1} + r_k c_{k-1}) \\ & - c_{k+1} p_k + r_k c_{k-1} - c_k q_k \\ & - \gamma h_k q_k a_k (1 + \sigma_k) (c_{k+1} + c_{k-1})), \end{aligned}$$

$$\begin{aligned} sb_k^{22} &= \sigma_k - \frac{\gamma h_k^3}{12} q_k (a_k b_{k-1} + c_k) \\ & + \frac{h_k}{12(1 + \sigma_k)} (h_k (\gamma q_k a_k a_{k+1} \sigma_k + (1 + \sigma_k) r_k b_{k-1}) \\ & + \sigma_k (p_k a_{k+1} - a_k q_k)), \end{aligned}$$

$$\begin{aligned} sp_k^{22} &= 1 + \frac{\gamma h_k^3}{12} q_k (a_k b_{k+1} + c_k) + \frac{h_k}{12\sigma_k(1 + \sigma_k)} \\ & \times (h_k (\gamma q_k a_k ((2\sigma_k + 1) a_{k+1} + a_{k-1}) \\ & + \sigma_k (\sigma_k + 1) p_k b_{k+1}) \\ & + (1 + 2\sigma_k) p_k a_{k+1} - r_k a_{k-1} + a_k q_k), \end{aligned}$$

$$\begin{aligned}
dg_k^{22} &= -1 - \sigma_k \\
&+ \frac{h_k}{12\sigma_k} ((\sigma_k (q_k a_k - p_k a_{k+1} + r_k a_{k-1}) \\
&\quad - p_k a_{k+1} + r_k a_{k-1} - a_k q_k) \\
&\quad - h_k q_k (\gamma a_k (\sigma_k + 1) (a_{k+1} + a_{k-1}) - b_k \sigma_k)), \\
R_k^{21} &= -\frac{\gamma h_k^3}{12} a_k q_k (g_{k+1} - g_{k-1}) \\
&\quad - \frac{h_k^2}{12} (q_k g_k + p_k g_{k+1} + r_k g_{k-1}).
\end{aligned} \tag{22}$$

Note that the scheme (21) fails when the solution is to be determined at $k = 1$. We overcome this difficulty by modifying the scheme in such a way that the solutions retain order and accuracy even in the vicinity of singularity $r = 0$. We consider the following approximations:

$$\begin{aligned}
a_{k+1} &= a_k + h_k \sigma_k a_k^{(1)} + \frac{h_k^2 \sigma_k^2}{2!} a_k^{(2)} + \frac{h_k^3 \sigma_k^3}{3!} a_k^{(3)} \\
&\quad + \frac{h_k^4 \sigma_k^4}{4!} a_k^{(4)} + O(h_k^5), \\
a_{k-1} &= a_k - h_k a_k^{(1)} + \frac{h_k^2}{2!} a_k^{(2)} - \frac{h_k^3}{3!} a_k^{(3)} \\
&\quad + \frac{h_k^4}{4!} a_k^{(4)} + O(h_k^5).
\end{aligned} \tag{23}$$

Using the similar approximations of $b_{k\pm 1}$, $c_{k\pm 1}$ and $g_{k\pm 1}$ and neglecting $O(h_k^5)$ terms, we can rewrite (21) with the help of compact operators as

