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Research Article

Generalized Pattern-Matching Conditions for $C_k \wr S_n$

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We derive several multivariable generating functions for a generalized pattern-matching condition on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n. In particular, we derive the generating functions for the number of matches that occur in elements of $C_k \wr S_n$ for any pattern of length 2 by applying appropriate homomorphisms from the ring of symmetric functions over an infinite number of variables to simple symmetric function identities. This allows us to derive several natural analogues of the distribution of rises relative to the product order on elements of $C_k \wr S_n$. Our research leads to connections to many known objects/structures yet to be explained combinatorially.

1. Introduction

The goal of this paper is to study pattern-matching conditions on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n. $C_k \wr S_n$ is the group of $k^n n!$ signed permutations where we allow k signs of the form $1 = \omega^0, \omega, \omega^2, \ldots, \omega^{k-1}$ for some primitive kth root of unity ω. We can think of the elements $C_k \wr S_n$ as pairs $\gamma = (\sigma, \epsilon)$ where $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and $\epsilon = \epsilon_1 \cdots \epsilon_n \in \{1, \omega, \ldots, \omega^{k-1}\}^n$. For ease of notation, if $\epsilon = (\omega^{w_1}, \omega^{w_2}, \ldots, \omega^{w_n})$ where $w_i \in \{0, 1, \ldots, k-1\}$ for $i = 1, 2, \ldots, n$, then we simply write $\gamma = (\sigma, w)$ where $w = w_1 w_2 \cdots w_n \in \{0, 1, \ldots, k-1\}^n$.

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the ith largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, we say a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a *τ-match starting at position i* provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let τ-mch(σ) be the number of τ-matches in the permutation σ. Similarly, we say that τ *occurs* in σ if there exist $1 \le i_1 < \cdots < i_j \le n$ such that red $(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that σ *avoids* τ if there are no occurrences of τ in σ.

We can define similar notions for words over a finite alphabet $[k] = \{0, 1, \ldots, k-1\}$. Given a word $w = w_1 \cdots w_n \in [k]^n$, let $\mathrm{red}(w)$ be the word found by replacing the ith largest integer that appears in w by $i - 1$. For example, if $w = 2\ 7\ 2\ 4\ 7$, then $\mathrm{red}(w) = 0\ 2\ 0\ 1\ 2$. Given a word $u \in [k]^j$ such that $\mathrm{red}(u) = u$, we say a word $w \in [k]^n$ has a *u-match starting at position i* provided $\mathrm{red}(w_i \cdots w_{i+j-1}) = u$. Let u-mch(w) be the number of u-matches in the word w. Similarly, we say that u *occurs* in a word w if there exist $1 \le i_1 < \cdots < i_j \le n$ such that $\mathrm{red}(w_{i_1} \cdots w_{i_j}) = u$. We say that w *avoids* u if there are no occurrences of u in w.

There are a number of papers on pattern matching and pattern avoidance in $C_k \wr S_n$ [1–4]. For example, the following pattern matching condition was studied in [2–4].

Definition 1. Let $(\tau, u) \in C_k \wr S_j$, Υ be a subset of $C_k \wr S_j$ and $(\sigma, w) \in C_k \wr S_n$.

(1) One says that (σ, w) has an *exact occurrence* of (τ, u) (resp., Υ) if there are $1 \le i_1 < i_2 < \cdots < i_j \le n$ such that $(\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}), w_{i_1} \cdots w_{i_j}) = (\tau, u)$ (resp., $(\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}), w_{i_1} \cdots w_{i_j}) \in \Upsilon$).

(2) One says that $(\sigma, w) \in C_k \wr S_n$ *avoids an exact occurrence of* (τ, u) (resp., Υ) if there are no exact occurrences of (τ, u) (resp., Υ) in (σ, w).

(3) One says that there is an *exact* (τ, u)*-match in* (σ, w) *starting at position* i (resp., *exact* Υ*-match in* (σ, w) *starting at position* i) if

$$\left(\operatorname{red}\left(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}\right), w_iw_{i+1}\cdots w_{i+j-1}\right) = (\tau, u)$$
$$\left(\text{resp.}, \left(\operatorname{red}\left(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}\right), w_iw_{i+1}\cdots w_{i+j-1}\right) \in \Upsilon\right). \tag{1}$$

That is, an exact occurrence or an exact match of $(\tau, u) \in C_k \wr S_j$ in an element $(\sigma, w) \in C_k \wr S_n$ is just an ordinary occurrence or match of τ in σ where the corresponding signs agree *exactly*. For example, Mansour [3] proved via recursion that for any $(\tau, u) \in C_k \wr S_2$, the number of elements in $C_k \wr S_n$ which avoid exact occurrences of (τ, u) is $\sum_{j=0}^{n} j!(k-1)^j\binom{n}{j}^2$. This generalized a result of Simion [5] who proved the same result for the hyperoctahedral group $C_2 \wr S_n$. Similarly, Mansour and West [4] determined the number of permutations in $C_2 \wr S_n$ that avoid all possible exact occurrences of 2 or 3 element sets of patterns of elements of $C_2 \wr S_2$. For example, let K_n^1 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all exact occurrences of the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 1\ 0)\}$, let K_n^2 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all exact occurrences of the patterns in the set $\{(1\ 2, 0\ 1), (1\ 2, 1\ 0), (2\ 1, 0\ 1)\}$, and let K_n^3 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all exact occurrences of the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 0\ 0)\}$. Then Mansour and West [4] proved that

$$K_n^1 = F_{2n+1},$$
$$K_n^2 = n!\sum_{j=0}^{n}\binom{n}{j}^{-1}, \tag{2}$$
$$K_n^3 = n! + n!\sum_{j=1}^{n}\frac{1}{j},$$

where F_n is the nth Fibonacci number.

An alternative matching condition arises when we drop the requirement of the exact matching of signs and replace it by the condition that the two sequences of signs match in the sense of words described above. That is, we will consider the following matching conditions.

Definition 2. Let $(\tau, u) \in C_k \wr S_j$ where $\operatorname{red}(u) = u$, Υ be a subset of $C_k \wr S_j$ where for all $(\tau, u) \in \Upsilon$, $\operatorname{red}(u) = u$, and $(\sigma, w) \in C_k \wr S_n$.

(1) One says that (σ, w) has an *occurrence* of (τ, u) (resp., Υ) if there are $1 \le i_1 < i_2 < \cdots < i_j \le n$ such that $(\operatorname{red}(\sigma_{i_1}\cdots\sigma_{i_j}), \operatorname{red}(w_{i_1}\cdots w_{i_j})) = (\tau, u)$ (resp., $(\operatorname{red}(\sigma_{i_1}\cdots\sigma_{i_j}), \operatorname{red}(w_{i_1}\cdots w_{i_j})) \in \Upsilon$).

(2) We say that $(\sigma, w) \in C_k \wr S_n$ *avoids* (τ, u) (resp., Υ) if there are no occurrences of (τ, u) (resp., Υ) in (σ, w).

(3) One says that there is a (τ, u)*-match in* (σ, w) *starting at position* i (resp., Υ*-matchin* (σ, w)*starting at position* i) if $(\operatorname{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}), w_iw_{i+1}\cdots w_{i+j-1}) = (\tau, u)$ (resp., $(\operatorname{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}), \operatorname{red}(w_iw_{i+1}\cdots w_{i+j-1})) \in \Upsilon$).

For example, suppose that $(\tau, u) = (1\ 2, 0\ 0)$ and $(\sigma, w) = (1\ 3\ 2\ 4, 1\ 2\ 2\ 2)$. Then there are no exact occurrences or exact matches of (τ, u) in (σ, w). However, there are two occurrences of (τ, u), one in positions 2 and 4 and one in positions 3 and 4. Thus, there are two occurrences of (τ, u) in (σ, w), and there is a (τ, u)-match in (σ, w) starting at position 3.

Finally, we will consider a more general matching condition which generalizes both occurrences and matches and exact occurrences and exact matches in $C_k \wr S_n$.

Definition 3. Let $(\tau, u) \in C_k \wr S_j$, let Υ be a subset of $C_k \wr S_j$, and let $\vec{A} = (A_1, \ldots, A_j)$ be a sequence of subsets of $[k]$, and $(\sigma, w) \in C_k \wr S_n$.

(1) One says that (σ, w) has an *occurrence* of $(\tau, u, \vec{A})$ (resp., $(\Upsilon, \vec{A})$) if there are $1 \le i_1 < i_2 < \cdots < i_j \le n$ such that $(\operatorname{red}(\sigma_{i_1}\cdots\sigma_{i_j}), \operatorname{red}(w_{i_1}\cdots w_{i_j})) = (\tau, \operatorname{red}(u))$ (resp., $(\operatorname{red}(\sigma_{i_1}\cdots\sigma_{i_j}), \operatorname{red}(w_{i_1}\cdots w_{i_j}))$ is equal to $(\tau, \operatorname{red}(u))$ for some (τ, u) in Υ) and $w_{i_s} \in A_s$ for $s = 1, 2, \ldots, j$.

(2) One says that $(\sigma, w) \in C_k \wr S_n$ *avoids* (τ, u) (resp., Υ) if there are no occurrences of (τ, u) (resp., Υ) in (σ, w).

(3) We say that there is a (τ, u)*-match in* (σ, w) *starting at position* i (resp., Υ*-match in* (σ, w) *starting at position* i) if $(\operatorname{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}), \operatorname{red}(w_iw_{i+1}\cdots w_{i+j-1})) = (\tau, \operatorname{red}(u))$ (resp., $(\operatorname{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}), \operatorname{red}(w_iw_{i+1}\cdots w_{i+j-1}))$ is equal to $(\tau, \operatorname{red}(u))$ for some (τ, u) in Υ) and $w_{i+s} \in A_{s+1}$ for $s = 0, 1, \ldots, j-1$.

Thus, a $(\tau, u, \vec{A})$-occurrence or $(\tau, u, \vec{A})$-match where $u = u_1\cdots u_j$ and $\vec{A} = (A_1, \ldots, A_j)$ is such that $A_i = \{u_i\}$ for $i = 1, \ldots, j$ is just an exact occurrence or exact match of (τ, u). Similarly, a $(\tau, u, \vec{A})$-occurrence or $(\tau, u, \vec{A})$-match where $u = u_1\cdots u_j$, $\operatorname{red}(u) = u$, and $\vec{A} = (A_1, \ldots, A_j)$ is such that $A_i = [k]$ for $i = 1, \ldots, j$ is just an occurrence or match of (τ, u).

Suppose we are given $(\tau, u) \in C_k \wr S_j$, $\Upsilon \subseteq C_k \wr S_j$, $\vec{A} = (A_1, \ldots, A_j)$ where $A_i \subseteq [k]$ for $i = 1, \ldots, j$, and $(\sigma, w) \in C_k \wr S_n$. We let (τ, u)-mch$((\sigma, w))$ (resp., (τ, u)-Emch$((\sigma, w))$) denote the number of (τ, u)-matches (resp., exact (τ, u)-matches) in (σ, w). We let Υ-mch$((\sigma, w))$ (resp., Υ-Emch$((\sigma, w))$) denote the number of Υ-matches (resp., exact Υ-matches) in (σ, w). We let $(\tau, u, \vec{A})$-mch$((\sigma, w))$ be the number of $(\tau, u, \vec{A})$-matches in (σ, w) and let $(\Upsilon, \vec{A})$-mch$((\sigma, w))$ be the number of $(\Upsilon, \vec{A})$-matches in (σ, w).

The main result of this paper is to derive a generating function for the distribution of $(\tau, u, \vec{A})$-matches where (τ, u) is any element of $C_k \wr S_2$. To state our main result, we first need some notation. We define the p, q-analogues of n, $n!$, $\binom{n}{k}$, and $\binom{n}{a_1, \ldots, a_m}$ by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1},$$
$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$
$$\begin{bmatrix} n \\ a_1,\ldots,a_m \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[a_1]_{p,q}!\cdots[a_m]_{p,q}!}, \tag{3}$$

respectively. We define the q-analogues of n, $n!$, $\binom{n}{k}$, and $\binom{n}{a_1,\ldots,a_m}$ by $[n]_{1,q}$, $[n]_{1,q}!$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{1,q}$, and $\left[\begin{smallmatrix} n \\ a_1,\ldots,a_m \end{smallmatrix}\right]_{1,q}$, respectively.

