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Research Article

The BALM Copula

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The class of probability distributions possessing the almost-lack-of-memory property appeared about 20 years ago. It reasonably took place in research and modeling, due to its suitability to represent uncertainty in periodic random environment. Multivariate version of the almost-lack-of-memory property is less known, but it is not less interesting. In this paper we give the copula of the bivariate almost-lack-of-memory (BALM) distributions and discuss some of its properties and applications. An example shows how the Marshal-Olkin distribution can be turned into BALM and what is its copula.

1. Introduction

The class of probability distributions called "almost-lack-of-memory (ALM) distributions" was introduced in Chukova and Dimitrov [1] as a counterexample of a characterization problem. Dimitrov and Khalil [2] found a constructive approach considering the waiting time up to the first success for extended in time Bernoulli trials. Similar approach was used in Dimitrov and Kolev [3] in sequences of extended in time and correlated Bernoulli trials. The fact that nonhomogeneous in time Poisson processes with periodic failure rates are uniquely related to the ALM distributions was established in Chukova et al. [4]. It gave impetus to several additional statistical studies on estimations of process parameters (see, e.g., [5, 6] to name a few) of these properties. Best collection of properties of the ALM distributions and related processes can be found in Dimitrov et al. [7]. Meanwhile, Dimitrov et al. [8] extended the ALM property to bivariate case and called the obtained class BALM distributions. For the BALM distributions, a characterization via a specific hyperbolic partial differential equation of order 2 was obtained in Dimitrov et al. [9]. Roy [10] found another interpretation of bivariate lack-of-memory (LM) property and gave a characterization of class of bivariate distributions via survival functions possessing that LM property for all choices of the participating in it four nonnegative arguments. One curious part of the BALM distributions is that the two components of the 2-dimensional vector satisfy the properties characterizing the bivariate exponential distributions with independent components only in the nodes of a rectangular grid in the first quadrant. However, inside the rectangles of that grid any kind of dependence between the two components may hold. In addition, the marginal distributions have periodic failure rates.

This picture makes the BALM class attractive for modeling dependences in investment portfolios, financial mathematics, risk studies, and more (see [11]) where bivariate models are used. Later, the copula approach in modeling dependences [12], its potential use in financial mathematics (as proposed in [13]), and some new ideas expressed in Kolev et al. [14] encouraged us to focus our attention on the construction of copula for the BALM distributions. The results on an extension of the multiplicative lack of memory by Dimitrov and von Collani [15] played a key role in the construction use in this paper.

Here, in the introduction part we give the basic definition and some of the main properties of the univariate ALM distributions needed to build a quick vision on this subject in the multivariate extensions too. Then we present our main results.

Definition 1. A nonnegative random variable X possesses the *almost-lack-of-memory (ALM) property* if there exists an infinite sequence of distinct numbers $\{c_m\}_{m=1}^{\infty}$, such that the lack-of-memory property

$$P(X > c_m + x \mid X > c_m) = P(X > x) \tag{1}$$

holds for every member c_m, $m = 1, 2, \ldots$, and all $x > 0$.

In general, ALM distributions may be discrete, continuous, or of mixed type. But in all the cases the sequence $\{c_m\}_{m=1}^{\infty}$ is a lattice of step $c > 0$. Certain explicit forms of ALM distributions related to random processes are based on an assumption of independence of the uncertainty over nonoverlapping time intervals. It is known (see [7]) that random variable (r.v.) X with the ALM property admits a representation as a sum of two independent componentsas follows:

$$X = W_c + cZ. \tag{2}$$

Here Z is a geometrically distributed r.v. with some parameter α, and W_c has an arbitrary distribution over the interval $[0, c)$. If we denote by $G_W(\cdot)$ the survival function (briefly, SF) of W_c in the presentation (2), then the notation ALM(α, c, G_W) of the class of distribution functions for the r.v. X above shows the main parameters of the family. These parameters are $\alpha \in (0,1)$, $c > 0$, and an SF $G_W(\cdot)$ with support on $[0, c)$. An important relationship here is expressed by the equation $\alpha = P(X > c)$. An explanation of the parameter c can be found in the fact that the failure rate function

$$r_X(t) = \frac{f_X(t)}{P(X > t)}, \quad t > 0 \tag{3}$$

is a periodic function of period c. The distribution function $G_W(\cdot)$ can be arbitrary; it just needs to have support on the interval $[0, c]$. Dimitrov and Khalil [2] in their constructive approach explained the parameter c by the duration of the extended in time Bernoulli trials (the time until a trial is completed), and W_c is the time within a trial when the success is noticed under the condition that success occurs.