$$\begin{aligned}
&\begin{bmatrix} sb_k^{11} & sb_k^{12} \\ \widetilde{sb}_k^{21} & \widetilde{sb}_k^{22} \end{bmatrix} \begin{bmatrix} U_{k-1} \\ V_{k-1} \end{bmatrix} + \begin{bmatrix} dg_k^{11} & dg_k^{12} \\ \widetilde{dg}_k^{21} & \widetilde{dg}_k^{22} \end{bmatrix} \begin{bmatrix} U_k \\ V_k \end{bmatrix} \\
&+ \begin{bmatrix} sp_k^{11} & sp_k^{12} \\ \widetilde{sp}_k^{21} & \widetilde{sp}_k^{22} \end{bmatrix} \begin{bmatrix} U_{k+1} \\ V_{k+1} \end{bmatrix} = \begin{bmatrix} R_k^{11} \\ \widetilde{R}_k^{21} \end{bmatrix},
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\widetilde{sb}_k^{21} &= ((12\gamma q_k a_k \sigma_k (\sigma_k + 1)^2 h_k^2 + 6(\sigma_k + 1) \sigma_k \\
&\quad \times (-r_k \sigma_k + \sigma_k p_k - q_k \sigma_k - 2r_k) h_k) c_k \\
&\quad + (-6\gamma q_k a_k \sigma_k (\sigma_k + 1)^2 h_k^3 + 6(\sigma_k + 1) \\
&\quad \times \sigma_k (p_k \sigma_k^2 + r_k \sigma_k + 2r_k) h_k^2) c_k^{(1)} - 3(\sigma_k + 1) \\
&\quad \times \sigma_k (p_k \sigma_k^2 + r_k) h_k^3 c_k^{(2)} - 3\sigma_k (\sigma_k + 2) \\
&\quad \times (2\gamma q_k a_k a_k^{(2)} + a_k^{(2)} r_k) h_k^3 \\
&\quad + 6\sigma_k (\sigma_k + 2) (\sigma_k + 1) (r_k a_k^{(1)} + \gamma a_k^2 q_k) h_k^2 \\
&\quad - 6r_k a_k \sigma_k (\sigma_k + 2) (\sigma_k + 1) h_k) \\
&\quad \times (72\sigma_k (\sigma_k + 1)^2)^{-1}, \\
sp_k^{21} &= (6q_k a_k \gamma \sigma_k (\sigma_k + 2) h_k^3 a_k^{(1)} \\
&\quad + (12a_k \gamma q_k (\sigma_k + 1)^2 h_k^2 + 6(\sigma_k + 1) \\
&\quad \times ((2\sigma_k + 1) p_k - r_k + q_k) h_k) c_k \\
&\quad + 6h_k^2 (\sigma_k + 1) (q_k a_k \gamma \sigma_k (\sigma_k + 1) h_k \\
&\quad + \sigma_k (2\sigma_k + 1) p_k + r_k) c_k^{(1)} \\
&\quad + 3(\sigma_k + 1) \sigma_k (p_k \sigma_k^2 + r_k) h_k^3 c_k^{(2)} + 3r_k a_k^{(2)} \sigma_k \\
&\quad \times (\sigma_k + 2) h_k^3) \times (72\sigma_k (1 + \sigma_k)^2)^{-1}, \\
\widetilde{dg}_k^{21} &= (\sigma_k (\sigma_k + 2) (\sigma_k + 1) (a_k^{(3)} r_k + 3\gamma q_k a_k a_k^{(2)}) h_k^4 \\
&\quad - 3\sigma_k (\sigma_k + 2) (\sigma_k + 1) (2\gamma q_k a_k a_k^{(1)} + a_k^{(2)} r_k) h_k^3 \\
&\quad - 6(\sigma_k + 1)^3 (p_k \sigma_k c_k^{(1)} + 2\gamma q_k a_k c_k + r_k c_k^{(1)}) h_k^2 \\
&\quad - 6c_k (\sigma_k + 1)^2 (-r_k \sigma_k + \sigma_k p_k - q_k \sigma_k - r_k + q_k + p_k) \\
&\quad \times h_k) \times (72\sigma_k (\sigma_k + 1)^2)^{-1}, \\
\widetilde{sb}_k^{22} &= 1 + (6\gamma q_k \sigma_k^2 (\sigma_k + 1) h_k^2 a_k^2 - 6h_k \sigma_k \\
&\quad \times (\gamma h_k^2 q_k ((\sigma_k^2 - 1 + \sigma_k) a_k^{(1)} + b_k (\sigma_k + 1)^2) \\
&\quad - \sigma_k (\sigma_k + 1) (p_k - q_k)) a_k + 6p_k \sigma_k^3 \\
&\quad \times (\sigma_k + 1) h_k^2 a_k^{(1)} - 3\sigma_k (p_k \sigma_k^3 + p_k \sigma_k^2 - r_k) \\
&\quad \times h_k^3 a_k^{(2)} - 6\sigma_k (\sigma_k + 1) (p_k \sigma_k^2 + r_k) h_k^3 b_k^{(1)} \\
&\quad + 6h_k^2 \sigma_k (\sigma_k + 1)^2 (r_k b_k - \gamma q_k c_k h_k)) \\
&\quad \times (72\sigma_k^2 (\sigma_k + 1)^2)^{-1},
\end{aligned}$$