Next suppose that $\Upsilon \subseteq C_k \wr S_2$ and $\vec{A} = (A_1, A_2)$ where $A_1, A_2 \subseteq [k]$. Then we will say that $(\sigma, w) \in C_k \wr S_n$ is a *maximum packing* for $(\Upsilon, \vec{A})$ if (σ, w) has $(\Upsilon, \vec{A})$-matches starting at positions $1, 2, \ldots, n-1$. We let $\mathcal{MP}_{(\Upsilon,\vec{A}),n}$ denote the set of $(\sigma, w) \in C_k \wr S_n$ which are maximum packings for $(\Upsilon, \vec{A})$. Given any word $w = w_1 \cdots w_n \in [k]^n$, we let $z(w) = \prod_{i=1}^n z_{w_i}$. For any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we let $\mathrm{inv}(\sigma)$ (resp., $\mathrm{coinv}(\sigma)$) equal the number of pairs (i, j) such that $1 \le i < j \le n$ and $\sigma_i > \sigma_j$ (resp., $\sigma_i < \sigma_j$). We then define

$$\begin{aligned} &\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1}) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{MP}_{(\Upsilon,\vec{A}),n}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} z(w). \end{aligned} \tag{4}$$

We shall also be interested in the specializations

$$\begin{aligned} &\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; 1, \ldots, 1) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{MP}_{(\Upsilon,\vec{A}),n}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)}, \\ &\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; 1, r, \ldots, r^{k-1}) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{MP}_{(\Upsilon,\vec{A}),n}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} r^{\|w\|}, \end{aligned} \tag{5}$$

where for any word $w = w_1 \cdots w_n \in [k]^n$, $||w|| = w_1 + \cdots + w_n$. In the special case, where $\Upsilon = \{(\tau, u)\}$ where $(\tau, u) \in C_k \wr S_2$, we will denote $\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1})$ as $\mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1})$. For example, if $\tau = 12$, $u = 01$, and $\vec{A} = ([k], [k])$, then $(\sigma, w) \in C_k \wr S_n$ is a maximum packing for $(\tau, u, \vec{A})$ if and only if $\sigma = 12 \cdots n$ is the identity permutation and $w = w_1 \cdots w_n$ is such that $0 \le w_1 < \cdots < w_n \le k-1$. Thus,

$$\mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1}) = p^{\binom{n}{2}} \left(\prod_{i=0}^{k-1} (1 + z_i t) \right)\Bigg|_{t^n}, \tag{6}$$

where for any power series $f(t) = \sum_{n\ge0} f_n t^n$, we write $f(t)|_{t^n} = f_n$ for the coefficient of t^n in $f(t)$. Then it is easy to prove that

$$\begin{aligned} \mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; 1, \ldots, 1) &= p^{\binom{n}{2}} \binom{k}{n}, \\ \mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; 1, r, \ldots, r^{k-1}) &= p^{\binom{n}{2}} r^{\binom{n}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_r. \end{aligned} \tag{7}$$

If $\tau = 12$, $u = 00$, and $\vec{A} = ([k], [k])$, then $(\sigma, w) \in C_k \wr S_n$ is a maximum packing for $(\tau, u, \vec{A})$ if and only if $\sigma = 12 \cdots n$ is the identity permutation and $w = i^n$ for some $0 \le i \le k-1$. Thus,

$$\mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1}) = p^{\binom{n}{2}} \sum_{i=0}^{k-1} z_i^n. \tag{8}$$

It then follows that

$$\begin{aligned} \mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; 1, \ldots, 1) &= k p^{\binom{n}{2}}, \\ \mathrm{mp}_{(\tau,u,\vec{A}),n}(p, q; 1, r, \ldots, r^{k-1}) &= [k]_{r^n} p^{\binom{n}{2}}. \end{aligned} \tag{9}$$

Similarly if $\Upsilon = \{(12, 00), (12, 01)\}$ and $\vec{A} = ([k], [k])$, then $(\sigma, w) \in C_k \wr S_n$ is a maximum packing for $(\Upsilon, \vec{A})$ if and only if $\sigma = 12 \cdots n$ is the identity permutation and $w = w_1 \cdots w_n$ is such that $0 \le w_1 \le \cdots \le w_n \le k-1$. Thus,

$$\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; z_0, \ldots, z_{k-1}) = p^{\binom{n}{2}} \left(\prod_{i=0}^{k-1} \frac{1}{1 - z_i t} \right)\Bigg|_{t^n}. \tag{10}$$

We shall then prove that

$$\begin{aligned} \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; 1, \ldots, 1) &= p^{\binom{n}{2}} \binom{n+k-1}{n}, \\ \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p, q; 1, r, \ldots, r^{k-1}) &= p^{\binom{n}{2}} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r. \end{aligned} \tag{11}$$

The main result of this paper is to prove the following.

Theorem 4. *For any $\Upsilon \subseteq C_k \wr S_2$ and $\vec{A} = (A_1, A_2)$ where $A_1, A_2 \subseteq [k]$,*

$$\begin{aligned} &\sum_{n\ge0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k\wr S_n} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))} z(w) \\ &\quad = 1 \times \Big(1 - \big((z_0 + \cdots + z_{k-1}) t \\ &\qquad + \sum_{n\ge2} \frac{t^n}{[n]_{p,q}!} (x-1)^{n-1} \mathrm{mp}_{(\Upsilon,\vec{A}),n} \\ &\qquad \times (p, q; z_0, \ldots, z_{k-1})\big)\Big)^{-1}. \end{aligned} \tag{12}$$

We note that (12) can be specialized to give natural analogues for generating functions of rises and descents in $C_k \wr S_n$ where we compare pairsusing the product order. That is, instead of thinking of an element of $C_k \wr S_n$ as a pair $(\sigma_1 \cdots \sigma_n, w_1 \cdots w_n)$, we can think of it as a sequence of pairs $(\sigma_1, w_1)(\sigma_2, w_2) \cdots (\sigma_n, w_n)$. We then define a partial order on such pairs by the usual product order. That is, $(i_1, j_1) \le (i_2, j_2)$

if and only if $i_1 \le i_2$ and $j_1 \le j_2$. Then we define the following sets and statistics for elements $(\sigma, w) \in C_k \wr S_n$:

$$\begin{aligned}
&\mathrm{Des}((\sigma,w)) = \{i : \sigma_i > \sigma_{i+1} \,\&\, w_i \ge w_{i+1}\},\\
&\mathrm{des}((\sigma,w)) = |\mathrm{Des}((\sigma,w))|,\\
&\mathrm{Ris}((\sigma,w)) = \{i : \sigma_i < \sigma_{i+1} \,\&\, w_i \le w_{i+1}\},\\
&\mathrm{ris}((\sigma,w)) = |\mathrm{Ris}((\sigma,w))|,\\
&W\mathrm{Des}((\sigma,w)) = \{i : \sigma_i > \sigma_{i+1} \,\&\, w_i = w_{i+1}\},\\
&w\,\mathrm{des}((\sigma,w)) = |W\mathrm{Des}((\sigma,w))|,\\
&W\mathrm{Ris}((\sigma,w)) = \{i : \sigma_i < \sigma_{i+1} \,\&\, w_i = w_{i+1}\},\\
&w\mathrm{ris}((\sigma,w)) = |W\mathrm{Ris}((\sigma,w))|,\\
&S\mathrm{Des}((\sigma,w)) = \{i : \sigma_i > \sigma_{i+1} \,\&\, w_i > w_{i+1}\},\\
&s\,\mathrm{des}((\sigma,w)) = |S\mathrm{Des}((\sigma,w))|,\\
&S\mathrm{Ris}((\sigma,w)) = \{i : \sigma_i < \sigma_{i+1} \,\&\, w_i < w_{i+1}\},\\
&s\mathrm{ris}((\sigma,w)) = |S\mathrm{Ris}((\sigma,w))|.
\end{aligned} \tag{13}$$

We shall refer to $\mathrm{Des}((\sigma,w))$ as the *descent set* of (σ,w), $W\mathrm{Des}((\sigma,w))$ as the *weak descent set* of (σ,w), and $S\mathrm{Des}((\sigma,w))$ as the *strict descent set* of (σ,w). Similarly, we will refer to $\mathrm{Ris}((\sigma,w))$ as the *rise set* of (σ,w), $W\mathrm{Ris}((\sigma,w))$ as the *weak rise set* of (σ,w), and $S\mathrm{Ris}((\sigma,w))$ as the *strict rise set* of (σ,w). It is easy to see that

$i \in W\mathrm{Des}((\sigma,w))$ if and only if there is a $(2\ 1, 0\ 0)$-match starting at position i,

$i \in S\mathrm{Des}((\sigma,w))$ if and only if there is a $(2\ 1, 1\ 0)$-match starting at position i, and

$i \in \mathrm{Des}((\sigma,w))$ if and only if there is a Υ-match starting at position i where $\Upsilon = \{(2\ 1, 0\ 0), (2\ 1, 1\ 0)\}$.

Similarly,

$i \in W\mathrm{Ris}((\sigma,w))$ if and only if there is a $(1\ 2, 0\ 0)$-match starting at position i,

$i \in S\mathrm{Ris}((\sigma,w))$ if and only if there is a $(1\ 2, 0\ 1)$-match starting at position i, and

$i \in \mathrm{Ris}((\sigma,w))$ if and only if there is a Υ-match starting at position i where $\Upsilon = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$.

If $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, then we define the reverse of σ, σ^r by $\sigma^r = \sigma_n \cdots \sigma_1$. Similarly, if $w = w_1 \cdots w_n \in [k]^n$, then we define $w^r = w_n \cdots w_1$. It is easy to see that

$$\begin{aligned}
\mathrm{ris}((\sigma,w)) &= \mathrm{des}((\sigma^r, w^r)),\\
w\mathrm{ris}((\sigma,w)) &= w\,\mathrm{des}((\sigma^r, w^r)),\\
s\mathrm{ris}((\sigma,w)) &= s\,\mathrm{des}((\sigma^r, w^r)).
\end{aligned} \tag{14}$$

Thus, we need to find the distributions for only one of the corresponding pairs. Substituting $z_0 = z_1 = \cdots = z_{k-1} = 1$ into (12) in these special cases will yield the following generating functions for rises, strict rises, and weak rises in $C_k \wr S_n$.

Theorem 5. *Consider the following:*

$$\begin{aligned}
&\sum_{n\ge 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k \wr S_n} x^{\mathrm{ris}((\sigma,w))}\\
&\quad = \frac{1-x}{1-x+\sum_{n\ge 1}\left(((x-1)t)^n/n!\right)\binom{n+k-1}{n}},\\
&\sum_{n\ge 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k \wr S_n} x^{w\mathrm{ris}((\sigma,w))}\\
&\quad = \frac{1-x}{1-x+k\left(e^{(x-1)t}-1\right)},\\
&\sum_{n\ge 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k \wr S_n} x^{s\mathrm{ris}((\sigma,w))}\\
&\quad = \frac{1-x}{1-x+\sum_{n\ge 1}\left(((x-1)t)^n/n!\right)\binom{k}{n}}.
\end{aligned} \tag{15, 16}$$

Other distribution results for (τ, u)-matches follow from these results. For example, if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, then we define the complement of σ, σ^c by

$$\sigma^c = (n+1-\sigma_1)\cdots(n+1-\sigma_n). \tag{17}$$

If $w = w_1 \cdots w_n \in [k]^n$, then we define the complement of w, w^c by

$$w^c = (k-1-w_1)\cdots(k-1-w_n). \tag{18}$$

We can then consider maps $\phi_{a,b} : C_k \wr S_n \to C_k \wr S_n$ where $\phi_{a,b}((\sigma,w)) = (\sigma^a, w^b)$ for $a, b \in \{r, c\}$. Such maps will easily allow us to establish that the distribution of (τ, u)-matches is the same for various classes of (τ, u)'s. For example, one can use such maps to show that the distributions of $(1\ 2, 0\ 1)$-matches, $(2\ 1, 0\ 1)$-matches, $(1\ 2, 1\ 0)$-matches, and $(2\ 1, 1\ 0)$-matches are all the same.

Another interesting case is when we let $\Upsilon = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$. In this case, we have a Υ-match in (σ,w) starting at i if and only if $\sigma_i < \sigma_{i+1}$ and $w_i \ne w_{i+1}$. In that case, (12) can be specialized to prove the following.

Theorem 6. *Consider the following:*

$$\begin{aligned}
&\sum_{n\ge 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k \wr S_n} x^{\Upsilon\text{-mch}((\sigma,w))}\\
&\quad = \frac{(k-1)(1-x)}{(k-1)(1-x)+k\left(e^{(k-1)(x-1)t}-1\right)}.
\end{aligned} \tag{19}$$

Substituting $z_i = r^i$ for $i = 0, \ldots, k-1$ into (12) will yield the following generating function:

$$\begin{aligned}
&D_k^{\Upsilon}(x,p,q,r,t)\\
&\quad = \sum_{n\ge 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{\|w\|} x^{\Upsilon\text{-mch}((\sigma,w))}.
\end{aligned} \tag{20}$$

Let

$$\begin{aligned}\Upsilon_{\mathbf{r}} &= \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\},\\ \Upsilon_{\mathbf{w}} &= \{(1\ 2, 0\ 0)\},\\ \Upsilon_{\mathbf{s}} &= \{(1\ 2, 0\ 1)\},\\ \Upsilon_{\mathbf{d}} &= \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}.\end{aligned} \tag{21}$$

Thus, $\Upsilon_{\mathbf{r}}$-matches correspond to rises, $\Upsilon_{\mathbf{w}}$-matches correspond to weak rises, and $\Upsilon_{\mathbf{s}}$-matches correspond to strict rises. We shall find $D_k^{\Upsilon_{\mathbf{a}}}(x, p, q, r, t)$ for $\mathbf{a} \in \{\mathbf{r}, \mathbf{w}, \mathbf{s}\}$ and find $D_k^{\Upsilon_{\mathbf{d}}}(x, p, q, 1, t)$. For example, we will show that the following generating function for the distribution of inversions, coinversions, and rises over $C_k \wr S_n$ is an immediate consequence of (12).

Lemma 7. *Consider the following:*

$$\begin{aligned}&D_k^{\Upsilon_{\mathbf{r}}}(x, p, q, r, t)\\ &\quad= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k\wr S_n} q^{\operatorname{inv}(\sigma)} p^{\operatorname{coinv}(\sigma)} r^{\|w\|} x^{\operatorname{ris}((\sigma,w))}\\ &\quad= \frac{1-x}{1-x+\sum_{n\geq 1}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)\left[{n+k-1 \atop n}\right]_r},\end{aligned} \tag{22}$$

which reduces to (15) *when one sets* $p = q = r = 1$.

We shall prove (12) by applying a ring homomorphism, defined on the ring Λ of symmetric functions over infinitely many variables $x_1, x_2, \ldots$, to a simple symmetric function identity. There has been a long line of research, [6–15], which shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on the ring of symmetric functions Λ over infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities. For example, the nth elementary symmetric function, e_n, and the nth homogeneous symmetric function, h_n, are defined by the generating functions

$$\begin{aligned}E(t) &= \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t),\\ H(t) &= \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.\end{aligned} \tag{23}$$

We let $P(t) = \sum_{n\geq 0} p_n t^n$ where $p_n = \sum_i x_i^n$ is the nth power symmetric function. A partition of n is a sequence $\mu = (\mu_1, \ldots, \mu_k)$ such that $0 < \mu_1 \leq \cdots \leq \mu_k$ and $\mu_1 + \cdots + \mu_k = n$. We write $\mu \vdash n$ if μ is a partition of n, and we let $\ell(\mu)$ denote the number of parts of μ. If $\mu \vdash n$, we set $h_\mu = \prod_{i=1}^{\ell(\mu)} h_{\mu_i}$, $e_\mu = \prod_{i=1}^{\ell(\mu)} e_{\mu_i}$, and $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$. Let Λ_n denote the space of homogeneous symmetric functions of degree n over infinitely many variables $x_1, x_2, \ldots$ so that $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$. It is well known that $\{e_\lambda : \lambda \vdash n\}$, $\{h_\lambda : \lambda \vdash n\}$, and $\{p_\lambda : \lambda \vdash n\}$ are all bases of Λ_n. It follows that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ and, hence, we can define a ring homomorphism $\xi : \Lambda \to R$ where R is a ring by simply specifying $\xi(e_n)$ for all $n \geq 0$.