The exponential and geometric distributions are specific particular cases when special relationships between parameters α, c, and $G_W(\cdot)$ are met.

In the multivariate, case there are several approaches to define bivariate lack-of-memory property. The options generate more ways to introduce the concept of multivariate lack-of-memory classes of probability distributions; see, for example, Roy [10] and the references therein. Dimitrov et al. [8] proposed a two-dimensional notion of the ALM property by using the simplest bivariate LM property that characterizes the bivariate exponential distributions with independent components. The following definition explains it.

Definition 2. A pair of nonnegative r.v.'s (X, Y) possesses the *bivariate almost-lack-of-memory (BALM) property* if there exist two infinite sequences of distinct numbers $\{a_m\}_{m=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$, such that the lack-of-memory property

$$\begin{aligned} &P(X > a_m + x,\ Y > b_n + y \mid X > a_m,\ Y > b_n) \\ &\quad = P(X > x,\ Y > y) \end{aligned} \tag{4}$$

holds for all members of the two sequences ($m, n = 1, 2, \ldots$ and for every $x > 0$, $y > 0$).

The properties of BALM distributions are similar to those of the univariate ALM class. It appears that in order to avoid the known exponential characterization, the sequences $\{a_m\}_{m=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ have the form $a_m = ma$ with some $a > 0$ and $b_n = nb$ with some $b > 0$. The points with coordinates (ma, nb) $m, n = 0, 1, 2, \ldots$ form nodes of a lattice in the first quadrant in the plane Oxy, and these nodes partition the first quadrant into rectangles equal to the rectangle $(0, a] \times (0, b]$. In particular, an analogous representation to () is obtained. Any two-dimensional random vectors (X, Y) with a BALM distribution can be presented as a sum of two independent bivariate vectors as follows:

$$(X, Y) = (V_a, W_b) + (Z_1, Z_2). \tag{5}$$

The first component (V_a, W_b) in (5) is defined by an arbitrary survival function $G_{V,W}(v, w)$ with a support on the rectangle $(0, a] \times (0, b]$. The coordinate variables in the second component (Z_1, Z_2) are independent geometrically distributed r.v.'s over the sets $\{0, a, 2a, 3a, \ldots\}$ and $\{0, b, 2b, 3b, \ldots\}$, with parameters $\alpha \in (0,1)$ and $\beta \in (0,1)$, respectively. The BALM property required by Definition 2 holds for all $x > 0$, $y > 0$, and for $a_m = ma$, $b_n = nb$ with nonnegative integers m and n.

Important fact here is that the marginal distributions of the BALM random vectors (X, Y) are univariate ALM distributions (see [8]). The two components X and Y may be dependent inside rectangles of the form $[(m-1)a,\ ma) \times [(n-1)b, nb)$, but are independent at the nodes (ma, nb) for any integers $m \ge 0$, $n \ge 0$. Dependence on the noted rectangles emulates the dependence between the pair (V_a, W_b) on the rectangle with the vertex at the origin. This statement can be noticed in the form of the BALM survival function which is given by

$$\begin{aligned} G_{X,Y}(x, y) = \alpha^{[x/a]} \beta^{[y/b]} \Big\{ &(1-\alpha)(1-\beta) G_{V_a, W_b} \\ &\times \left(x - \left[\frac{x}{a}\right] a, y - \left[\frac{y}{b}\right] b \right) \\ &+ \beta (1-\alpha) G_{V_a} \left(x - \left[\frac{x}{a}\right] a \right) \\ &+ \alpha (1-\beta) G_{W_b} \left(y - \left[\frac{y}{b}\right] b \right) \\ &+ \alpha\beta \Big\}, \quad x \ge 0,\ y \ge 0. \end{aligned} \tag{6}$$

Here $[x]$ is notation for the integer part of any nonnegative number x inside the brackets. The α, β in (6) are parameters with values on (0,1). Actually, it occurs that these parameters are related to the components of the random vector $\vec{X} = (X, Y)$ by the equalities

$$\alpha = P(X > a), \qquad \beta = P(Y > b). \tag{7}$$

$G_{V_a,W_b}(v,w)$ is the SF of a random vector (V_a,W_b) with probability 1 located on the rectangle $(0,a]\times(0,b]$, and $G_{V_a}(v) = G_{V_a,W_b}(v,0)$ and $G_{W_b}(w) = G_{V_a,W_b}(0,w)$ are marginal survival functions of the components V and W obtained from the concordance conditions.