$$\begin{aligned}
\widetilde{sp}_k^{22} &= 1 + \left(12q_k \gamma (\sigma_k + 1)^2 h_k^2 a_k^2 \right. \\
&\quad + \left(6q_k \gamma \sigma_k (\sigma_k^2 - 1 + \sigma_k) h_k^3 a_k^{(1)} \right. \\
&\quad + 6q_k \gamma b_k \sigma_k (\sigma_k + 1)^2 h_k^3 \\
&\quad + 6(\sigma_k + 1)(2\sigma_k p_k - r_k + q_k + p_k) h_k \Big) a_k \\
&\quad + 6(\sigma_k + 1)(2p_k \sigma_k^2 + \sigma_k p_k + r_k) \\
&\quad \times h_k^2 a_k^{(1)} + 3\sigma_k (p_k \sigma_k^3 + p_k \sigma_k^2 - r_k) h_k^3 a_k^{(2)} \\
&\quad + 6(\sigma_k + 1) \sigma_k (p_k \sigma_k^2 + r_k) h_k^3 b_k^{(1)} \\
&\quad + 6q_k c_k \gamma \sigma_k (\sigma_k + 1)^2 h_k^3 + 6p_k \sigma_k b_k \\
&\quad \times (\sigma_k + 1)^2 h_k^2 \Big) \times (72\sigma_k (\sigma_k + 1)^2)^{-1}, \\
\widetilde{dg}_k^{22} &= -1 - \sigma_k \\
&\quad + \left(-12\gamma (\sigma_k + 1)^3 q_k h_k^2 a_k^2 \right. \\
&\quad + \left(6\gamma \sigma_k (\sigma_k + 2) (\sigma_k + 1) q_k h_k^3 a_k^{(1)} \right. \\
&\quad - 3\gamma \sigma_k (\sigma_k + 2) (\sigma_k + 1) q_k h_k^4 a_k^{(2)} \\
&\quad + 6\gamma \sigma_k (\sigma_k + 1)^3 q_k h_k^4 b_k^{(1)} \\
&\quad + \left(6(\sigma_k - 1)(\sigma_k + 1)^2 q_k + 6(\sigma_k + 1)^3 \right. \\
&\quad \times (r_k - p_k) \Big) h_k \Big) a_k - 6(\sigma_k + 1)^3 \\
&\quad \times (\sigma_k p_k + r_k) h_k^2 a_k^{(1)} + 3\sigma_k (\sigma_k + 2) (\sigma_k + 1) \\
&\quad \times r_k h_k^3 a_k^{(2)} + 6(\sigma_k + 1)^2 \sigma_k (\sigma_k p_k - r_k) h_k^3 b_k^{(1)} \\
&\quad + 3\sigma_k (\sigma_k + 1)^2 (\sigma_k^2 p_k + r_k) h_k^4 b_k^{(2)} \\
&\quad - a_k^{(3)} \sigma_k (\sigma_k + 2) (\sigma_k + 1) r_k h_k^4 \\
&\quad + 6\sigma_k b_k (\sigma_k + 1)^2 q_k h_k^2 \Big) \times (72\sigma_k (\sigma_k + 1)^2)^{-1}, \\
\widetilde{R}_k^{21} &= -\frac{h_k^4}{24} \left(2(\sigma_k + 1) \gamma q_k a_k g_k^{(1)} + (\sigma_k^2 p_k + r_k) g_k^{(2)} \right) \\
&\quad - \frac{h_k^3}{12} (\sigma_k p_k - r_k) g_k^{(1)} - \frac{h_k^2}{12} (q_k + r_k + p_k) g_k.
\end{aligned} \tag{25}$$

The modified scheme (24) is free from the terms $1/(k \pm 1)$, hence easily solved for $k = 1(1)n$. The difference equation (24) along with the boundary conditions (4) gives a $2n \times 2n$ linear system of equations for the unknowns $U_k, V_k, k = 1(1)n$. The resulting block tridiagonal system can be easily solved using block Gauss-Seidel algorithm.

4. Extension to Sixth-Order Differential Equations

The proposed method can be easily extended to the sixth-order differential equations:

$$\begin{aligned}
u^{(6)}(r) &= f(r, u(r), u^{(1)}(r), u^{(2)}(r), u^{(3)}(r), u^{(4)}(r), \\
&\quad u^{(5)}(r)), \quad \alpha < r < \beta
\end{aligned} \tag{26}$$

subject to the necessary boundary conditions:

$$\begin{aligned}
u(\alpha) &= \alpha_0, & u^{(2)}(\alpha) &= \alpha_1, & u^{(4)}(\alpha) &= \alpha_2, \\
u(\beta) &= \beta_0, & u^{(2)}(\beta) &= \beta_1, & u^{(4)}(\beta) &= \beta_2
\end{aligned} \tag{27}$$

or equivalently,

$$\begin{aligned}
u^{(2)}(r) &= v(r), \\
v^{(2)}(r) &= w(r), \\
w^{(2)}(r) &= f(r, u, u^{(1)}(r), v(r), v^{(1)}(r), w(r), w^{(1)}(r)), \\
&\quad \alpha < r < \beta,
\end{aligned} \tag{28}$$

subject to the natural boundary conditions:

$$\begin{aligned}
u(\alpha) &= \alpha_0, & v(\alpha) &= \alpha_1, & w(\alpha) &= \alpha_2, \\
u(\beta) &= \beta_0, & v(\beta) &= \beta_1, & w(\beta) &= \beta_2.
\end{aligned} \tag{29}$$

We outline the similar algorithm for (28) as follows:

$$\begin{aligned}
\widetilde{U}'_k &= \frac{U_{k+1} - (1 - \sigma_k^2) U_k - \sigma_k^2 U_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\widetilde{V}'_k &= \frac{V_{k+1} - (1 - \sigma_k^2) V_k - \sigma_k^2 V_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\widetilde{W}'_k &= \frac{W_{k+1} - (1 - \sigma_k^2) W_k - \sigma_k^2 W_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\widetilde{U}'_{k+1} &= \frac{(1 + 2\sigma_k) U_{k+1} - (1 + \sigma_k)^2 U_k + \sigma_k^2 U_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\widetilde{V}'_{k+1} &= \frac{(1 + 2\sigma_k) V_{k+1} - (1 + \sigma_k)^2 V_k + \sigma_k^2 V_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\widetilde{W}'_{k+1} &= \frac{(1 + 2\sigma_k) W_{k+1} - (1 + \sigma_k)^2 W_k + \sigma_k^2 W_{k-1}}{h_k \sigma_k (1 + \sigma_k)},
\end{aligned}$$