Now it is well known that

$$H(t) = \frac{1}{E(-t)}, \tag{24}$$

$$P(t) = \frac{\sum_{n\geq 1} (-1)^{n-1} n e_n t^n}{E(-t)}. \tag{25}$$

A surprisingly large number of results on generating functions for various permutation statistics in the literature and large number of new generating functions can be derived by applying homomorphisms on Λ to simple identities such as (24) and (25). We shall show that (12) can be proved by applying appropriate ring homomorphisms to identity (24).

The outline of this paper is as follows. In Section 2, we will provide the necessary background in symmetric functions that we will need to derive our generating functions. In Section 3, we will prove (12) and give a number of cases where we can compute $\mathrm{mp}_{(\Upsilon,\vec{A})}(p, q; z_0, \ldots, z_{k-1})$ or its specializations which will prove all of the formulas in the special cases described above. We shall also show that if $NM_{(\Upsilon,\vec{A}),n}$ denotes the number of $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has no $(\Upsilon, \vec{A})$-matches, then the sequence $(NM_{(\Upsilon,\vec{A}),n})_{n\geq 0}$ appears in the OIES [16] in several special cases. In Section 4, we will study $NM_{(\Upsilon,\vec{A}),n}$ as a function of k, the size of the underlying cyclic group C_k. We will show that in several cases, $NM_{(\Upsilon,\vec{A}),n}$ is a polynomial in k whose coefficients have interesting combinatorial properties. Finally, in Section 5, we will discuss some related results and directions for future research.

2. Symmetric Functions

In this section, we give the necessary background on symmetric functions needed for our proofs of the generating functions.

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \ldots$ with coefficients in the field of complex numbers $\mathbb{C}$. The nth elementary symmetric function e_n in the variables $x_1, x_2, \ldots$ is defined by

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t), \tag{26}$$

and the nth homogeneous symmetric function h_n in the variables $x_1, x_2, \ldots$ is defined by

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}. \tag{27}$$

Thus,

$$H(t) = \frac{1}{E(-t)}. \tag{28}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition, that is, λ is a finite sequence of weakly increasing positive integers. Let $\ell(\lambda) = l$ denote the number of parts of λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For

Figure 1: A brick tabloid of shape (12) and type (1, 1, 2, 3, 5).

4	6	12	1	5	7	8	10	11	2	3	9

Figure 2: An element of IF(3, 6, 3).

any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well known fundamental theorem of symmetric functions says that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ or that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ is also a basis for Λ. Since $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ, we can specify a ring homomorphism θ on Λ by simply defining $\theta(e_n)$ for all $n \geq 0$.

Since the elementary symmetric functions e_λ and the homogeneous symmetric functions h_λ are both bases for Λ, it makes sense to talk about the coefficient of the homogeneous symmetric functions when written in terms of the elementary symmetric function basis. These coefficients have been shown to equal the sizes of certain sets of combinatorial objects up to a sign. A *brick tabloid* of shape (n), and type $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a filling of a row of n squares of cells with bricks of lengths $\lambda_1, \ldots, \lambda_k$ such that bricks do not overlap. One brick tabloid of shape (12) and type (1, 1, 2, 3, 5) is displayed in Figure 1.

Let $\mathscr{B}_{\lambda,n}$ denote the set of all λ-brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathscr{B}_{\lambda,n}|$. Through simple recursions stemming from (28), Eğecioğlu and Remmel proved in [17] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{29}$$

We end this section with two lemmas that will be needed in later sections. Both of the lemmas follow from simple codings of a basic result of Carlitz [18] that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in \mathscr{R}(1^k 0^{n-k})} q^{\operatorname{inv}(r)}, \tag{30}$$

where $\mathscr{R}(1^k 0^{n-k})$ is the number of rearrangements of k 1's and $n-k$ 0's. We start with a lemma from [19]. Fix a brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathscr{B}_{\mu,n}$. Let IF$(T)$ denote the set of all fillings of the cells of $T = (b_1, \ldots, b_{\ell(\mu)})$ with the numbers $1, \ldots, n$ so that the numbers increase within each brick reading from left to right. We then think of each such filling as a permutation of S_n by reading the numbers from left to right in each row. For example, Figure 2 pictures an element of IF(3, 6, 3) whose corresponding permutation is 4 6 12 1 5 7 8 10 11 2 3 9.

Then the following lemma which is proved in [19] gives a combinatorial interpretation to $p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \left[{n \atop b_1, \ldots, b_{\ell(\mu)}} \right]_{p,q}$.

Lemma 8. *If $T = (b_1, \ldots, b_{\ell(\mu)})$ is a brick tabloid in $\mathscr{B}_{\mu,n}$, then*

$$p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \begin{bmatrix} n \\ b_1, \ldots, b_{\ell(\mu)} \end{bmatrix}_{p,q} = \sum_{\sigma \in IF(T)} q^{\operatorname{inv}(\sigma)} p^{\operatorname{coinv}(\sigma)}. \tag{31}$$

Another well known combinatorial interpretation for $\left[{n+k-1 \atop k-1} \right]_q$ ([13]) is that it is equal to the sum of the sizes of the partitions that are contained in an $n \times (k-1)$ rectangle. Thus one has the following lemma.

Lemma 9. *Consider the following:*

$$\sum_{0 \leq a_1 \leq \cdots \leq a_n \leq k-1} q^{a_1 + \cdots + a_n} = \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_q. \tag{32}$$

3. Generating Functions

The main goal of this section is to prove (12). That is, we will prove the following theorem.

Theorem 10. *Let $\Upsilon \subseteq C_k \wr S_2$ and $\vec{A} = (A_1, A_2)$ where $A_1, A_2 \subseteq [k]$. For all $k \geq 2$,*

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} p^{\operatorname{coinv}(\sigma)} q^{\operatorname{inv}(\sigma)} x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))} z(w) \\ = 1 \times \Big(1 - \big((z_0 + \cdots + z_{k-1})\, t \\ + \sum_{n \geq 2} \frac{t^n}{[n]_{p,q}!} (x-1)^{n-1} \operatorname{mp}_{(\Upsilon,\vec{A}),n} \\ \times (p, q; z_0, \ldots, z_{k-1})\big)\Big)^{-1}. \end{aligned} \tag{33}$$

Proof. Define a ring homomorphism $\Gamma : \Lambda \to \mathbb{Q}(p, q, z_0, \ldots, z_{k-1}, x)$ by setting

$$\begin{aligned} \Gamma(e_0) &= 1, \\ \Gamma(e_1) &= z_0 + \cdots + z_{k-1}, \\ \Gamma(e_n) &= (-1)^{n-1} (x-1)^{n-1} \frac{1}{[n]_{p,q}!} \operatorname{mp}_{(\Upsilon,\vec{A}),n} \\ &\quad \times (p, q; z_0, \ldots z_{k-1}) \quad \text{for } n \geq 1. \end{aligned} \tag{34}$$

Then we claim that

$$\begin{aligned} [n]_{p,q}! \Gamma(h_n) \\ = \sum_{(\sigma,w) \in C_k \wr S_n} q^{\operatorname{inv}(\sigma)} p^{\operatorname{coinv}(\sigma)} z(w)\, x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}, \end{aligned} \tag{35}$$

for all $n \geq 1$. That is,

$$\begin{aligned} [n]_{p,q}! \Gamma(h_n) \\ = [n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma(e_\mu) \\ = [n]_{p,q}! \sum_{\mu \vdash \text{ÂŘ} n} (-1)^{n-\ell(\mu)} \end{aligned}$$

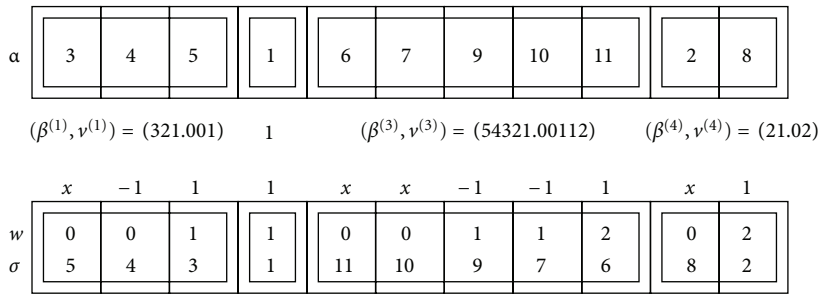

Figure 3: A composite object $C \in \mathscr{F}_{12}$.

$$\begin{aligned}
&\times \sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1}(x-1)^{b_j-1} \\
&\times \Big((z_0+\cdots+z_{k-1})\,\chi\big(b_j=1\big) \\
&\qquad + \frac{1}{[b_j]_{p,q}!}\mathrm{mp}_{(\Upsilon,\vec{A}),b_j}(p,q;z_0,z_1,\ldots,z_{k-1}) \\
&\qquad \times\chi\big(b_j>1\big)\Big) \\
&= \sum_{\mu\vdash \hat{A}\hat{R}n} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}} p^{\sum_{j=1}^{\ell(\mu)}\binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} \\
&\times \prod_{j=1}^{\ell(\mu)} \Big((z_0+\cdots+z_{k-1})\,\chi\big(b_j=1\big) \\
&\qquad + \frac{p^{-\binom{b_j}{2}}}{[b_j]_{p,q}!}\,\mathrm{mp}_{(\Upsilon,\vec{A}),b_j}(p,q;z_0,z_1,\ldots,z_{k-1}) \\
&\qquad \times\chi\big(b_j>1\big)\Big).
\end{aligned} \tag{36}$$

Next we want to give a combinatorial interpretation to (36). By Lemma 8, for each brick tabloid $T = (b_1,\ldots,b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)}\binom{b_j}{2}}\left[{n \atop b_1,\ldots,b_{\ell(\mu)}}\right]_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\alpha \in S_n$ such that α is increasing in each brick, and we weight α with $q^{\mathrm{inv}(\alpha)}p^{\mathrm{coinv}(\alpha)}$. If $b_j = 1$ and covers cell r, then we will interpret the factor $z_0+\cdots+z_{k-1}$ as picking an element $w_r \in [k]$ to put on top of α_r. Next suppose that $b_j > 1$ and covers cells $r, r+1,\ldots,r+s$. Let $\alpha_r < \alpha_{r+1} < \cdots < \alpha_{r+s}$ be the elements of α in cells $r, r+1,\ldots,r+s$, respectively. Then we interpret $p^{-\binom{n}{2}}\mathrm{mp}_{(\Upsilon,\vec{A}),b_j}(p,q;z_0,\ldots,z_{k-1})$ as the numbers of ways of picking a maximum packing $(\beta^{(j)}, v^{(j)}) \in C_k \wr S_{b_j}$ for $(\Upsilon,\vec{A})$ where we weight $(\beta^{(j)}, v^{(j)})$ by $p^{-\binom{b_j}{2}}p^{\mathrm{coinv}(\beta^{(j)})}q^{\mathrm{inv}(\beta^{(j)})}z(v^{(j)})$. We then reorder by the proper contribution after reordering which is $\mathrm{coinv}(\beta^{(j)})$ to the coinversion count of the resulting permutation and $\mathrm{inv}(\beta^{(j)})$ to the inversion count of the resulting permutation. Finally, we interpret $\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. For example, suppose that $\Upsilon = \{(21,00),(21,01)\}$, $k = 3$, and $\vec{A} = ([3],[3])$. Thus (σ, w) is a maximum packing for $(\Upsilon,\vec{A})$ only if σ is strictly decreasing and w is a weakly increasing word over the alphabet $\{0,1,2\}$. Then at the top of Figure 3, we have pictured the brick tabloid $T = (3,1,5,2)$ along with a permutation α which is increasing within bricks. Below that, we have picked our choices of $(\beta^{(j)}, v^{(j)})$ for the bricks b_j for $j = 1,3,4$ and choice of 1 for brick b_2 which is of length 1. These choices result in the filled brick tabloid pictured at the bottom of Figure 3. We have also specified a labeling of the cells with either x, -1, or 1 so that the last cell of each brick is labeled with 1 and the remaining cells are labeled with either x of -1. We shall call all such objects created in this way filled labeled brick tabloids, and let $\mathscr{F}_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus, a $C \in \mathscr{F}_n$ consists of a brick tabloid $T = (b_1,\ldots,b_t)$, an element $(\sigma, w) \in C_k \wr S_n$, and a labeling L of the cells of T with elements from $\{x, 1, -1\}$ such that

(1) if $b_j = 1$ and covers cell r, then w_r is an arbitrary element of $[k]$,

(2) if $b_j > 1$ and covers cells $r, r+1,\ldots,r+s$, then $(\mathrm{red}(\sigma_r\sigma_{r+1}\cdots\sigma_{r+s}), w_rw_{r+1}\cdots w_{r+s})$ is an element of $\mathscr{MP}_{(\Upsilon,\vec{A}),b_j}$,

(3) the final cell of each brick is labeled with 1,

(4) each cell which is not a final cell of a brick is labeled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)$ times the product of all the x labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $C = (T,(\sigma,w),L)$ is the filled labeled brick tabloid pictured at the bottom of Figure 3, then $w(C) = q^{28}p^{27}z_0^5z_1^4z_2^2x^4$ and $\mathrm{sgn}(C) = -1$. It follows that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C\in\mathscr{F}_n} \mathrm{sgn}(C)\,w(C). \tag{37}$$

Next we define a weight-preserving sign-reversing involution $I : \mathscr{F}_n \to \mathscr{F}_n$. To define $I(C)$, we scan the cells of $C = (T,(\sigma,w),L)$ from left to right looking for the leftmost cell t

such that either (i) t is labeled with -1 or (ii) t is at the end of a brick b_j, and the brick b_{j+1} immediately following b_j has the property that if b_j together with b_{j+1} cover cells $r, r+1, \ldots, r+s$, then $(\mathrm{red}(\sigma_r\sigma_{r+1}\cdots\sigma_{r+s}), w_r w_{r+1}\cdots, w_{r+s})$ is a maximum packing for $(\Upsilon, \vec{A})$. In case (i), $I(C) = (T', (\sigma', w'), L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, $w = w'$, and L' is the labeling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I(C) = (T', (\sigma', w'), L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, $w = w'$, and L' is the labeling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if C is the element of $\mathscr{F}_{12}$ pictured in Figure 3, then $I(C)$ is pictured in Figure 4.