The components X and Y have $\mathrm{ALM}(\alpha,a,G_{V_a})(v)$ and $\mathrm{ALM}(\beta,b,G_{W_b}(w))$ distributions correspondingly. As possible area of applications of the class of BALM variables, we see the financial portfolio with two components X and Y, where the actualization (renewals, updates) is performed when the value of investment X becomes multiple to a quantity $a>0$ and the value of investment Y becomes multiple to a quantity $b>0$.

In parallel to the works on ALM distributions, researchers successfully were developing unifying approaches to model dependencies between components of multivariate distributions, namely, the use of copula. The copula approach enjoyed an incredible evolution during the last decades, motivated by its application in probability modeling of dependences in finances, insurance, economics (see [11–13] and references therein). The new investigations are related mainly to finding the copula for random vectors with components possessing given marginal distributions, whose mutual dependence follows certain correlation structure. A systematic exposition of the copula approach and its applications, as well as several modern copula concepts and graphical tools for studying the dependence phenomena, are presented in Nelsen [12]. In Section 2 we obtain the copula representation of the BALM distributions. In Section 3 we present some properties of the copula itself. To assess better the features of this copula and the possible use of the BALM class of probability distributions, in the next section we list some of the properties of this class.

2. The Copula of the BALM Distributions

The structure of ALM distributions with dependent components is given by properties in Definition 2 and (5)-(6). The possible dependence between X and Y comes through the potential dependence between the components of (V_a,W_b), acting on the rectangle $[0,\ a)\times[0,\ b)$. The next statement gives the corresponding copula representation for the class of BALM probability distributions.

Theorem 3. *Let the random vector (X,Y) have the BALM distribution with SF as in (6), where $\alpha = P(X>a)$ and $\beta = P(Y>b)$ are numbers strictly between 0 and 1. The survival copula corresponding to SF $G_{X,Y}(x,y)$ in (6) is given by*

$$\overline{C}_{X,Y}(u,v) = \alpha^k\beta^m(1-\alpha)\times(1-\beta)\,\overline{C}_{V_a,W_b}\left(\frac{u-\alpha^{k+1}}{\alpha^k-\alpha^{k+1}},\frac{v-\beta^{m+1}}{\beta^m-\beta^{m+1}}\right), \tag{8}$$

valid for all $u,v\in[0,1]$, when they satisfy

$$\alpha^{k+1}<u\le\alpha^k,\qquad \beta^{m+1}<v\le\beta^m,\quad k,m=0,1,2,\ldots. \tag{9}$$

In (8) k and m are integers that satisfy the inequalities (9).

$\overline{C}_{V_a,W_b}(u,v)$ is the survival copula on $(\alpha,1]\times(\beta,1]$ associated to the survival function $G_{V_a,W_b}(v,w)$ whose support is on the rectangle $[0,a)\times[0,b)$.

Proof. The explicit survival function $G_{X,Y}(x,y)$ of $(X,\ Y)$ is written in (6). Supposedly that it is fulfilled

$$ka<x\le(k+1)a,\quad mb<y\le(m+1)b; \tag{10}$$

this SF can be written shorter as

$$\begin{aligned} G_{X,Y}(x,y) = \alpha^k\beta^m\{&(1-\alpha)(1-\beta)G_{V_a,W_b}\\ &\times(x-ka,y-mb)\\ &+(1-\alpha)\beta G_{V_a}(x-ka)\\ &+(1-\beta)\alpha G_{W_b}(y-mb)+\alpha\beta\}. \end{aligned} \tag{11}$$

The corresponding marginal survival functions found by the concordance equations are

$$\begin{aligned} G_X(x) &= P(X>x)=\alpha^{k+1}+\alpha^k(1-\alpha)G_{V_a}(x-ka),\\ G_Y(y) &= P(Y>y)=\beta^{m+1}+\beta^m(1-\beta)G_{W_b}(y-mb). \end{aligned} \tag{12}$$