$$\begin{aligned}
\bar{U}'_{k-1} &= \frac{-U_{k+1} + (1 + \sigma_k)^2 U_k - \sigma_k (2 + \sigma_k) U_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\bar{V}'_{k-1} &= \frac{-V_{k+1} + (1 + \sigma_k)^2 V_k - \sigma_k (2 + \sigma_k) V_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\bar{W}'_{k-1} &= \frac{-W_{k+1} + (1 + \sigma_k)^2 W_k - \sigma_k (2 + \sigma_k) W_{k-1}}{h_k \sigma_k (1 + \sigma_k)}, \\
\bar{f}_{k\pm 1} &= f(r_{k\pm 1}, U_{k\pm 1}, \bar{U}'_{k\pm 1}, V_{k\pm 1}, \bar{V}'_{k\pm 1}, W_{k\pm 1}, \bar{W}'_{k\pm 1}), \\
\bar{U}'_k &= \bar{U}'_k + \gamma h_k (V_{k+1} - V_{k-1}), \\
\bar{V}'_k &= \bar{V}'_k + \gamma h_k (W_{k+1} - W_{k-1}), \\
\bar{W}'_k &= \bar{W}'_k + \gamma h_k (\bar{f}_{k+1} - \bar{f}_{k-1}),
\end{aligned} \tag{30}$$

where the values of γ are same as obtained in Section 2.

Then, the $O(h_k^3)$ -approximations for (26) or (28) can be obtained by the following relations for $k = 1(1)n$:

$$\begin{aligned}
U_{k+1} - (1 + \sigma_k) U_k + \sigma_k U_{k-1} \\
+ \frac{h^2}{12} (p_k V_{k+1} + q_k V_k + r_k V_{k-1}) &= O(h_k^5), \\
V_{k+1} - (1 + \sigma_k) V_k + \sigma_k V_{k-1} \\
+ \frac{h^2}{12} (p_k W_{k+1} + q_k W_k + r_k W_{k-1}) &= O(h_k^5), \\
W_{k+1} - (1 + \sigma_k) W_k + \sigma_k W_{k-1} \\
+ \frac{h_k^2}{12} (p_k \bar{f}_k + q_k \bar{f}_k + r_k \bar{f}_{k-1}) &= O(h_k^5).
\end{aligned} \tag{31}$$

The boundary conditions (27) are used to obtain values at $k \pm 1$ for $k = 1$ and n , respectively. The numerical scheme may be implemented by neglecting $O(h_k^5)$ terms from (31).

5. Convergence Analysis

In this section, we derive the difference scheme of singular problem and investigate the convergence property of the proposed scheme. Consider the model problem

$$u^{(4)}(r) = a(r) u(r) + g(r), \quad \alpha < r < \beta \tag{32}$$

or equivalently

$$\begin{aligned}
u^{(2)}(r) &= v(r), \\
v^{(2)}(r) &= a(r) u(r) + g(r), \quad \alpha < r < \beta
\end{aligned} \tag{33}$$

along with the boundary conditions (2).

If $a(r), g(r) \in C^2[\alpha, \beta]$ and $a(r) > 0$ on $[\alpha, \beta]$, then the boundary value problem (33) has a unique solution. Under these assumptions (see, [27]), there are constants $\underline{a} > 0$ and \underline{g} such that $\underline{a} = \min_k \{a_k, a_k^{(1)}, a_k^{(2)}\}$, $\underline{g} = \min_k \{g_k, g_k^{(1)}, g_k^{(2)}\}$.

For the convergence, the coefficients p_k , q_k , and r_k associated with (5) and (15) must be negative (see, [22]), from which we obtain the condition $|\sigma_k - \sqrt{5}/2| < 1/2$.

Now applying the methods (5) and (15) to (33) and using the similar technique discussed in Section 3 for singular coefficients $a(r)$ and $g(r)$, we obtain the following system of difference equations:

$$\begin{aligned}
\mathbf{P}_k \mathbf{Z}_{k-1} + \mathbf{Q}_k \mathbf{Z}_k + \mathbf{R}_k \mathbf{Z}_{k+1} &= \mathbf{S}_k + \mathbf{T}_k(h_k), \\
k &= 1(1)n,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\mathbf{P}_k &= \begin{bmatrix} -\sigma_k & \frac{h_k^2}{12} \sigma_k (1 + \sigma_k - \sigma_k^2) \\ \frac{h_k^2}{12} \sigma_k (1 + \sigma_k - \sigma_k^2) & -\sigma_k \\ \times (a_k + h_k a_k^{(1)}) & \end{bmatrix}, \\
\mathbf{Q}_k &= (\sigma_k + 1) \\
&\times \begin{bmatrix} 1 & \frac{h_k^2}{12} (\sigma_k^2 + 3\sigma_k + 1) \\ \frac{h_k^2}{24} (2(\sigma_k^2 + 3\sigma_k + 1)a_k \\ + 4h_k \sigma_k (\sigma_k - 1)a_k^{(1)} \\ + h_k^2 \sigma_k (\sigma_k^2 - \sigma_k \\ + 1)a_k^{(2)}) & 1 \end{bmatrix}, \\
\mathbf{R}_k &= \begin{bmatrix} -1 & \frac{h_k^2}{12} (\sigma_k^2 + \sigma_k - 1) \\ \frac{h_k^2}{12} ((\sigma_k^2 + \sigma_k - 1)a_k \\ + h_k \sigma_k (\sigma_k^2 - \sigma_k + 1)a_k^{(1)}) & -1 \end{bmatrix}, \\
\mathbf{S}_k &= \left[0, \frac{h_k^2}{12} ((\sigma_k^3 - \sigma_k^2 - \sigma_k) g_k^{(2)} - (\sigma_k^2 + \sigma_k - 1) g_k^{(1)} \right. \\
&\quad \left. - (\sigma_k^3 + 4\sigma_k^2 + 4\sigma_k + 1) g_k \right]^T, \\
\mathbf{Z}_k &= [U_k, V_k]^T, \quad \mathbf{T}_k(h_k) = O(h_k^5).
\end{aligned} \tag{35}$$