It is easy to see that I is a weight-preserving sign-reversing involution and hence I shows that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C\in\mathscr{F}_n, I(C)=C} \mathrm{sgn}(C)\, w(C). \tag{38}$$

Thus, we must examine the fixed points $C = (T, \sigma, w, L)$ of I. First there can be no -1 labels in L so that $\mathrm{sgn}(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it cannot be the case that there is $(\Upsilon, \vec{A})$-match starting at position t in (σ, w) since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I is pictured in Figure 5 where $\sigma = 5\ 4\ 3\ 1\ 11\ 10\ 9\ 7\ 6\ 8\ 2$ and $w = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 2\ 0\ 2$.

It follows that if cell t is at the end of a brick, then t is not the start of a $(\Upsilon, \vec{A})$-match. However, if v is a cell which is not at the end of a brick, then our definitions force v to be the start of a $(\Upsilon, \vec{A})$-match. Since each such cell v must be labeled with an x, it follows that $\mathrm{sgn}(C)w(C) = q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, (\sigma, w), L)$ by having the bricks in T end at cells of the form t where t is not the start of a $(\Upsilon, \vec{A})$-match, labeling each cell t which is the start of a $(\Upsilon, \vec{A})$-match with x, and labeling the remaining cells with 1. Thus, we have shown that

$$\begin{aligned}[n]_{p,q}!\Gamma(h_n) &= \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)\, x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}\end{aligned} \tag{39}$$

as desired.

	x	1	1	1	x	x	−1	−1	1	x	1
w	0	0	1	1	0	0	1	1	2	0	2
σ	5	4	3	1	11	10	9	7	6	8	2

Figure 4: $I(C)$ for C in Figure 3.

	x	x	x	1	x	x	x	x	1	x	1
w	0	0	1	1	0	0	1	1	2	0	2
σ	5	4	3	1	11	10	9	7	6	8	2

Figure 5: A fixed point of I.

Applying Γ to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned}
&\sum_{n\geq 0}\Gamma(h_n)\,t^n \\
&\quad= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!} \\
&\qquad\times \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)\,x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))} \\
&\quad= \frac{1}{1+\sum_{n\geq 1}(-t)^n\Gamma(e_n)} \\
&\quad= 1\times\Bigg(1+(z_0+\cdots+z_{k-1})(-t)+\sum_{n\geq 1}(-1)^n t^n \\
&\qquad\times\frac{(-1)^{n-1}(x-1)^{n-1}}{[n]_{p,q}!} \\
&\qquad\times \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1})\Bigg)^{-1} \\
&\quad= 1\times\Big(1-\Big((z_0+\cdots+z_{k-1})\,t \\
&\qquad+\sum_{n\geq 2}\left((x-1)^{n-1}t^n/[n]_{p,q}!\right) \\
&\qquad\times\ \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1})\Big)\Big)^{-1},
\end{aligned} \tag{40}$$

which proves (33). □

To be able to use Theorem 10, we need to be able to compute $\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1})$. In fact, this is easy to do in most cases. Thus, we end this section by giving several such examples. We will examine the cases when

(i) $\Upsilon = \Upsilon_{\mathbf{s}} = \{(12, 01)\}$ and $\vec{A} = ([k], [k])$,

(ii) $\Upsilon = \Upsilon_{\mathbf{r}} = \{(12, 00), (12, 01)\}$ and $\vec{A} = ([k], [k])$,

(iii) $\Upsilon = \Upsilon_{\mathbf{w}} = \{(12, 00)\}$ and $\vec{A} = ([k], [k])$,

(iv) $\Upsilon = \Upsilon_{\mathbf{d}} = \{(12, 01), (12, 10)\}$ and $\vec{A} = ([k], [k])$,

(v) $\Upsilon = \{(12, ab)\}$ and $\vec{A} = (\{a\}, \{b\})$,

(vi) $\Upsilon = \{(12, 00), (12, 01)\}$ and $\vec{A} = ([k], \{k-1\})$,

(vii) $\Upsilon = \{(12,00),(12,01)\}$ and $\vec{A} = ([k],\{k-2,k-1\})$,

(viii) $\Upsilon = \{(12,01)\}$ and $\vec{A} = ([k],\{k-2,k-1\})$.

Example 11. Let $\Upsilon = \Upsilon_{\mathbf{s}} = \{(12,01)\}$ and $\vec{A} = ([k],[k])$.

In this case for any $(\sigma,w) \in C_k \wr S_n$, $(\Upsilon,\vec{A})$-mch$((\sigma,w)) = s$ris$((\sigma,w))$. It is easy to see that for $n \geq 2$, $(\sigma,w) \in \mathscr{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ is the identity permutation and $w = w_1\cdots w_n$ where $0 \leq w_1 < \cdots < w_n \leq k-1$. As we pointed out in the introduction, it follows that

$$\begin{aligned} \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1}) &= p^{\binom{n}{2}}\left(\prod_{i=0}^{k-1}(1+z_it)\right)\Bigg|_{t^n}, \\ \mathrm{mp}_{(\Upsilon_{\mathbf{s}},\vec{A}),n}(p,q;1,\ldots,1) &= p^{\binom{n}{2}}\binom{k}{n}. \end{aligned} \tag{41}$$

Note that if $0 \leq w_1 < \cdots < w_n \leq k-1$ and $a_i = w_i - (i-1)$ for $i = 1,\ldots,n$, then $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq k-n$ and $\sum_{i=1}^n w_i = \binom{n}{2} + \sum_{i=1}^n a_i$. Hence,

$$\begin{aligned} \mathrm{mp}_{(\Upsilon_{\mathbf{s}},\vec{A}),n}(p,q;1,r\ldots,r^{k-1}) &= p^{\binom{n}{2}} \sum_{0\leq w_1<\cdots<w_n\leq k-1} r^{w_1+\cdots+w_n} \\ &= p^{\binom{n}{2}} r^{\binom{n}{2}} \sum_{0\leq a_1\leq\cdots\leq a_n\leq k-n} r^{\sum_{i=1}^n a_i} \\ &= p^{\binom{n}{2}} r^{\binom{n}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_r. \end{aligned} \tag{42}$$

Here the last equality follows from Lemma 9. Thus, it follows that for all $k \geq 2$,

$$\begin{aligned} &D_k^{\Upsilon_{\mathbf{s}}}(x,p,q,r,t) \\ &\quad= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{\|w\|}x^{\mathrm{sris}((\sigma,w))} \\ &\quad= 1\times\Bigg(1-\Bigg(\left(1+r+\cdots+r^{k-1}\right)t \\ &\qquad\qquad+\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!} \\ &\qquad\qquad\times \mathrm{mp}_{(\Upsilon_{\mathbf{s}},A),n}\left(p,q;1,r,\ldots,r^{k-1}\right)\Bigg)\Bigg)^{-1} \\ &\quad= \frac{1-x}{1-x+\sum_{n\geq 1}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)r^{\binom{n}{2}}\left[{k \atop n}\right]_r}. \end{aligned} \tag{43}$$

Example 12. Let $\Upsilon = \Upsilon_{\mathbf{r}} = \{(12,00),(12,01)\}$ and $\vec{A} = ([k],[k])$.

In this case for any $(\sigma,w) \in C_k \wr S_n$, $(\Upsilon,\vec{A})$-mch$((\sigma,w)) = \mathrm{ris}((\sigma,w))$. It is easy to see that for $n \geq 2$, $(\sigma,w) \in \mathscr{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ is the identity permutation and $w = w_1\cdots w_n$ where $0 \leq w_1 \leq \cdots \leq w_n \leq k-1$. As we pointed out in the introduction, it follows that

$$\begin{aligned} \mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1}) &= p^{\binom{n}{2}}\left(\prod_{i=0}^{k-1}\frac{1}{(1-z_it)}\right)\Bigg|_{t^n}, \\ \mathrm{mp}_{(\Upsilon_r,\vec{A}),n}(p,q;1,r\ldots,r^{k-1}) &= p^{\binom{n}{2}} \sum_{0\leq w_1\leq\cdots\leq w_n\leq k-1} r^{w_1+\cdots+w_n} \\ &= p^{\binom{n}{2}}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r, \\ \mathrm{mp}_{(\Upsilon_{\mathbf{r}},\vec{A}),n}(p,q;1,\ldots,1) &= p^{\binom{n}{2}}\binom{n+k-1}{n}. \end{aligned} \tag{44}$$

Hence, for all $k \geq 2$,

$$\begin{aligned} &D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) \\ &\quad= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{\|w\|}x^{\mathrm{ris}((\sigma,w))} \\ &\quad= 1\times\Bigg(1-\Bigg(\left(1+r+\cdots+r^{k-1}\right)t \\ &\qquad\qquad+\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}\mathrm{mp}_{(\Upsilon_r,A),n} \\ &\qquad\qquad\times\left(p,q;1,r,\ldots,r^{k-1}\right)\Bigg)\Bigg)^{-1} \\ &\quad= \frac{1-x}{1-x+\sum_{n\geq 1}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)\left[{n+k-1 \atop n}\right]_r}. \end{aligned} \tag{45}$$

Example 13. Let $\Upsilon = \Upsilon_{\mathbf{w}} = \{(12,00)\}$ and $\vec{A} = ([k],[k])$.

In this case for any $(\sigma,w) \in C_k \wr S_n$, $(\Upsilon,\vec{A})$-mch$((\sigma,w)) = w$ris$((\sigma,w))$. It is easy to see that for $n \geq 2$, $(\sigma,w) \in \mathscr{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ is the identity permutation and $w = w_1\cdots w_n$ where $0 \leq w_1 = \cdots = w_n \leq k-1$. As we pointed out in the introduction, it follows that

$$\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1}) = p^{\binom{n}{2}}\sum_{i=0}^{k-1}z_i^n. \tag{46}$$

Hence,

$$\begin{aligned} \mathrm{mp}_{(\Upsilon_{\mathbf{w}},\vec{A}),n}(p,q;1,\ldots,1) &= kp^{\binom{n}{2}}, \\ \mathrm{mp}_{(\Upsilon_{\mathbf{w}},\vec{A}),n}(p,q;1,r,\ldots,r^{k-1}) &= p^{\binom{n}{2}} = p^{\binom{n}{2}}[k]_{r^n}. \end{aligned} \tag{47}$$

Thus, for all $k \geq 2$,

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{w}}}(x,p,q,r,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{\|w\|}x^{w\mathrm{ris}((\sigma,w))} \\ &= 1\times\Big(1-\Big(\big(1+r+\cdots+r^{k-1}\big)t \\ &\qquad +\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}\mathrm{mp}_{(\Upsilon_r,A),n} \\ &\qquad \times\big(p,q;1,r,\ldots,r^{k-1}\big)\Big)\Big)^{-1} \\ &= \frac{1-x}{1-x+\sum_{n\geq 1}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)[k]_{r^n}}. \end{aligned} \tag{48}$$

Example 14. Let $\Upsilon = \Upsilon_{\mathbf{d}} = \{(12,01),(12,10)\}$ and $\vec{A} = ([k],[k])$.

In this case, it is easy to see that for $n \geq 2$, $(\sigma,w) \in \mathcal{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ and $w = w_1\cdots w_n$ where $w_i \neq w_{i+1}$ for $i = 1,\ldots,n$. In this case, one cannot find simple compact expressions for $\mathrm{mp}_{(\Upsilon_{\mathbf{r}},A),n}(p,q;z_0,\ldots,z_{k-1})$ or $\mathrm{mp}_{(\Upsilon_{\mathbf{r}},A),n}(p,q;1,r,\ldots,r^{k-1})$. However, it is clear that $\mathrm{mp}_{(\Upsilon_{\mathbf{r}},A),n}(p,q;1,\ldots,1) = k(k-1)^{n-1}$. Thus, it follows that, for all $k \geq 2$,

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}x^{w\mathrm{ris}((\sigma,w))} \\ &= 1\times\left(1-\left(kt+\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}\right.\right. \\ &\qquad \left.\left.\times\ \mathrm{mp}_{(\Upsilon_{\mathbf{d}},A),n}(p,q;1,1,\ldots,1)\right)\right)^{-1} \\ &= \frac{1-x}{1-x+\sum_{n\geq 1}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)k(k-1)^{n-1}}. \end{aligned} \tag{49}$$

Example 15. Let $\Upsilon = \{(12,ab)\}$ and $\vec{A} = (\{a\},\{b\})$.

In this case, $(\Upsilon,\vec{A})$-mch$((\sigma,w))$ $=(\tau,u)$-Emch$((\sigma,w))$ where $(\tau,u) = (12,ab)$. There are two cases that we have to consider depending on whether a equals b or not. If $a = b$, then for $n \geq 2$, $(\sigma,w) \in C_k \wr S_n$ is an element of $\mathcal{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ and $w = a^n$. Thus, $\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1}) = p^{\binom{n}{2}}z_a^n$. It follows that if $(\tau,u) = (12,aa)$, then

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)\,x^{(\tau,u)\text{-Emch}((\sigma,w))} \\ &\quad = 1\times\left(1-\left(\left(z_0+\cdots+z_{k-1}\right)t\right.\right. \\ &\qquad \left.\left.+\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}p^{\binom{n}{2}}z_a^n\right)\right)^{-1}. \end{aligned} \tag{50}$$

In particular,

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}x^{(\tau,u)\text{-Emch}((\sigma,w))} \\ &\quad = \frac{1-x}{1-x+k(x-1)t+\sum_{n\geq 2}\left(p^{\binom{n}{2}}((x-1)t)^n/[n]_{p,q}!\right)}. \end{aligned} \tag{51}$$

If $a \neq b$, then the only possible maximum packings for $(12,ab)$ are when $n = 2$ in which case $(\sigma,w) \in C_k \wr S_2$ is an element of $\mathcal{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $(\sigma,w) = (12,ab)$. Thus, $\mathrm{mp}_{(\Upsilon,\vec{A}),2}(p,q;z_0,\ldots,z_{k-1}) = pz_az_b$. It follows that if $(\tau,u) = (12,ab)$ with $a \neq b$, then

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)\,x^{(\tau,u)\text{-Emch}((\sigma,w))} \\ &\quad = \frac{1}{1-\left(\left(z_0+\cdots+z_{k-1}\right)t+\left((x-1)t^2/[2]_{p,q}!\right)pz_az_b\right)}. \end{aligned} \tag{52}$$

In particular,

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}x^{(\tau,u)\text{-Emch}((\sigma,w))} \\ &\quad = \frac{1-x}{1-x+k(x-1)t+\left(p(x-1)^2t^2/[2]_{p,q}!\right)}. \end{aligned} \tag{53}$$

Example 16. Let $\Upsilon = \{(12,00),(12,01)\}$ and $\vec{A} = ([k],\{k-1\})$.