Both functions belong to the class of ALM marginal distributions. Moreover, it is fulfilled

$$P(X>ka)=\alpha^k,\qquad P(Y>mb)=\beta^m,\qquad k,m=0,1,2,\ldots. \tag{13}$$

Let $C_{X,Y}(u,v)$ be the copula of the pair (X,Y), and let $C_{V_a,W_b}(u,v)$ be the copula of the pair (V_a,W_b), with $u\in[0,1]$ and $v\in[0,1]$. Taking into account Sklar's theorem [12], we get

$$C_{X,Y}(F_X(x),F_Y(y)) = G_{X,Y}(x,y)-G_X(x)-G_Y(y)+1, \tag{14}$$

and also

$$C_{V_a,W_b}\left(F_{V_a}(v),F_{W_b}(w)\right) = G_{V_a,W_b}(v,w)-G_{V_a}(v)-G_{W_b}(w)+1. \tag{15}$$

Relations (9)–(11) show that using

$$u=F_X(x),\qquad v=F_Y(y),\qquad \alpha^k=G_X(ka),\qquad \beta^m=G_Y(mb), \tag{16}$$

we obtain the corresponding copula

$$C_{X,Y}(u,v) = \alpha^k \beta^m \{(1-\alpha)(1-\beta) \times \left[1-u-v+C_{V_a,W_b}\left(u\alpha^k, v\beta^m\right)\right] + (1-\alpha)\beta(1-u) + (1-\beta)\alpha(1-v) + \alpha\beta\}. \quad (17)$$

Here $C_{V_a,W_b}(u,v)$ is the copula associated with the joint distribution of variables (V_a, W_b) whose support is on the rectangle $[0,a] \times [0,b]$. The expression $C_{V_a,W_b}(u\alpha^k, v\beta^m)$ corresponds to $G_{V_a,W_b}(x-ka, y-mb)$. Under the conditions (9) the arguments in this copula on the right hand side of (8) are always between $[0,1]$. After some algebra, the last relation gives

$$C_{X,Y}(u,v) = \alpha^k \beta^m (1-\alpha)(1-\beta) C_{V_a,W_b}(u,v) + \alpha^k (1-\alpha)(1-\beta^m) u + (1-\alpha)\left(1-\alpha^k\right) v + \left(1-\alpha^k\right)(1-\beta^m). \quad (18)$$

Finally, the survival copula $\overline{C}_{X,Y}(u,v)$ producing the survival function $G_{X,Y}(x,y)$ by use of Sklar's theorem like $\overline{C}_{X,Y}(G_X(x), G_Y(y)) = G_{X,Y}(x,y)$ in the statement is obtained by applying $\overline{C}_{X,Y}(u,v) = u + v - 1 + C_{X,Y}(1-u, 1-v)$. Also the result of Theorem 4 (ii) below is taken into account. □

The usefulness of the copula of BALM distributions follows from numerous reasons. We list the following two possible situations.

(1) Components in financial investments can be dependent and subject to the influence of periodic random environment, possibly with different periodicity a, b. Then bivariate ALM models are appropriate in modeling such event. Copula may be used to model dependence between the two components.

(2) In environmental processes as pollution, spread of diseases is two-dimensional process. The growth of dimensions of the infected area may be modeled by appropriate BALM random variable. Dependences on rectangles can be modeled by appropriate copula, and the marginal distributions then will make the transfer of the picture from copula to reality work.

The use of copula is as usually explained in the books (see references in Section 1).

Here especially, when evaluating dependent components inside one rectangle $[0,a) \times [0,b)$ of interaction between X and Y, this dependence is transferred from the properties of the survival copula $\overline{C}_{X,Y}(u,v)$ on the rectangle $(\alpha = F(a), 1] \times (\beta = G_Y(b), 1]$. It will allow to get models of dependencies of periodic nature. This seems important in actuarial and financial practice.

3. Some Properties of the BALM Copula

The properties of the BALM distributions are discussed mainly in Dimitrov et al. [8]. Most important facts are presented in Section 1. It is useful and recommended to the interested reader to review these and to put them in correspondence to the properties of the BALM copula. Main feature here is the fact that the additive LM property is transferred into multiplicative LM property (from the points (ka, mb) to the points (α^k, β^m)). This idea of Galambos and Kotz [16] is used in Dimitrov and von Collani, [15], to transfer the ALM class of univariate distributions into the class of distributions with multiplicative almost-lack-of-memory distributions. The BALM property transfer into bivariate multiplicative almost-lack-of-memory property has not been discussed. It is partially described next.