Incorporating the boundary values $U_0 = \alpha_0$, $V_0 = \alpha_1$, $U_{n+1} = \beta_0$, and $V_{n+1} = \beta_1$, the system of difference equations (34) in the matrix-vector form can be written as

$$\mathbf{M}\mathbf{Z} = \mathbf{J} + \mathbf{T}_k(h_k), \tag{36}$$

where $\mathbf{M} = [\mathbf{P}_k \ \mathbf{Q}_k \ \mathbf{R}_k]$ is the block tridiagonal matrix, $\mathbf{J} = [\mathbf{S}_1 - \mathbf{P}_1 \alpha, \mathbf{S}_2, \dots, \mathbf{S}_{n-1}, \mathbf{S}_n - \mathbf{R}_n \beta]^T$, $\alpha = [\alpha_0, \alpha_1]^T$, and $\beta = [\beta_0, \beta_1]^T$.

Let $z_k = [u_k, v_k]^T$, $k = 1(1)n$, and $\mathbf{z} = [z_1, \dots, z_n] \equiv \mathbf{Z}$, which satisfies

$$\mathbf{M}\mathbf{z} = \mathbf{J}. \tag{37}$$

Let $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]^T$ be the discretization error vector, and let $\varepsilon_k = Z_k - z_k$, $k = 1(1)n$ be the discretization errors at the node r_k . Subtracting (37) from (36), we obtain the error equation:

$$\mathbf{M}\boldsymbol{\varepsilon} = \mathbf{T}_k(h_k). \quad (38)$$

Also, we obtain

$$\begin{aligned} \|\mathbf{P}_k\|_\infty &= \max_{2 \leq k \leq n} \begin{cases} \sigma_k + \frac{h_k^2}{12} \sigma_k |1 + \sigma_k - \sigma_k^2|, \\ \sigma_k + \frac{h_k^2}{12} \sigma_k |1 + \sigma_k - \sigma_k^2| \cdot |a_k + h_k a_k^{(1)}|, \end{cases} \\ \|\mathbf{R}_k\|_\infty &= \max_{1 \leq k \leq n-1} \begin{cases} 1 + \frac{h_k^2}{12} |\sigma_k^2 + \sigma_k - 1|, \\ 1 + \frac{h_k^2}{12} |(\sigma_k^2 + \sigma_k - 1) a_k \\ + h_k \sigma_k (\sigma_k^2 - \sigma_k + 1) a_k^{(1)}|. \end{cases} \end{aligned} \quad (39)$$

Thus for sufficiently small h_k or equivalently as $h_k \rightarrow 0$, we obtain the relations $\|\mathbf{P}_k\|_\infty = \sigma_k$, $k = 2(1)n$ and $\|\mathbf{R}_k\|_\infty = 1$, $k = 1(1)n - 1$. Hence, the graph $\mathbf{G}(\mathbf{M})$ of the matrix \mathbf{M} is strongly connected, and thus the matrix \mathbf{M} is irreducible (see, [28]).

Further, let Σ_l be the sum of the l th row sum of the matrix \mathbf{M} ; then we have the following.

For $l = 1$,

$$\begin{aligned} \Sigma_l &= \sigma_l + \frac{h_l^2}{12} \sigma_l (\sigma_l^2 + 5\sigma_l + 5), \\ \Sigma_{l+1} &= \sigma_l + \frac{h_l^2}{12} a_l \sigma_l (\sigma_l^2 + 5\sigma_l + 5) \\ &\quad + \frac{h_l^3}{12} a_l^{(1)} \sigma_l (3\sigma_l^2 - \sigma_l - 1) + \frac{h_l^4}{24} a_l^{(2)} \sigma_l (\sigma_l^3 + 1). \end{aligned} \quad (40)$$

For $l = 3(2)2n - 3$,

$$\begin{aligned} \Sigma_l &= \frac{h_l^2}{2} \sigma_l (\sigma_l + 1), \\ \Sigma_{l+1} &= \frac{h_l^2}{2} a_l \sigma_l (\sigma_l + 1) + \frac{h_l^3}{6} a_l^{(1)} \sigma_l (\sigma_l^2 - 1) \\ &\quad + \frac{h_l^4}{24} a_l^{(2)} \sigma_l (\sigma_l^3 + 1). \end{aligned} \quad (41)$$

For $l = 2n - 1$,

$$\begin{aligned} \Sigma_l &= 1 + \frac{h_l^2}{12} (5\sigma_l^2 + 5\sigma_l + 1), \\ \Sigma_{l+1} &= 1 + \frac{h_l^2}{12} a_l (5\sigma_l^2 + 5\sigma_l + 1) + \frac{h_l^3}{12} a_l^{(1)} \sigma_l (\sigma_l^2 + \sigma_l - 3) \\ &\quad + \frac{h_l^4}{24} a_l^{(2)} \sigma_l (\sigma_l^3 + 1). \end{aligned} \quad (42)$$

This implies the following.