In this case, it is easy to see that for all $n \geq 2$, $(\sigma,w) \in \mathcal{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma = 12\cdots n$ and $w = a(k-1)^{n-1}$ where $a \in [k]$. It follows that for all $n \geq 2$,

$$\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;z_0,\ldots,z_{k-1}) = p^{\binom{n}{2}}\left(z_0+\cdots+z_{k-1}\right)z_{k-1}^{n-1}. \tag{54}$$

Hence,

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}z(w)\,x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}\\ = 1\times\Big(1-\big((z_0+\cdots+z_{k-1})\,t\\ +\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}\\ \times p^{\binom{n}{2}}(z_0+\cdots+z_{k-1})\,z_{k-1}^{n-1}\Big)\Big)^{-1}.\end{aligned}\tag{55}$$

Setting $p=q=z_0=\cdots=z_{k-1}=1$ in (55), we obtain that

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}\\ =\frac{1}{1-k\sum_{n\geq 1}\left((x-1)^{n-1}t^n/n!\right)}\\ =\frac{1-x}{1-x+k\sum_{n\geq 1}\left(((x-1)\,t)^n/n!\right)}\\ =\frac{1-x}{1-x+k\left(e^{(x-1)t}-1\right)}\\ =\frac{1-x}{1-k-x+ke^{(x-1)t}}.\end{aligned}\tag{56}$$

Let $NM_{(\{(12,00),(12,01)\},([k],\{k-1\}))n}$ denote the number of $\sigma\in C_k\wr S_n$ that have no $(\{(12,00),(12,01)\},([k],\{k-1\}))$-matches. Then setting $x=0$ in (56), we see that

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{n!}NM_{(\{(12,00),(12,01)\},([k],\{k-1\}))n}\\ =\frac{1}{1-k+ke^{-t}}=\frac{e^t}{k-(k-1)\,e^t}.\end{aligned}\tag{57}$$

One can easily calculate that the sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1\},\{1\}))n})_{n\geq 0}$ starts out with

$$1,2,6,26,150,1082,9366,94586,1091670,\ldots\tag{58}$$

which is sequence A000629 in the OEIS and which also counts the number of necklaces of partitions of $n+1$ labeled beads. Similarly, the sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1,2\},\{2\}))n})_{n\geq 0}$ starts out with

$$1,3,15,111,1095,13503,199815,3449631,68062695,\ldots,\tag{59}$$

and the sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1,2,3\},\{3\}))n})_{n\geq 0}$ starts out with

$$\begin{aligned}1,4,28,292,4060,70564,1471708,35810212,\\ 995827420,\ldots.\end{aligned}\tag{60}$$

Neither of these two sequences appear in the OEIS.

Example 17. Let $\Upsilon=\{(12,00),(12,01)\}$ and $\vec{A}=([k],\{k-2,k-1\})$.

In this case, it is easy to see that for all $k\geq 2$ and $n\geq 2$, $(\sigma,w)\in\mathscr{MP}_{(\Upsilon,\vec{A}),n}$ if and only if $\sigma=12\cdots n$ and w is either of the form $w=a(k-2)^i(k-1)^j$ where $0\leq a\leq k-3$ and $i+j=n-1$ or $w=(k-2)^r(k-1)^s$ where $r+s=n$. Hence, for all $n\geq 2$,

$$\mathrm{mp}_{(\Upsilon,\vec{A}),n}(p,q;1,\ldots,1)=((k-1)\,n+1)\,p^{\binom{n}{2}}.\tag{61}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}\\ = 1\times\Big(1-\big((z_0+\cdots+z_{k-1})\,t\\ +\sum_{n\geq 2}\frac{(x-1)^{n-1}t^n}{[n]_{p,q}!}\\ \times((k-1)\,n+1)\,p^{\binom{n}{2}}\Big)\Big)^{-1}.\end{aligned}\tag{62}$$

Setting $p=q=1$ in (62), we obtain that

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))}\\ =\frac{1}{1-\sum_{n\geq 1}((k-1)\,n+1)\left((x-1)^{n-1}t^n/n!\right)}\\ =\frac{1-x}{1-x+\sum_{n\geq 1}((k-1)\,n+1)\left(((x-1)\,t)^n/n!\right)}\\ =1-x\Bigg(1-x+(k-1)\,(x-1)\,t\\ \times\left(\sum_{n\geq 1}\frac{((x-1)\,t)^{n-1}}{(n-1)!}\right)\\ +\sum_{n\geq 1}\frac{((x-1)\,t)^n}{n!}\Bigg)^{-1}\\ =\frac{1-x}{1-x+(k-1)\,(x-1)\,te^{(x-1)t}+e^{(x-1)t}-1}\\ =\frac{1-x}{-x+(1+(k-1)\,(x-1)\,t)\,e^{(x-1)t}}.\end{aligned}\tag{63}$$

Let $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\}))n}$ denote the number of $\sigma\in C_k\wr S_n$ that have no $(\{(12,00),(12,01)\},([k],\{k-2,k-1\}))$-matches. Then setting $x=0$ in (63), we see that

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{n!}NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\}))n}\\ =\frac{1}{(1-(k-1)\,t)\,e^{-t}}=\frac{e^t}{1-(k-1)\,t}.\end{aligned}\tag{64}$$

Note that it follows that for $n \geq 1$,

$$\begin{aligned} NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n} &= n!\sum_{j=0}^{n}\frac{(k-1)^j}{(n-j)!} \\ &= 1+n!\sum_{j=1}^{n}\frac{(k-1)^j}{(n-j)!} \\ &= 1+(k-1)\,n\left((n-1)!\sum_{i=0}^{n-1}\frac{(k-1)^i}{(n-1-i)!}\right) \\ &= 1+(k-1)\,nNM_{(\{12,00),(12,01)\},([k],\{k-2,k-1\}),n-1}. \end{aligned} \tag{65}$$

It would be interesting to find a direct combinatorial proof of this recursion, which we leave as an open problem.

One can easily calculate that the sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1\},\{0,1\})),n})_{n\geq 0}$ starts out with

$$1, 2, 5, 16, 65, 326, 1957, 13700, 109601, 986410\ldots, \tag{66}$$

which is sequence A000522 in the OEIS having multiple combinatorial interpretations including the number of permutations $\sigma = \sigma_1\cdots\sigma_{n+1}$ with no $1 \leq i < j \leq n$ such that $\sigma_i < \sigma_j < \sigma_{n+1}$, that is, in the *Babson-Steingrimsson notation*, the number of σ in S_{n+1} which avoid the pattern 1-2-3] (the same pattern is denoted 123⌋ in the recently introduced notation [20]).

The sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1,2\},\{1,2\})),n})_{n\geq 0}$ starts out with

$$1, 3, 13, 79, 633, 6331, 75973, 1063623, 18017969, 306323443\ldots, \tag{67}$$

which is sequence A010844 of the OEIS. This sequence does not have a combinatorial interpretation listed in the OEIS so that we have obtained a simple combinatorial interpretation of this sequence. However, if $a(n)$ is the nth term of this sequence where $a(0) = 1$, then $a(n)$ is equal to 2^n times the permanent of the $n\times n$ matrix with 3/2 on the main diagonal and 1s everywhere else.

The sequence $(NM_{(\{(12,00),(12,01)\},(\{0,1,2,3\},\{2,3\})),n})_{n\geq 0}$ starts out with

$$1, 4, 25, 226, 2713, 40696, 732529, 15383110, 369194641, 9968255308,\ldots \tag{68}$$

is sequence A010844 of the OEIS. This sequence does not have a combinatorial interpretation listed in the OEIS. Thus, we have obtained a new combinatorial interpretation of this sequence. In this case, if $a(n)$ is the nth term of the sequence where $a(0) = 1$, then $a(n)$ is equal to 3^n times the permanent of the $n \times n$ matrix with 4/3 on the main diagonal and 1's everywhere else.

Similarly, the sequences $(NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n})_{n\geq 0}$ for $k = 5, 6$, and 7 appear in the OEIS as sequences as A056545, A056546, and A056547, respectively.

Example 18. Let $\Upsilon = \{(12, 01)\}$ and $\vec{A} = ([k], \{k-2, k-1\})$.

In this case, it is easy to see that $\mathcal{MP}_{(\Upsilon,\vec{A}),n} = \emptyset$ for $k > 3$. That is, for $(\sigma, w) \in \mathcal{MP}_{(\Upsilon,\vec{A}),n}$, we must have $\sigma = 12\cdots n$ and $w = w_1\cdots w_n$ where $0 \leq w_1 < \cdots < w_n \leq k-1$ and $w_i \in \{k-2, k-1\}$ for $i > 1$. Now if $n = 2$, our choices for w are either $a(k-2)$ or $a(k-1)$ where $a \in \{0,\ldots,k-3\}$ or $w = (k-2)(k-1)$ so that $\mathrm{mp}_{(\Upsilon,\vec{A}),2}(p, q; 1,\ldots,1) = p(2(k-2)+1) = p(2k-3)$. Now if $n = 3$, our choices for w are $a(k-2)(k-1)$ where $a \in \{0,\ldots,k-3\}$, so that $\mathrm{mp}_{(\Upsilon,\vec{A}),3}(p, q; 1,\ldots,1) = p^3(k-2)$.

It follows that

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))} \\ &\quad = 1\times\left(1-\left(kt+p\,(2k-3)\,(x-1)\,\frac{t^2}{[2]_{p,q}!}\right.\right. \\ &\qquad\left.\left.+p^3(x-1)^2\,(k-2)\,\frac{t^3}{[3]_{p,q}!}\right)\right)^{-1}. \end{aligned} \tag{69}$$

Setting $p = q = 1$ in (69), we obtain that for $k \geq 2$

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{(\Upsilon,\vec{A})\text{-mch}((\sigma,w))} \\ &\quad = 1\times\left(1-\left(kt+(2k-3)\,(x-1)\left(\frac{t^2}{2}\right)\right.\right. \\ &\qquad\left.\left.+(x-1)^2\,(k-2)\left(\frac{t^3}{6}\right)\right)\right)^{-1}. \end{aligned} \tag{70}$$

Let $NM_{(12,01),([k],\{k-2,k-1\}),n}$ denote the number of $\sigma \in C_k \wr S_n$ that have no $((12, 01), ([k], \{k-2, k-1\}))$-matches. Then setting $x = 0$ in (70), we see that

$$\begin{aligned} &\sum_{n\geq 0}\frac{t^n}{n!}NM_{(12,01),([k],\{k-2,k-1\}),n} \\ &\quad = \frac{1}{1-kt+(2k-3)\left(t^2/2\right)-(k-2)\left(t^3/6\right)}. \end{aligned} \tag{71}$$

One can easily calculate the initial terms of the sequences $(NM_{(12,01),([k],\{k-2,k-1\}),n})_{n\geq 0}$. For example, the sequence $(NM_{(12,01),([2],\{0,1\}),n})_{n\geq 0}$ starts out

$$1, 2, 7, 36, 246, 2100, 21150, 257040, 3510360, 53933040,\ldots. \tag{72}$$

The sequence $(NM_{(12,01),([3],\{1,2\}),n})_{n\geq 0}$ starts out

$$1, 3, 15, 109, 1050, 12630, 182270, 3068730, 59046120, 1278133080,\ldots. \tag{73}$$

The sequence $(NM_{(12,01),([4],\{2,3\}),n})_{n\geq 0}$ starts out

$$1, 4, 27, 266, 3478, 56800, 1112990, 25443180, 664724760, 19537301289,\ldots. \tag{74}$$

However, none of these sequences appear in the OEIS.

Table 1: $\mathrm{NM}_{n,k}^{\Upsilon_{\mathbf{r}}}$ for $k, n \le 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	5	16	65	326
$k=3$	1	3	12	64	441	3771
$k=4$	1	4	22	164	1589	19136
$k=5$	1	5	35	335	4180	64876

4. Expressions for $NM_{(\Upsilon,\vec{A}),n}$ as Polynomials in k and Connections to Other Objects

Recall that for any $\Upsilon \subseteq C_k \wr S_2$ and $\vec{A} = (A_1, A_2)$ where $A_i \subseteq [k]$, $NM_{(\Upsilon,\vec{A}),n}$ is the number of $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has no $(\Upsilon, \vec{A})$-matches. We start out by considering the special case where $\Upsilon \subseteq C_2 \wr S_2$ and $\vec{A} = ([k],[k])$. In that case, we will simply write $A_{n,k}^{\Upsilon}$ for $NM_{(\Upsilon,\vec{A}),n}$. Thus, $A_{n,k}^{\Upsilon}$ is the number of permutations $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has no Υ-matches. By setting $p = q = z_0 = \cdots = z_{k-1} = 1$ in (33), we see that

$$\sum_{n\ge 0} A_{n,k}^{\Upsilon} \frac{t^n}{n!} = 1 \times \left(1 - \left(kt + \sum_{n\ge 2} \mathrm{mp}_{(\Upsilon,([k],[k])),n} \times (1,1;1,\ldots,1)\left((-1)^{n-1} t^n/n!\right)\right)\right)^{-1}. \tag{75}$$

We can easily compute the initial sequences of values for these generating functions using any computer algebra system such as Mathematica or Maple. In this section, we will demonstrate these values in several cases.