Here we combine some results about the univariate distributions which possesses the multiplicative-almost-lack-of-memory (MALM) property as discussed by Dimitrov and von Collani [15] and the BALM distributions. The goal is to present some useful properties of the copula of BALM distributions. These can be easily verified by the use of the specific form (8) of the BALM copula.

We use notations α for $G_X(a)$ and β for $G_Y(b)$ just to simplify the text. Denote by (U_1, U_2) a bivariate random vector with survival function as the BALM copula $\overline{C}_{X,Y}(u,v)$ in (8), with some $\alpha \in (0,1)$, $\beta \in (0,1)$, and arbitrary joint survival copula $\overline{C}_{V_a,W_b}(u,v)$ on the rectangle $[\alpha, 1] \times [\beta, 1]$.

Theorem 4. *The random vector (U_1, U_2) satisfies the equations as follows.*

(i) *The marginal distribution of each component (U_1, U_2) is uniform on the interval $[0,1]$, when $\alpha = P(U_1 \le \alpha)$ and $\beta = P(U_2 \le \beta)$.*

(ii) *The pair (U_1, U_2) possesses the bivariate multiplicative lack-of-memory property at the points (α^m, β^n) as shown by*

$$P\left\{\left(U_1 \le u\alpha^k, U_2 \le v\beta^m\right) \mid \left(U_1 \le \alpha^k, U_2 \le \beta^m\right)\right\} = P\{U_1 \le u, U_2 \le v\}, \quad (19)$$

for every integers $k, m = 0, 1, 2, \dots$ and every $u \in (0,1)$ and $v \in (0,1)$; that is, the pair (U_1, U_2) possesses the bivariate multiplicative lack-of-memory property at the points (α^k, β^m). This is the meaning of the bivariate MALM property when α and β are free. (But in the BALM copula they are engaged.)

(iii) *The random vector (U_1, U_2) has the same distribution as the pair $(U_V \alpha^{Z_1}, U_W \beta^{Z_2})$, where (U_V, U_W) is the pair of r.v. with survival copula $\overline{C}_{V_a W_b}(u,v)$, and Z_1, Z_2 are independent geometrically distributed random variables of parameters α and β correspondingly that is, it is true that*

$$(U_1, U_2) =^d \left(U_V \alpha^{Z_1}, U_W \beta^{Z_2}\right), \quad P(Z_1 = k, Z_2 = m) = \alpha^k (1-\alpha) \beta^m (1-\beta). \quad (20)$$

(iv) *On the line segments* $U_1 = \alpha^k$, $k = 1, 2, \ldots$ *and* $U_2 = \beta^m$, $m = 1, 2, \ldots$ *the two components are mutually independent; that is, it is fulfilled*

$$P\left\{U_1 \le a^k, U_2 \le \beta^m\right\} = P\left\{U_1 \le \alpha^k\right\} P\left\{U_2 \le \beta^m\right\}, \quad (21)$$

for arbitrary $k, m = 0, 1, 2, \ldots$.

(v) *Each of the components* U_1 *and* U_2 *of the copula possesses the univariate multiplicative almost-lack-of-memory property*

$$\begin{gathered} P\left(U_1 \le u\alpha^k \mid U_1 \le \alpha^k\right) = P\left(U_1 \le u\right) \\ \forall 0 \le u < 1, \ k = 1, 2, \ldots, \\ P\left(U_2 \le v\beta^m \mid U_2 \le \beta^m\right) = P\left(U_2 \le v\right) \\ \forall 0 \le u < 1, \ m = 1, 2, \ldots. \end{gathered} \quad (22)$$

Proof. The properties listed here follow from the specific choice of the parameters in the copula components and from the form (8) of the copula itself. Statement (ii) is shown as Example 3.1.(a) in Dimitrov and von Collani [15]. (iv) Corresponds to Corollary 2 in Dimitrov et al. [8] and also from (refnodes). The others need to write the conditional probability explicitly according to the rules, and then use the analytic presentations of respective probabilities. The presentation in property (iii) follows from appropriate explicit expansion of the generating functions of the random variables on both sides in (20) to confirm their coincidence. We prove it here.