For $l = 1$,

$$\begin{aligned} \Sigma_l &= \sigma_l + \frac{h_l^2}{12} \sigma_l (\sigma_l^2 + 5\sigma_l + 5), \\ \Sigma_{l+1} &\geq \sigma_l + \frac{h_l^2}{12} a_l \sigma_l (\sigma_l^2 + 5\sigma_l + 5) \\ &\quad + \frac{h_l^3}{12} a_l \sigma_l (3\sigma_l^2 - \sigma_l - 1) + \frac{h_l^4}{24} a_l \sigma_l (\sigma_l^3 + 1). \end{aligned} \quad (43)$$

For $l = 3(2)2n - 3$,

$$\begin{aligned} \Sigma_l &= \frac{h_l^2}{2} \sigma_l (\sigma_l + 1), \\ \Sigma_{l+1} &\geq \frac{h_l^2}{2} a_l \sigma_l (\sigma_l + 1) + \frac{h_l^3}{6} a_l \sigma_l (\sigma_l^2 - 1) \\ &\quad + \frac{h_l^4}{24} a_l \sigma_l (\sigma_l^3 + 1). \end{aligned} \quad (44)$$

For $l = 2n - 1$,

$$\begin{aligned} \Sigma_l &= 1 + \frac{h_l^2}{12} (5\sigma_l^2 + 5\sigma_l + 1), \\ \Sigma_{l+1} &\geq 1 + \frac{h_l^2}{12} a_l (5\sigma_l^2 + 5\sigma_l + 1) \\ &\quad + \frac{h_l^3}{12} a_l \sigma_l (\sigma_l^2 + \sigma_l - 3) + \frac{h_l^4}{24} a_l \sigma_l (\sigma_l^3 + 1), \end{aligned} \quad (45)$$

For sufficiently small value of h_k , that is, in the limiting case as $h_k \rightarrow 0$, we obtain

$$\begin{aligned} \Sigma_l &\geq \sigma_l > 0, \quad l = 1, 2, \\ \Sigma_l &\geq 0, \quad l = 3(1)2n - 2, \\ \Sigma_l &\geq 1 > 0, \quad l = 2n - 1, 2n. \end{aligned} \quad (46)$$

Hence we find that \mathbf{M} is monotone (see, [29, 30]). Consequently \mathbf{M}^{-1} exists, and $\mathbf{M}^{-1} \geq 0$.

Let $\mathbf{M}_{i,l}^{-1}$ be the (i, l) th element of \mathbf{M}^{-1} , and we define

$$\|\mathbf{M}^{-1}\| = \max_{1 \leq i \leq 2n} \sum_{l=1}^{2n} |\mathbf{M}_{i,l}^{-1}|, \quad (47)$$

$$\|\mathbf{T}\| = \max_{1 \leq l \leq 2n} \sum_{i=1}^{2n} |\mathbf{T}_l(h_l)| = O(h_l^3).$$

From the theory of matrix, we know that

$$\sum_{l=1}^{2n} \mathbf{M}_{i,l}^{-1} \Sigma_l = 1, \quad 1 \leq i \leq 2n. \quad (48)$$

Thus the following bounds can be estimated with the help of series expansions.

For $l = 1$,

$$\begin{aligned} \mathbf{M}_{i,l}^{-1} &\leq \Sigma_l^{-1} = \frac{1}{\sigma_l} - \frac{h_l^2}{12\sigma_l} (\sigma_l^2 + 5\sigma_l + 5) + O(h_l^4), \\ \mathbf{M}_{i,l+1}^{-1} &\leq \Sigma_{l+1}^{-1} \leq \frac{1}{\sigma_l} - \frac{h_l^2 a}{12\sigma_l} (\sigma_l^2 + 5\sigma_l + 5) \\ &\quad - \frac{h_l^3 a}{12\sigma_l} (3\sigma_l^2 - \sigma_l - 1) + O(h_l^4). \end{aligned} \quad (49)$$

For $l = 3(2)2n - 3$,

$$\begin{aligned} \mathbf{M}_{i,l}^{-1} &\leq \min_l \Sigma_l^{-1} = \frac{2}{h_l^2 \sigma_l (\sigma_l + 1)}, \\ \mathbf{M}_{i,l+1}^{-1} &\leq \min_l \Sigma_{l+1}^{-1} \leq \frac{2}{h_l^2 a \sigma_l (\sigma_l + 1)} \\ &\quad - \frac{2(\sigma_l - 1)}{3h_l a \sigma_l (\sigma_l + 1)} + \frac{\sigma_l^2 - 5\sigma_l + 1}{18a \sigma_l (\sigma_l + 1)} \\ &\quad + O(h_l). \end{aligned} \quad (50)$$