4.1. $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$. For example, if $\Upsilon = \Upsilon_{\mathbf{r}} = \{(1\ \ 2, 0\ \ 0), (1\ \ 2, 0\ \ 1)\}$, $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$ equals the number of $(\sigma, \epsilon) \in C_k \wr S_n$ such that $\mathrm{ris}((\sigma, \epsilon)) = 0$. Table 1 gives initial values of $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$.

Several of these sequences appear in the OEIS [16]. In fact, we can easily calculate $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$ as a polynomial in k. For example, we have

$$NM_{0,k}^{\Upsilon_{\mathbf{r}}} = 1,$$
$$NM_{1,k}^{\Upsilon_{\mathbf{r}}} = k,$$
$$NM_{2,k}^{\Upsilon_{\mathbf{r}}} = \frac{1}{2}k(3k-1),$$
$$NM_{3,k}^{\Upsilon_{\mathbf{r}}} = \frac{1}{6}k\left(19k^2 - 15k + 2\right),$$
$$NM_{4,k}^{\Upsilon_{\mathbf{r}}} = \frac{1}{24}k\left(211k^3 - 270k^2 + 89k - 6\right),$$
$$NM_{5,k}^{\Upsilon_{\mathbf{r}}} = \frac{1}{120}k\left(3651k^4 - 6490k^3 + 3585k^2 - 650k + 24\right). \tag{76}$$

We point out that $NM_{2,k}^{\Upsilon_{\mathbf{r}}}$ forms the familiar sequence of pentagonal numbers (A000326 in [16]). Other previously documented sequences appearing in Table 1 include the structured octagonal antiprism numbers (A100184 in [16]) for $NM_{3,k}^{\Upsilon_{\mathbf{r}}}$, as well as $NM_{n,2}^{\Upsilon_{\mathbf{r}}}$ (A000522 in [16]), for which there are many known combinatorial interpretations, including the total number of arrangements of all subsets of $[n]$.

We conjecture that for $n \ge 1$ and $k \ge 2$, $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$ is always of the form $(1/n!)kP_n(k)$ where $P_n(k)$ is a polynomial of degree $n-1$ whose leading coefficient is positive and such that signs of the remaining coefficients alternate. Now we can prove that $NM_{n,k}^{\Upsilon_{\mathbf{r}}}$ is always of the form $(1/n!)kP_n(k)$ where $P_n(k)$ is a polynomial of degree $n-1$ and the term of degree 1 in k is $(-1)^{n-1}(n-1)!$. That is, for any $k \ge 2$, if we set $p = q = z_0 = \cdots = z_{k-1} = 1$ and $x = 0$ in the proof of Theorem 10 and use the fact that $\mathrm{mp}_{(\Upsilon_{\mathbf{r}},([k],[k])),n}(1,1;1,\ldots,1) = \binom{n+k-1}{n}$, we see that

$$n!\overline{\Gamma}(h_n) = NM_{n,k}^{\Upsilon_{\mathbf{r}}}, \tag{77}$$

where

$$\overline{\Gamma}(e_n) = \frac{\binom{n+k-1}{n}}{n!} = \frac{(k)\uparrow_n}{(n!)^2}. \tag{78}$$

Here we let $(q)\uparrow_0 = 1$ and $(q)\uparrow_n = q(q+1)\cdots(q+n-1)$ for $n \ge 1$. But then

$$\begin{aligned} n!\overline{\Gamma}(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)} B_{\mu,n}\overline{\Gamma}(e_\mu) \\ &= \frac{1}{n!}\sum_{\mu\hat{A}\check{R}\vdash n}(-1)^{n-\ell(\mu)} B_{\mu,n}\binom{n}{\mu_1,\ldots,\mu_{\ell(\mu)}}\prod_{i=1}^{2\ell(\mu)}(k)\uparrow_{\mu_i} \\ &= \frac{1}{n!}\sum_{\mu\hat{A}\check{R}\vdash n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}}\binom{n}{b_1,\ldots,b_{\ell(\mu)}}\prod_{i=1}^{2\ell(\mu)}(k)\uparrow_{b_i}. \end{aligned} \tag{79}$$

It is easy to see that the right-hand side of (79) is a polynomial of degree n, and the lowest degree term comes from the term $(-1)^{n-1}k(k+1)\cdots(k+n-1)$ corresponding to $\mu = (n)$ which is of the form $(-1)^{n-1}(n-1)!k + O(k^2)$. On the other hand, the highest-degree term arises when we pick the highest power of k from the products $\prod_{i=1}^{\ell(\mu)}(k)\uparrow_{b_i}$ which is clearly just k^n. It follows that the highest power of k in $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ is just

$$\frac{1}{n!}\sum_{\mu\vdash\hat{A}\check{R}n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}}\binom{n}{b_1,\ldots,b_{\ell(\mu)}}^2. \tag{80}$$

−1	−1	1	−1	−1	−1	1	−1	−1	−1	1
1	4	7	2	8	9	11	3	5	6	10
2	3	6	4	7	9	10	1	5	8	11

Figure 6: A brick tabloid of shape (12) and type (1, 1, 2, 3, 5).

1	−1	1	−1	−1	−1	1	−1	−1	−1	1
1	4	7	2	8	9	11	3	5	6	10
2	3	6	4	7	9	10	1	5	8	11

Figure 7: $I_r(O)$ for O in Figure 6.

We can give a combinatorial interpretation to

$$F_n = \sum_{\mu \vdash \hat{A}\check{R} n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathscr{B}_{\mu,n}} \binom{n}{b_1,\ldots,b_{\ell(\mu)}}^2. \quad (81)$$

That is, we will show that F_n counts the number of pairs of permutations $(\alpha, \beta) \in S_n \times S_n$ such that if $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1, \ldots, \beta_n$, then there is no $1 \le i \le n-1$ such that $\alpha_i < \alpha_{i+1}$ and $\beta_i < \beta_{i+1}$. That is, F_n is the number of pairs $(\alpha, \beta) \in S_n \times S_n$ that have no common rises. To prove our claim, we can give a combinatorial interpretation to the right-hand side of (81). That is, if we start with a brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)})$, then by Lemma 8, we can interpret $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}^2$ as all ways to fill the cells of T with pairs of permutations (α, β) such that both α and β are increasing in each brick. Then we interpret $(-1)^{n-\ell(\mu)}$ as labeling each cell of T which is not at the end of brick with -1 and labeling each cell which is at the end of the brick with 1. We let $\mathscr{OR}_n$ denote the set of all filled brick tabloids constructed in this way. For example, Figure 6 pictures a typical element of $\mathscr{OR}_{11}$. If $O \in \mathscr{OR}_n$, we define the sign of O, sgn(O), as the product of labels at the top of the cells of O.

We now can define a sign-reversing involution I_r on $\mathscr{OR}_n$.

To define I_r, suppose that $O \in \mathscr{OR}_n$ and α and β are the two permutations in O with α on the bottom and β on the top. Then we scan the cells of O from left to right looking for the leftmost cell t such that either (i) t is labeled with -1 or (ii) t is at the end of a brick b_j, and the brick b_{j+1} immediately following b_j has the property that both α and β are strictly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_r(O)$ is the result of replacing the brick b in O containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and changing the label of cell t from -1 to 1. In case (ii), $I_r(O)$ is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b and changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I(O) = O$. For example, if O is the element of $\mathscr{F}_{12}$ pictured in Figure 6, then $I_r(O)$ is pictured in Figure 7.

It is easy to see that I_r is a weight-preserving sign-reversing involution and, hence, I_r shows that

$$F_n = \sum_{O \in \mathscr{OR}_n, I_r(O)=O} \operatorname{sgn}(O). \quad (82)$$

1	1	1	1	1	1	1	1	1	1	1
4	7	1	11	9	8	2	3	5	10	6
3	2	6	4	7	9	10	8	5	1	11

Figure 8: A fixed point of I_r.

Table 2: $\mathrm{NM}_{n,k}^{\Upsilon_s}$ for $k, n \le 5$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$k = 2$	1	2	7	36	246	2100
$k = 3$	1	3	15	109	1050	12630
$k = 4$	1	4	26	244	3031	47000
$k = 5$	1	5	40	460	6995	132751

Thus, we must examine the fixed points O of I_r. First there can be no -1 labels in O so that all bricks in O must be of size 1 and $\operatorname{sgn}(O) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it cannot be the case that $\alpha_t < \alpha_{t+1}$ and $\beta_t < \beta_{t+1}$ where α and β are the two permutations in O since otherwise we could combine b_j and b_{j+1}. Thus, (α, β) is a pair of permutations in S_n with no common rises. Vice versa, if (α, β) is a pair of permutations in S_n with no common rises, we can create a fixed point O of I_r by having α be the bottom permutation of O, having β be the top permutation of O, and having all bricks be of size 1. For example, Figure 8 pictures a fixed point of I_r.

It follows that F_n is the number of pairs $(\alpha, \beta) \in S_n^2$ with no common rises so that the leading coefficient of $A_{k,r}^{\Upsilon_r}$ is $F_n/n!$.

4.2. $NM_{n,k}^{\Upsilon_s}$. For $\Upsilon_s = \{(1\ \ 2, 0\ \ 1)\}$, $NM_{n,k}^{\Upsilon_s}$ equals the number of $(\sigma, \epsilon) \in C_k \wr S_n$ such that $\operatorname{sris}((\sigma, \epsilon)) = 0$. Table 2 gives initial values of $NM_{n,k}^{\Upsilon_s}$.

In fact, we can easily calculate $NM_{n,k}^{\Upsilon_s}$ as a polynomial in k. For example, we have

$$\begin{aligned}
NM_{0,k}^{\Upsilon_s} &= 1, \\
NM_{1,k}^{\Upsilon_s} &= k, \\
NM_{2,k}^{\Upsilon_s} &= \frac{1}{2} k (3k+1), \\
NM_{3,k}^{\Upsilon_s} &= \frac{1}{6} k \left(19k^2 + 15k + 2\right), \\
NM_{4,k}^{\Upsilon_s} &= \frac{1}{24} k \left(211k^3 + 270k^2 + 89k + 6\right), \\
NM_{5,k}^{\Upsilon_s} &= \frac{1}{120} k \left(3651k^4 + 6490k^3 + 3585k^2 + 650k + 24\right).
\end{aligned} \quad (83)$$

We point out that $NM_{2,k}^{\Upsilon_s}$ forms the familiar sequence of the second pentagonal numbers (A005449 in [16]). None of the other rows or columns in Table 2 matched any previously known sequences in [16].

Table 3: $\mathrm{NM}_{n,k}^{\Upsilon_{\mathbf{w}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	6	26	150	1082
$k=3$	1	3	15	111	1095	13503
$k=4$	1	4	28	292	4060	70564
$k=5$	1	5	45	605	10845	243005

We conjecture that for $n \geq 1$ and $k \geq 2$, $NM_{n,k}^{\Upsilon_s}$ is always of the form $(1/n!)kR_n(k)$ where $R_n(k)$ is a polynomial of degree $n-1$ with positive coefficients. In fact, we see that the coefficients of $P_{n,k}$ and $R_{n,k}$ are the same up to a sign for all n. we can prove this. That is, for any $k \geq 2$, if we set $p = q = z_0 = \cdots = z_{k-1} = 1$ and $x = 0$ in the proof of Theorem 10 and use the fact that $\mathrm{mp}_{(\Upsilon_s,([k],[k]))n} = \binom{k}{r}$, we see that

$$n!\overline{\Gamma}_s(h_n) = NM_{n,k}^{\Upsilon_s}, \tag{84}$$

where

$$\overline{\Gamma}_s(e_n) = \frac{\binom{k}{n}}{n!} = \frac{(k)\downarrow_n}{(n!)^2}. \tag{85}$$

Here we let $(q)\downarrow_0 = 1$ and $(q)\downarrow_n = q(q-1)\cdots(q-n+1)$ for $n \geq 1$. But then

$$\begin{aligned} n!\overline{\Gamma}_s(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,n}\overline{\Gamma}(e_\mu) \\ &= \frac{1}{n!}\sum_{\mu\text{ÂŘ}\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,n}\binom{n}{\mu_1,\ldots,\mu_{\ell(\mu)}}\prod_{i=1}^{2\ell(\mu)}(k)\downarrow_{\mu_i} \\ &= \frac{1}{n!}\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}}\binom{n}{b_1,\ldots,b_{\ell(\mu)}}\prod_{i=1}^{2\ell(\mu)}(k)\downarrow_{b_i}. \end{aligned} \tag{86}$$

Since for any $n \geq 1$, $(k)\downarrow_n = (-1)^n(k)\uparrow_n$, it is easy to see that the right-hand side of (86) is obtained from the right-hand side of (79) by replacing k by $-k$ and multiplying by $(-1)^n$. Thus, conjecture that $R_n(k)$ has positive coefficients is equivalent to the conjecture that the signs of the coefficients of $P_n(k)$ alternate.

4.3. $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$. For $\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$, $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$ equals the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $w\mathrm{ris}((\sigma,\epsilon)) = 0$. Table 3 gives initial values of $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$.