Consider the generating function on the left hand side in (iii) as follows:

$$\begin{aligned} \varphi_{U_1 \cdot U_2}(s,t) &= \mathbf{E}\left[s^{U_1} t^{U_2}\right] \\ &= \int_0^1 \int_0^1 s^u\, t^v\, dC_{U_1,U_2}(u,v) \\ &= \sum_k \sum_m \alpha^k (1-\alpha) \beta^m (1-\beta) \\ &\quad \times \int_{\alpha^{k+1}}^{\alpha^k} \int_{\beta^m+1}^{\beta^m} s^u t^v\, dC_{V_a,W_b} \\ &\quad \times \left(\frac{u-\alpha^{k+1}}{\alpha^k-\alpha^{k+1}}, \frac{v-\beta^{m+1}}{\beta^m-\beta^{m+1}}\right) \\ &= \sum_k \sum_m \alpha^k (1-\alpha) \beta^m (1-\beta) \\ &\quad \times \int_\alpha^1 \int_\beta^1 s^{\alpha^k u} t^{\beta^m v}\, dC_{V_a,W_b}(u,v) \\ &= \sum_k \sum_m \int_\alpha^1 \int_\beta^1 \alpha^k (1-\alpha) \beta^m (1-\beta) \\ &\quad \times s^{\alpha^k u} t^{\beta^m v} dC_{V_a,W_b}(u,v) \\ &= \sum_k \sum_m P\left(Z_1 = k, Z_2 = m\right) \\ &\quad \times \mathbf{E}\left[s^{\alpha^{Z_1} U_V} t^{\beta^{Z_2} U_W} \mid Z_1 = k, Z_2 = m\right] \\ &= \mathbf{E}\left[s^{\alpha^{Z_1} U_V} t^{\beta^{Z_2} U_W}\right]. \end{aligned} \quad (23)$$

At the end we got the generating function of the right hand side of (iii). □

We omit further details.

Notice that property (iii) is suitable for simulation purposes.

4. The Contorted Marshal-Olkin BALM Distribution and Its Survival Copula

As an example in this section we construct the contorted Marshal-Olkin (MO) BALM distribution, following the ideas in Section 1, and show what its respective survival copula is.

It is known that the bivariate Marshal-Olkin distribution has survival functions

$$S_{\mathrm{MO}}(x,y) = e^{-\lambda x - \mu y - \nu \max(x,y)}, \quad x > 0, \ y > 0. \quad (24)$$

Here $\lambda > 0$, $\mu > 0$, and $\nu > 0$ are parameters of the MO distribution. Its survival copula is presented by the expression

$$\overline{C}_{\mathrm{MO}}(u,v) = uv \min\left(u^{-\nu/(\lambda+\nu)}, v^{-\nu/(\mu+\nu)}\right) \quad u, v \in [0,1]. \quad (25)$$

Define the random vector (V_a, W_b) which has the survival function

$$G_{V_a,W_b}(x,y) = \frac{1}{1 - S_{\mathrm{MO}}(a,b)} \left(S_{\mathrm{MO}}(x,y) - S_{\mathrm{MO}}(a,b)\right), \quad (x,y) \in (0,a] \times (0,b]. \quad (26)$$

This random vector (V_a, W_b) with probability 1 is located on the rectangle $(0,a] \times (0,b]$. Let

$$\alpha = e^{-(\lambda+\nu)a}, \qquad \beta = e^{-(\mu+\nu)b} \quad (27)$$

in (6), where the SF $G_{V_a,W_b}(x,y)$ is the one from (26). The obtained then in (6) survival copula $G_{X,Y}(x,y)$ defines a pair of random variables (X,Y) which has the contorted Marshal-Olkin distribution on the first quadrant of the plane.

The survival copula of the contorted MO distribution is determined by Theorem 3, (8), where the survival copula of the pair (V_a, W_b) is given by the expression

$$\overline{C}_{V_a \cdot W_b}(u,v) = \alpha u \beta v \min\left((\alpha u)^{-\nu/(\lambda+\nu)}, (\beta v)^{-\nu/(\mu+\nu)}\right), \quad u, v \in [0,1]. \quad (28)$$

The construction of the MO contorted BALM distribution at the beginning of this section shows one of the ways a class of known probability distributions with dependences over the entire positive quadrant can be converted into BALM distributions.

5. Conclusions

The copula of BALM distributions can be used in risk modeling when the risk components are dependent, and each component could be a subject to the influence of some specific periodic random environment. Portfolio with multivariate ALM properties is real. Modeling dependent risks with two components is a right step towards multi-component periodic risks studies. For instance, the construction of the MO contorted BALM distribution shows the ways a class of known probability distributions with dependences over the entire positive quadrant can be converted into BALM distributions. ALM distributions in dimensions higher than two are not studied yet. Results presented here may serve well for possible extensions in such direction.

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