For $l = 2n - 1$,

$$\begin{aligned} \mathbf{M}_{i,l}^{-1} &\leq \Sigma_l^{-1} = 1 - \frac{h_l^2}{12} (5\sigma_l^2 + 5\sigma_l + 1) + O(h_l^4), \\ \mathbf{M}_{i,l+1}^{-1} &\leq \Sigma_{l+1}^{-1} \leq 1 - \frac{h_l^2 a}{12} (5\sigma_l^2 + 5\sigma_l + 1) \\ &\quad - \frac{h_l^3 a}{12} \sigma_l (\sigma_l^2 + \sigma_l - 3) + O(h_l^4). \end{aligned} \quad (51)$$

With the help of (48), we obtain the following bounds:

$$\begin{aligned} \|\mathbf{M}^{-1}\| &\leq \frac{2(a+1)}{h_l^2 a \sigma_l (\sigma_l + 1)} - \frac{2(\sigma_l - 1)}{3h_l a \sigma_l (\sigma_l + 1)} \\ &\quad + \frac{(\sigma_l^2 - 5\sigma_l + 1 + 36a(\sigma_l + 1)^2)}{18a \sigma_l (\sigma_l + 1)} + O(h_l). \end{aligned} \quad (52)$$

From (38) and (52), we obtain the following error estimates:

$$\|\epsilon\| \leq \|\mathbf{M}^{-1}\| \cdot \|\mathbf{T}_l(h_l)\| \leq O(h_l^3). \quad (53)$$

This proves the third-order convergence of the proposed method. We generalize the above results in the following theorem.

Theorem 1. *The method given by (15) for the numerical solution of fourth-order singular differential equation (1) with sufficiently small h_k and $|\sigma_k - \sqrt{5}/2| < 1/2$, $\sigma_k \neq 1$ gives a third-order convergent solution.*

6. Computational Illustrations

To illustrate the geometric mesh finite difference method, we have solved both linear and nonlinear problems. The

boundary conditions may be obtained from the analytical solution as a test procedure. The numerical accuracy of results are tested using maximum absolute errors and root mean square errors with the error tolerance being $\leq 10^{-15}$. For the simplicity in computation, we choose $\sigma_k = \sigma = \text{constant}$, for $k = 1(1)n$ and define the geometric mesh as follows ([24]):

$$\begin{aligned} x_0 &= \alpha, \\ h_1 &= \begin{cases} \frac{(\beta - \alpha)(1 - \sigma)}{(1 - \sigma^{n+1})}, & \sigma < 1, \\ \frac{(\beta - \alpha)(\sigma - 1)}{(\sigma^{n+1} - 1)}, & \sigma > 1. \end{cases} \end{aligned} \quad (54)$$

The subsequent mesh spacing is determined by $h_{k+1} = \sigma h_k$, $k = 1(1)n$. If the boundary value problems exhibit layer behaviour near the left boundary (see, [21]), the solution value can be captured by choosing $\sigma > 1$. If the layer occurs at the right boundary, we choose $\sigma < 1$. If the layer occurs in the interior region, then mesh in the first half of the interval may be arranged by choosing $\sigma > 1$ and second half of the interval by choosing $\sigma < 1$.

All the numerical computations are performed using long double length arithmetic in C under Linux operating system with 2 GB operational memory.

Example 1. Consider the fourth-order linear problem (see, [31]) in the polar form:

$$\begin{aligned} \Delta^4 u(r) &\equiv \left(\frac{d^2}{dr^2} + \frac{\lambda}{r} \frac{d}{dr} \right)^2 u(r) \\ &= \left(1 + \frac{2\lambda}{r} + \frac{\lambda(\lambda - 2)}{r^2} - \frac{\lambda(\lambda - 2)}{r^3} \right) e^r, \end{aligned} \quad (55)$$

$0 < r < 1.$

The analytical solution is $u(r) = e^r$. The errors estimates for various values of n are reported in Tables 1 and 2 for uniform mesh ($\sigma = 1$) and geometric mesh ($\sigma \neq 1$), respectively.

Example 2. Consider the boundary value problems that arise from time-dependent Navier-Stokes equation (see, [32]) for axis symmetric flow of an incompressible fluid contained between infinite disks

$$\begin{aligned} u^{(4)}(r) &= \lambda u(r) u^{(2)}(r) - \lambda(r^2 - 1) \\ &\quad \times (1 + 4r + r^2) e^{2r} - (11 + 8r + r^2) e^r, \end{aligned} \quad (56)$$

$0 < r < 1.$

The analytical solution is $u(r) = (1 - r^2)e^r$. The errors estimates are reported in Table 3 for various values of n and $\lambda = 10^3$.