In fact, we can easily calculate $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$ as a polynomial in k. For example, we have

$$\begin{aligned} NM_{0,k}^{\Upsilon_{\mathbf{w}}} &= 1, \\ NM_{1,k}^{\Upsilon_{\mathbf{w}}} &= k, \\ NM_{2,k}^{\Upsilon_{\mathbf{w}}} &= k(2k-1), \\ NM_{3,k}^{\Upsilon_{\mathbf{w}}} &= k\left(6k^2-6k+1\right), \\ NM_{4,k}^{\Upsilon_{\mathbf{w}}} &= k\left(24k^3-36k^2+14k-1\right), \\ NM_{5,k}^{\Upsilon_{\mathbf{w}}} &= k\left(120k^4-240k^3+150k^2-30k+1\right). \end{aligned} \tag{87}$$

We point out that $NM_{2,k}^{\Upsilon_{\mathbf{w}}}$ forms the familiar sequence of hexagonal numbers (A000384 in [16]). Additionally, $NM_{n,2}^{\Upsilon_{\mathbf{w}}}$ matches the sequence counting the number of necklaces on the set of labeled beads (A000629 in [16]). In fact, in this case, we can give a completely combinatorial interpretation of the coefficients of $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$ as a polynomial in k. Let $\mathrm{Ostpn}(n)$ denote the set of ordered set partitions of $\{1,2,\ldots,n\}$. For any set partition $\pi \in \mathrm{Ostpn}(n)$, let $\ell(\pi)$ denote the number of parts of π. Then we claim that

$$NM_{n,k}^{\Upsilon_{\mathbf{w}}} = \sum_{\pi\in\mathrm{Ostpn}(n)}(-1)^{n-\ell(\pi)}k^{\ell(\pi)}, \tag{88}$$

so that the coefficient of k^j in $NM_{n,k}^{\Upsilon_{\mathbf{w}}}$ is equal to $(-1)^{n-j}j!S_{n,j}$ where $S_{n,j}$ is the Stirling number of the second kind which is the number of set partitions of $\{1,\ldots,n\}$ into j parts. That is, for any $k \geq 2$, if we set $p = q = z_0 = \cdots = z_{k-1} = 1$ and $x = 0$ in Theorem 10 and use the fact that $\mathrm{mp}_{\Upsilon_{\mathbf{w}},n} = k$ for all $n \geq 2$, then we see that

$$n!\overline{\Gamma}_w(h_n) = NM_{n,k}^{\Upsilon_{\mathbf{w}}}, \tag{89}$$

where

$$\overline{\Gamma}_w(e_n) = \frac{k}{n!}. \tag{90}$$

But then

$$\begin{aligned} n!\overline{\Gamma}_s(h_n) &= n!\sum_{\mu\text{ÂŘ}\vdash n}(-1)^{n-\ell(\mu)}\overline{\Gamma}_s(e_\mu) \\ &= n!\sum_{\mu\vdash\text{ÂŘ}n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}}\prod_{i=1}^{\ell(\mu)}\frac{k}{b_i!} \\ &= \sum_{\mu\vdash\text{ÂŘ}n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathscr{B}_{\mu,n}}\binom{n}{b_1,\ldots,b_{\ell(\mu)}}k^{\ell(\mu)}. \end{aligned} \tag{91}$$

Since $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}$ counts the number of ordered set partitions $\pi = (\pi_1,\ldots,\pi_{\ell(\mu)})$ such that $|\pi_j| = b_j$, it is easy to see that the right-hand side of (91) equals the right-hand side of (88).

Table 4: $\mathrm{NM}_{n,k}^{\Upsilon_{\mathbf{d}}}$ for $k, n \le 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	6	26	150	1082
$k=3$	1	3	12	66	480	4368
$k=4$	1	4	20	132	1140	12324
$k=5$	1	5	30	230	2280	28280

L	1	$-k$	1	k	1	$-k$	k	1	1	k	$-k$	k
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 9: A composite object $C \in \mathscr{D}_{12}$.

4.4. $NM_{n,k}^{\Upsilon_{\mathbf{d}}}$. Recall that $\Upsilon_{\mathbf{d}} = \{(1\ \ 2, 0\ \ 1), (1\ \ 2, 1\ \ 0)\}$. Table 4 gives initial values of $NM_{n,k}^{\Upsilon_{\mathbf{d}}}$.

In fact, we can easily calculate $NM_{n,k}^{\Upsilon_{\mathbf{d}}}$ as a polynomial in k. For example, we have

$$\begin{aligned} NM_{0,k}^{\Upsilon_{\mathbf{d}}} &= 1, \\ NM_{1,k}^{\Upsilon_{\mathbf{d}}} &= k, \\ NM_{2,k}^{\Upsilon_{\mathbf{d}}} &= k^2 + k, \\ NM_{3,k}^{\Upsilon_{\mathbf{d}}} &= k^3 + 4k^2 + k, \\ NM_{4,k}^{\Upsilon_{\mathbf{d}}} &= k^4 + 11k^3 + 11k^2 + k, \\ NM_{5,k}^{\Upsilon_{\mathbf{d}}} &= k^5 + 26k^4 + 66k^3 + 26k^2 + k. \end{aligned} \tag{92}$$

In this case, we will show that $NM_{n,k}^{\Upsilon_{\mathbf{d}}}$ is just the Eulerian polynomial

$$NM_{n,k}^{\Upsilon_{\mathbf{d}}} = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)+1}. \tag{93}$$

That is, for any $k \ge 2$, if we set $p = q = z_0 = \cdots = z_{k-1} = 1$ and $x = 0$ in the proof of Theorem 10 and the fact that $\mathrm{mp}_{(\Upsilon_{\mathbf{d}},([k],[k])),n} = k(k-1)^{n-1}$, we see that

$$n!\overline{\Gamma}_U(h_n) = NM_{n,k}^{\Upsilon_{\mathbf{d}}}, \tag{94}$$

where

$$\overline{\Gamma}_U(e_n) = \frac{k(k-1)^{n-1}}{n!}. \tag{95}$$

But then

$$\begin{aligned} n!\overline{\Gamma}_U(h_n) &= n! \sum_{\mu \vdash \hat{A}\check{R} n} (-1)^{n-\ell(\mu)} B_{u,n} \overline{\Gamma}_U(e_\mu) \\ &= n! \sum_{\mu \vdash \hat{A}\check{R} n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathscr{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \frac{k(k-1)^{b_i-1}}{b_i!} \\ &= \sum_{\mu \hat{A}\check{R} \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathscr{B}_{\mu,n}} \binom{n}{b_1,\ldots,b_{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} k(1-k)^{b_i-1}. \end{aligned} \tag{96}$$

L	1	k	1	k	1	$-k$	k	1	1	k	$-k$	k
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 10: $J(C)$ for C in Figure 9.

Next we want to give a combinatorial interpretation to (96). For any brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathscr{B}_{\mu,n}$, we can interpret $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}$ as the set of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick. We then interpret $\prod_{j=1}^{\ell(\mu)} k(1-k)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an 1 or a $-k$ and letting the label of the last cell of each brick be k. We let $\mathscr{D}_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus, a $C \in \mathscr{D}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, and a labeling L of the cells of T with elements from $\{k, -k, 1\}$ such that

(1) σ is strictly increasing in each brick,

(2) the final cell of each brick is labeled with k,

(3) each cell which is not a final cell of a brick is labeled with 1 or $-k$.

We then define the weight $w(C)$ of C to be the product of all the k labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 3, 3, 2)$, then Figure 9 pictures such a composite object $C \in \mathscr{D}_{12}$ where $w(C) = k^7$ and $\mathrm{sgn}(C) = -1$.

Thus,

$$n!\overline{\Gamma}_U(h_n) = \sum_{C \in \mathscr{D}_n} \mathrm{sgn}(C)\, w(C). \tag{97}$$

Next we define a weight-preserving sign-reversing involution $J : \mathscr{D}_n \to \mathscr{D}_n$. To define $J(C)$, we scan the cells of $C = (T, \sigma, L)$ from left to right looking for the leftmost cell t such that either (i) t is labeled with $-k$ or (ii) t is at the end of a brick b_j, and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $J(C) = (T', \sigma', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, and L' is the labeling that results from L by changing the label of cell t from $-k$ to k. In case (ii), $J(C) = (T', \sigma', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, and L' is the labeling that results from L by changing the label of cell t from k to $-k$. If neither case (i) or case (ii) applies, then we let $J(C) = C$. For example, if C is the element of $\mathscr{D}_{12}$ pictured in Figure 9, then $J(C)$ is pictured in Figure 10.

It is easy to see that J is a weight-preserving sign-reversing involution, and hence J shows that

$$n!\overline{\Gamma}_U(h_n) = \sum_{C \in \mathscr{D}_n, J(C)=C} \mathrm{sgn}(C)\, w(C). \tag{98}$$

Thus, we must examine the fixed points $C = (T, \sigma, L)$ of J. First there can be no $-k$ labels in L so that $\mathrm{sgn}(C) = 1$.

L	1	1	1	k	1	1	k	1	1	k	1	k
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 11: A fixed point of J.

Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it cannot be the case that $\sigma_t < \sigma_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I is pictured in Figure 11 where

$$\sigma = 2\ 3\ 4\ 11\ 6\ 9\ 10\ 1\ 8\ 12\ 5\ 7. \tag{99}$$

It follows that if cell t is at the end of a brick which is not the last brick, then $\sigma_t > \sigma_{t1}$. However, if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$. Since each such cell v must be labeled with an 1, it follows that $\operatorname{sgn}(C)w(C) = k^{\operatorname{des}(\sigma)+1}$ where the $+1$ comes from the fact that the last cell of the last brick is also labeled with k. Vice versa, if $\sigma \in S_n$, then we can create a fixed point $C = (T, \sigma, L)$ by having the bricks in T end at cells of the form t where $\sigma_t > \sigma_{t+1}$ and labeling each such cell with k, labeling the last cell with k, and labeling the remaining cells with 1. Thus, we have shown that

$$n!\overline{\Gamma}_U(h_n) = \sum_{\sigma \in S_n} k^{\operatorname{des}(\sigma)+1} \tag{100}$$

as desired.

4.5. $NM_{(\Upsilon,\vec{A}),n}$. Next we consider another case where we can show that as a polynomial in k, $NM_{(\Upsilon,\vec{A}),n}$ has coefficients whose signs alternate. That is, it follows from (64) that

$$\begin{aligned} NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n} &= n!\sum_{r=0}^{n}\frac{(k-1)^r}{(n-r)!} \\ &= \sum_{r=0}^{n}(n)\downarrow_r (k-1)^r. \end{aligned} \tag{101}$$

We can easily calculate $NM_{(\{(12,00),(12,01)\},([k],\{k-2mk-1\})),n}$ as a polynomial in k. For example, we have

$$\begin{aligned} NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),0} &= 1, \\ NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),1} &= k, \\ NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),2} &= 2k^2 - 2k + 1, \\ NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),3} &= 6k^3 - 12k^2 + 9k - 2, \\ NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),4} &= 24k^4 - 72k^3 + 84k^2 - 44k + 9, \\ NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),5} &= 120k^5 - 480k^4 + 780k^3 + -640k^2 + 265k - 44. \end{aligned} \tag{102}$$

Thus, we see that the signs of the polynomial $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}$ seem to alternate and that the leading term seems to be $n!k^n$. We can prove both of these facts. Taking the coefficient of k^s on both sides of (101), we see that

$$\begin{aligned} &NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}\big|_{k^s} \\ &\quad = \sum_{r=s}^{n}(n)\downarrow_r \binom{r}{s}(-1)^{r-s}. \end{aligned} \tag{103}$$

Now we can give a combinatorial interpretation to the right-hand side of (103) as follows. First for each r, we interpret $(n)\downarrow_r$ as the number of ways to pick a sequence $a_1 \cdots a_r$ of pairwise distinct elements from $[n]$. Then we interpret $\binom{r}{s}$ as picking s of the elements which we indicate by putting an x above each of the s elements picked. Finally, we interpret $(-1)^{r-s}$ as placing -1 above the elements in the sequence. We let $M_{n,s,r}$ denote the set of all labeled sequences constructed in this way. For example, if $n = 10$, $s = 3$, and $r = 6$, then the following labeled sequence would be in $M_{10,4,6}$:

$$(a, L) = \begin{matrix} -1 & x & -1 & -1 & x & x \\ 6 & 3 & 10 & 4 & 5 & 8 \end{matrix}. \tag{104}$$

Given a labeled sequence $(a, L) \in M_{n,s,r}$, we let the sign of (a, L), $\operatorname{sgn}(a, L)$, be the product of the -1 labels in L. We let $M_{n,s} = \bigcup_{r=s}^{n} M_{n,s,r}$. Then it is easy to see that

$$\begin{aligned} &NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}\big|_{k^s} \\ &\quad = \sum_{r=s}^{n}(n)\downarrow_r \binom{r}{s}(-1)^{r-s} = \sum_{(a,L)\in M_{n,s}} \operatorname{sgn}(a, L). \end{aligned} \tag{105}$$

Given a sequence $(a, L) = (a_1 \cdots a_r, L_1 \cdots L_r)$, we say that the minimal sequence of (a, L) is the longest string $a_t a_{t+1} \cdots a_r$ such that for all $j = t, \ldots, r$, $a_j = \min([n] - \{a_1, \ldots, a_{j-1}\})$ and $L_i = -1$ for $j = t, \ldots, r$. The minimal sequence for (a, L) might be empty. For example, it could be the last label $L_r = x$ or it could be that $a_r \neq \min([n] - \{a_1, \ldots, a_{r-1}\})$. We let $\operatorname{ms}(a, L)$ denote the length of the minimal sequence of (a, L). For example, the minimal sequence of

$$(a, L) = \begin{matrix} -1 & x & x & x & -1 & -1 \\ 6 & 3 & 10 & 4 & 5 & 8 \end{matrix} \tag{106}$$

is empty so that $\operatorname{ms}(a, L) = 0$. The minimal sequence of

$$(a', L') = \begin{matrix} -1 & x & x & x & -1 & -1 \\ 1 & 3 & 10 & 4 & 2 & 5 \end{matrix} \tag{107}$$

is 25 so that $\operatorname{ms}(a, L) = 2$.

Now suppose that $(a, L) \in M_{n,s,r}$ is such that $a = a_1 \cdots a_r$, $L = L_1 \cdots L_r$, and its minimal sequence is empty.

Let $b_1 < \cdots < b_{n-r}$ be the elements of $[n] - \{a_1, \ldots, a_r\}$. Then we define the minimal sequence chain of (a, L) to be $(a^{(r)}, L^{(r)}), (a^{(r+1)}, L^{(r+1)}), \ldots, (a^{(n)}, L^{(n)})$ where $(a^{(r)}, L^{(r)}) = (a, L)$ and $(a^{(r+p)}, L^{(r+p)}) = (a_1 \cdots a_r b_1 \cdots b_p, L_1 \cdots L_r(-1)^p)$ for $1 \le p \le n - r$. Note that the signs of the elements of a minimal sequence chain of (a, L) alternate. Thus, if the minimal sequence chain of (a, L) is of even length, then $\mathrm{ms}(a^{(n)}, L^{(n)})$ is odd, and the sum of the signs of elements in the minimal sequence chain of (a, L) is 0. If the minimal sequence chain of (a, L) is of odd length, then $\mathrm{ms}(a^{(n)}, L^{(n)})$ is even, and the sum of the signs of elements in the minimal sequence chain of (a, L) is $\mathrm{sgn}(a^{(n)}, L^{(n)})$. Clearly every element of $M_{n,r}$ is an element of a unique minimal sequence chain for some $(a, L) \in M_{n,s}$ whose minimal sequence is empty so that

$$\begin{aligned}\sum_{(a,L)\in M_{n,s}} &\mathrm{sgn}\,(a, L)\\ &= \sum_{(a,L)\in M_{n,s,n},\mathrm{ms}(a,L)\text{ is even}} \mathrm{sgn}\,(a, L)\\ &= (-1)^{n-s}\left|\{(a, L) \in M_{n,s,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{108}$$

Thus, we have shown that

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^s}\\ &\quad= (-1)^{n-s}\left|\{(a, L) \in M_{n,s,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{109}$$

This shows that as a polynomial in k, $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}$ is sign alternating starting with the leading term $n!k^n$. That is, by (109),

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^n}\\ &\quad= \left|\{(a, L) \in M_{n,n,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{110}$$

But the elements of $M_{n,n,n}$ have no -1 labels since every element of $M_{n,n,n}$ has an empty minimal sequences and, hence, $|\{(a, L) \in M_{n,n,n} : \mathrm{ms}(a, L) \text{ is even}\}| = n!$.

We can also compute several other coefficients of $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}$ as a polynomial in k. For example, we can explicitly compute the coefficient of k^{n-1} in $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}$. Note that

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^{n-1}}\\ &\quad= -\left|\{(a, L) \in M_{n,n-1,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{111}$$

Now an element $(a, L) \in M_{n,n-1,n}$ has exactly one label which is -1. If $L_n = -1$, then the last element of a is automatically its minimal sequence so that it is not counted in $|\{(a, L) \in M_{n,n-1,n} : \mathrm{ms}(a, L) \text{ is even}\}|$. Thus, $|\{(a, L) \in M_{n,n-1,n} : \mathrm{ms}(a, L) \text{ is even}\}|$ counts all $(a, L) \in M_{n,n-1,n}$ such that $L_n = x$ which is clearly $(n-1)n!$.

We claim that

$$NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^0} = (-1)^n D_n, \tag{112}$$

where D_n is the number of derangements of S_n, that is, the number of permutations with no fixed points. It is well known that $D_n = n!\sum_{k=0}^{n}(-1)^k/k!$ for $n \ge 1$. But by (105),

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^0}\\ &\quad= \sum_{r=0}^{n}(n)\downarrow_r(-1)^r = (-1)^n n!\sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!} = D_n.\end{aligned} \tag{113}$$

One can also use our combinatorial interpretation of $NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^0}$ to give a combinatorial proof of this fact. That is, it is well known that $D_1 = 0, D_2 = 1$ for $n \ge 2 D_{n+1} = nD_n + nD_{n-1}$. In our case, we know that

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^0}\\ &\quad= (-1)^n\left|\{(a, L) \in M_{n,0,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{114}$$

Thus, we must show that $D_n = |\{(a, L) \in M_{n,0,n} : \mathrm{ms}(a, L) \text{ is even}\}|$. Now if $(a, L) \in M_{n,0,n}$, then all the labels in L are -1 so that we can identify each element $(a, L) \in M_{n,0,n}$ with just a permutation $a = \sigma = \sigma_1 \cdots \sigma_n \in S_n$. Now consider a $\sigma \in S_{n+1}$ whose minimal sequence has even length. If $\sigma_1 \ne 1$, then clearly the minimal sequence of $\mathrm{red}(\sigma_2 \cdots \sigma_{n+1}) \in S_n$ has the same length as the minimal sequence of σ. Vice versa, given $\tau_1 \cdots \tau_n \in S_n$, let $\tau^{(i)} = i\tau_1^* \cdots \tau_n^*$ be the permutation of S_{n+1} such that $\mathrm{red}(\tau_1^* \cdots \tau_n^*) = \tau$. Clearly for $i = 2, \ldots, n+1$, $\tau^{(i)}$ is a permutation in S_{n+1} whose minimal sequences have the same length as the minimal sequence of τ. It follows that

$$\begin{aligned}&\left|\{(a, L) \in M_{n+1,0,n+1} : \mathrm{ms}\,(\sigma, L) \text{ is even } \&\ \sigma_1 \ne 1\}\right|\\ &\quad= n\left|\{(a, L) \in M_{n,0,n} : \mathrm{ms}\,(a, L) \text{ is even}\}\right|.\end{aligned} \tag{115}$$

Next suppose that $\sigma \in S_{n+1}$ is such that $\sigma_1 = 1$, and its minimal sequence has even length. Then we claim that $\mathrm{red}(\sigma_3 \cdots \sigma_{n+1})$ has a minimal sequence whose length has the same parity as the length of the minimal sequence of σ. That is, if 1 is an element of the minimal sequence of σ, then it must be the case that $\sigma = 12\cdots(n+1)$ is the identity so that $\mathrm{red}(\sigma_3 \cdots \sigma_{n+1}) = 1 \cdots n-1$ has minimal sequence of length $n-1$. If 1 is not in the minimal sequence of σ, then it cannot be the case that σ_2 is in the minimal sequence of σ so that the lengths of the minimal sequences of σ and $\mathrm{red}(\sigma_3 \cdots \sigma_{n+1})$ are the same. Vice versa, given $\tau_1 \cdots \tau_{n-1} \in S_{n-1}$, let $\tau^{(1,i)} = 1i\tau_1^* \cdots \tau_{n-1}^*$ be the permutation of S_{n+1} such that $\mathrm{red}(\tau_1^* \cdots \tau_{n-1}^*) = \tau$. Clearly for $i = 2, \ldots, n+1$, $\tau^{(1,i)}$ is a permutation in S_{n+1} whose lengths of the minimal sequences have the same parity as the length of the minimal sequence of τ. It follows that

$$\begin{aligned}&\left|\{(a, L) \in M_{n+1,0,n+1} : \mathrm{ms}\,(\sigma, L) \text{ is even and } \sigma_1 = 1\}\right|\\ &\quad= n\left|\{(a, L) \in M_{n-1,0,n-1} : \mathrm{ms}\,(a, L)\text{, is even}\}\right|\end{aligned} \tag{116}$$

Hence, $|\{(a, L) \in M_{n+1,0,n+1} : \mathrm{ms}(\sigma, L) \text{ is even}|$ satisfies the same recursion as the recursion for D_n.

Finally, it is not difficult to see that

$$\begin{aligned}&NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n}|_{k^1}\\ &\quad= NM_{(\{(12,00),(12,01)\},([k],\{k-2,k-1\})),n+1}|_{k^0}.\end{aligned} \tag{117}$$

To prove this, we will construct a bijection $\theta : M_{n,1,n} \to M_{n+1,0,n+1}$ which has the property that for all $(a, L) \in M_{n,1,n}$, $\mathrm{ms}(a, L)$ and $\mathrm{ms}(\theta(a, L))$ have the same parity. Now suppose we are given an $(a, L) \in M_{n,1,n}$ where $a = a_1 \cdots a_n$ and $L = L_1 \cdots L_n$. Then is must be the case that there is a unique j such that $L_j = x$. Let $\theta(a, L) = (a', L') = (a_1 + 1, \ldots, a_{j-1} + 1, 1, a_j + 1, \ldots, a_n + 1, (-1)^{n+1})$. That is, (a', L') arises from (a, L) by inserting 1 just before the element of a that is marked with x and adding 1 to all the elements of a and removing the x and replacing it with -1 and labeling 1 with -1. It is easy to see that θ is a bijection. Thus, all we need to do is to show that for all $(a, L) \in M_{n,1,n}$, $\mathrm{ms}(a, L)$ and $\mathrm{ms}(\theta(a, L))$ have the same parity. The minimal sequence of (a, L) does not contain a_j. In $(a\prime, L\prime)$, the minimal sequence does not contain $a'_j = a_{j-1} + 1 > 1$ since $1 = \min([n + 1] - \{a'_1, \ldots, a'_{j-2}\})$. Now if the minimal sequence of (a, L) does not contain a_{j+1}, then it is easy to see that the minimal sequence of (a', L') does not contain a'_{j+2} which means that the lengths of the minimal sequences of (a, L) and (a', L') are the same. So suppose that the minimal sequence of (a, L) is $a_{j+1} \cdots a_n$. Then there are two cases. Namely, either (i) $a_j = \min([n] - \{a_1, \ldots, a_{j-1}\})$ in which case the minimal sequence of (a', L') is $1, a_j + 1, a_{j+1} + 1, \ldots, a_n + 1$ or (ii) $a_j \neq \min([n] - \{a_1, \ldots, a_{j-1}\})$ in which case the minimal sequence of (a', L') is $a_{j+1} + 1, \ldots, a_n + 1$. Hence in all cases, the lengths of the minimal sequences of (a, L) and (a', L') have the same parity. Thus, θ shows that

$$\begin{aligned} &\left|\{(a, L) \in M_{n,1,n} : \mathrm{ms}(\sigma, L) \text{ is even}\}\right| \\ &\quad = \left|\{(a, L) \in M_{n+1,0,n+1} : \mathrm{ms}(\sigma, L) \text{ is even}\}\right|, \end{aligned} \tag{118}$$

which by (109) establishes (117).

5. Further Research

An obvious question is to ask whether the results of this paper can be extended to patterns of length ≥ 3. There are some partial results in this direction due to Duane and Remmel [21]. Given a word $u \in \{0, 1, \ldots, k-1\}^j$ such that $\mathrm{red}(u) = u$ and $\tau \in S_j$, Duane and Remmel say that (τ, u) has the $C_k \wr S_n$-*minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with (τ, u)-$\mathrm{mch}(\sigma, w) = 2$ is $2j-1$. This means that in a k-colored permutation, (σ, w), two (τ, u)-matches in (σ, w) can share at most one pair of letters, and this pair of letters must occur at the end of the first (τ, u)-match and at the start of the second (τ, u)-match. Similarly, we say that (τ, u) has the $C_k \wr S_n$-*exact match minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with (τ, u)-$\mathrm{Emch}(\sigma, w) = 2$ is $2j - 1$. Now if (τ, u) has the $C_k \wr S_n$-minimal overlapping property, then the shortest k-colored permutations, (σ, w), such that (τ, u)-$\mathrm{mch}(\sigma, w) = n$, have length $n(j-1) + 1$. We let $\mathcal{MP}^k_{(\tau,u),n(j-1)+1}$ equal the set of k-colored permutations, $(\sigma, w) \in C_k \wr S_{n(j-1)+1}$, such that (τ, u)-$\mathrm{mch}(\sigma, w) = n$. We refer to the k-colored permutations in $\mathcal{MP}^k_{(\tau,u),n(j-1)+1}$ as maximum packings for (τ, u). We let

$$\begin{aligned} &\mathrm{mp}^k_{(\tau,u),n(j-1)+1} = \left|\mathcal{MP}^k_{(\tau,u),n(j-1)+1}\right|, \\ &\mathrm{mp}^k_{(\tau,u),n(j-1)+1}(p, q, r) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{MP}^k_{(\tau,u),n(j-1)+1}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} r^{\sum w} \\ &\mathrm{mp}^k_{(\tau,u),n(j-1)+1}(p, q, z_0, \ldots, z_{k-1}) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{MP}^k_{(\tau,u),n(j-1)+1}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} z(w). \end{aligned} \tag{119}$$

Similarly, we let $\mathcal{EMP}^k_{(\tau,u),n(j-1)+1}$ equal the set of k-colored permutations, $(\sigma, w) \in C_k \wr S_{n(j-1)+1}$, such that (τ, u)-$\mathrm{Emch}(\sigma, w) = n$, and refer to the k-colored permutations in $\mathcal{EMP}^k_{(\tau,u),n(j-1)+1}$ as exact match maximum packings for (τ, u). We let

$$\begin{aligned} &\mathrm{emp}^k_{(\tau,u),n(j-1)+1} = \left|\mathcal{EMP}^k_{(\tau,u),n(j-1)+1}\right|, \\ &\mathrm{emp}^k_{(\tau,u),n(j-1)+1}(p, q, r) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{EMP}^k_{(\tau,u),n(j-1)+1}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} r^{\sum w}, \\ &\mathrm{emp}^k_{(\tau,u),n(j-1)+1}(p, q, z_0, \ldots, z_{k-1}) \\ &\quad = \sum_{(\sigma,w)\in\mathcal{EMP}^k_{(\tau,u),n(j-1)+1}} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} z(w). \end{aligned} \tag{120}$$

Then Duane and Remmel [21] proved that (I) if $\mathrm{red}(u) = u$ and (τ, u) has the $C_k \wr S_n$-minimal overlapping property, then

$$\begin{aligned} &\sum_{n\geq 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k\wr S_n} x^{(\tau,u)\text{-}\mathrm{mch}(\sigma,u)} p^{\mathrm{coinv}(\sigma)} q^{\mathrm{inv}(\sigma)} z(w) \\ &\quad = 1 \times \Big(1 - \Big((z_0 + \cdots + z_{k-1})t \\ &\qquad + \sum_{n\geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!}(x-1)^n \\ &\qquad \times \mathrm{mp}^k_{(\tau,u),n(j-1)+1}(p, q, z_0, \ldots, z_{k-1})\Big)\Big)^{-1}, \end{aligned} \tag{121}$$

(II) if $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$-exact match minimal overlapping property, then

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{(\tau,u)\text{-Emch}(\sigma,u)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w)$$
$$= 1\times\Big(1-\Big((z_0+\cdots+z_{k-1})t + \sum_{n\geq 1}\frac{t^{n(j-1)+1}}{[n(j-1)+1]_q!}(x-1)^n \times \text{emp}^k_{(\tau,u),n(j-1)+1}(p,q,z_0,\ldots,z_{k-1})\Big)\Big)^{-1}. \quad (122)$$

Duane and Remmel's proof of (I) and (II) works equally well for $(\Upsilon, \vec{A})$-matches.

There are, however, several examples in the literature of generating functions for the number of τ-matches in permutations where τ does not have the minimal overlapping property, and we have found some analogues of such results for $C_k \wr S_n$. In addition, there are many formulas in the literature for the number of permutations that avoid certain patterns in permutations. We have found several examples of such formulas for the matching conditions for $C_k \wr S_n$ discussed in this paper. Such results will appear in subsequent papers.

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