TABLE 1: The maximum absolute errors for Example 1.

n	$\lambda = 1$		$\lambda = 2$	
	σ			
	1	0.9937	1	0.996759
8	$0.866e-04$	$0.772e-04$	$0.290e-05$	$0.271e-05$
16	$0.223e-04$	$0.176e-04$	$0.190e-05$	$0.170e-05$
32	$0.123e-04$	$0.730e-05$	$0.640e-08$	$0.246e-08$
64	$0.615e-06$	$0.348e-08$	$0.365e-09$	$0.882e-10$

TABLE 2: The root mean square errors for Example 1.

n	$\lambda = 1$		$\lambda = 2$	
	σ			
	1	0.9937	1	0.996759
8	$0.664e-04$	$0.590e-04$	$0.219e-05$	$0.205e-05$
16	$0.166e-04$	$0.131e-04$	$0.139e-05$	$0.125e-05$
32	$0.877e-05$	$0.517e-05$	$0.419e-08$	$0.239e-08$
64	$0.417e-06$	$0.214e-08$	$0.214e-09$	$0.483e-10$

TABLE 3: The maximum absolute errors and root mean square errors for Example 2.

n	σ			
	1	0.998	1	0.998
8	$0.302e-05$	$0.255e-05$	$0.235e-05$	$0.199e-05$
16	$0.238e-06$	$0.169e-06$	$0.179e-06$	$0.128e-06$
32	$0.167e-07$	$0.751e-08$	$0.124e-07$	$0.557e-08$
64	$0.111e-08$	$0.919e-10$	$0.819e-09$	$0.649e-10$

TABLE 4: The maximum absolute errors for Example 3.

n	$\lambda = 1$		$\lambda = 2$	
	σ			
	1	1.11	1	0.994
8	$0.338e-03$	$0.189e-03$	$0.322e-06$	$0.268e-06$
16	$0.102e-03$	$0.292e-04$	$0.261e-07$	$0.183e-07$
32	$0.337e-04$	$0.481e-05$	$0.186e-08$	$0.847e-09$
64	$0.111e-04$	$0.662e-06$	$0.125e-09$	$0.126e-10$

TABLE 5: The root mean square errors for Example 3.

n	$\lambda = 1$		$\lambda = 2$	
	σ			
	1	1.11	1	0.994
8	$0.255e-03$	$0.140e-03$	$0.245e-06$	$0.202e-06$
16	$0.742e-04$	$0.197e-04$	$0.193e-07$	$0.134e-07$
32	$0.240e-04$	$0.270e-05$	$0.136e-08$	$0.589e-09$
64	$0.782e-05$	$0.243e-06$	$0.900e-10$	$0.834e-11$

TABLE 6: The maximum absolute errors and root mean square errors for Example 4.

n	σ			
	1	0.9991	1	0.9991
8	$0.380e-07$	$0.302e-07$	$0.291e-07$	$0.230e-07$
16	$0.301e-08$	$0.186e-08$	$0.222e-08$	$0.134e-08$
32	$0.210e-09$	$0.587e-10$	$0.153e-09$	$0.373e-10$
64	$0.127e-10$	$0.913e-11$	$0.912e-11$	$0.654e-11$

Example 3. Consider the sixth-order linear singular problem:

$$\begin{aligned} \Delta^6 u(r) &\equiv \left(\frac{d^2}{dr^2} + \frac{\lambda}{r} \frac{d}{dr} \right)^3 u(r) \\ &= \left(1 + \frac{3\lambda(\lambda-2)}{r^2} + \frac{3\lambda(\lambda-2)(\lambda-4)}{r^4} \right) \sinh(r) \\ &\quad + \frac{\lambda}{r} \left(3 + \frac{(\lambda-2)(\lambda-7)}{r^2} \right. \\ &\quad \left. + \frac{3(\lambda-2)(\lambda-4)}{r^4} \right) \cosh(r), \quad 0 < r < 1. \end{aligned} \quad (57)$$

The analytical solution is $u(r) = \sinh(r)$. The errors estimates for various values of n are reported in Tables 4 and 5 for uniform mesh ($\sigma = 1$) and geometric mesh ($\sigma \neq 1$), respectively.

Example 4. Consider the nonlinear problem

$$\begin{aligned} u^{(6)}(r) &= \lambda \left(u(r) u^{(2)}(r) + u^{(2)}(r) u^{(4)}(r) \right. \\ &\quad \left. + u(r) u^{(4)}(r) \right) + \sinh(r) (1 - 3\lambda \sinh(r)), \\ &\quad 0 < r < 1. \end{aligned} \quad (58)$$

The analytical solution is $u(r) = \sinh(r)$. The errors estimates are reported in Table 6 for various values of n and $\lambda = 10^3$.

7. Conclusion

The numerical results confirm that the proposed geometric mesh finite difference scheme converges and applicable to both singular and nonsingular differential equations. The numerical accuracy obtained using geometric mesh shows superiority over corresponding uniform mesh. The optimum mesh ratio parameter within the specified convergent region may be obtained by simulations. We have employed block Gauss-Seidel method to solve the block matrix systems. The method can be extended to general even-order nonlinear differential equations. Application to the proposed scheme to nonlinear singular elliptic problems is an open problem.

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