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Research Article

CR-Submanifolds of Generalized $f.p.k.$-Space Forms

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We study sectional curvature, Ricci tensor, and scalar curvature of submanifolds of generalized $f.p.k.$-space forms. Then we give an upper bound for foliate ξ_α-horizontal (and vertical) CR-submanifold of a generalized $f.p.k.$-space form and an upper bound for minimal ξ_α-horizontal (and vertical) CR-submanifold of a generalized $f.p.k.$-space form. Finally, we give the same results for special cases of generalized $f.p.k.$-space forms such as S-space forms, generalized Sasakian space forms, Sasakian space forms, Kenmotsu space forms, cosymplectic space forms, and almost $C(\alpha)$-manifolds.

Dedicated to my sister and my parents for their endless support, kind and sacrifices.

1. Introduction

In 1978, Bejancu introduced and studied CR-submanifolds of a Kähler manifold [1, 2]. Since then, many papers appeared on this topic with ambient manifold such as Sasakian space form [3], cosymplectic space form [4], and Kenmotsu space form [5, 6]. Recently Falcitelli and Pastore [7] introduced generalized globally framed f-space forms. Globally framed f-manifolds are studied from the point of view of the curvature and are introduced and the interrelation with generalized Sasakian and generalized complex space forms is pointed out. In this paper, we study CR-submanifolds of generalized f-space forms.

The theory of a submanifold of a Sasaki manifold was investigated from two different points of view: one is the case where submanifolds are tangent to the structure vector and the other is the case where those are normal to the structure vector [8].

In the class of f-structures introduced in 1963 by Yano [9], the so-called f-structures with complemented frames, also called globally framed f-structures or f-structures with parallelizable kernel (briefly $f.p.k.$-structures) [10–13] are particulary interesting. An $f.p.k.$-manifold is a manifold M^{2n+s} on which an f-structure is defined, that is a $(1,1)$-tensor field φ satisfying $\varphi^3+\varphi=0$, of rank $2n$, such that the subbundle $\ker\varphi$ is parallelizable. Then, there exists a global frame $\{\xi_i\}$, $i\in\{1,\dots,s\}$, for the subbundle $\ker\varphi$, with dual 1-form η^i, satisfying $\varphi^2=-I+\eta^i\otimes\xi_i$, $\eta^i(\xi_j)=\delta^i_j$, from which $\varphi\xi_i=0$, $\eta^i\circ\varphi=0$ follow. An $f.p.k.$-structure on a manifold M^{2n+s} is said to be normal if the tensor field $N=[\varphi,\varphi]+2d\eta^i\otimes\xi_i$ vanishes, $[\varphi,\varphi]$ denoting the Nijenhuis torsion of φ. It is known that one can consider a Riemannian metric g on M^{2n+s} associated with an $f.p.k.$-structure (φ,ξ_i,η^i), such that $g(\varphi X,\varphi Y)=g(X,Y)-\sum_{i=1}^{s}\eta^i(X)\eta^i(Y)$, for any $X,Y\in\Gamma(T^{2n+s}M)$, and the structure (φ,ξ_i,η^i,g) is then called a metric $f.p.k.$-structure. Therefore, $T^{2n+s}M$ splits as complementary orthogonal sum of its subbundles $\operatorname{Im}\varphi$ and $\ker\varphi$. We denote their respective differentiable distributions by D and $D^\perp$.

Let Ω denote the 2-form on M^{2n+s} defined by $\Omega(X,Y)=g(X,\varphi Y)$, for any $X,Y\in\Gamma(T^{2n+s}M)$.

Several subclasses have been studied from different points of view [10, 11, 14–16], also dropping the normality condition, and, in this case, the term almost precedes the name of the considered structures or manifolds. As in [10], a metric

$f.p.k.$-structure is said a $\mathcal{K}$-structure if it is normal and the fundamental 2-form Ω is closed; a manifold with a $\mathcal{K}$-structure is called a $\mathcal{K}$-manifold. In particular, if $d\eta^i = \Omega$, for all $i \in \{1, \ldots, s\}$, the $\mathcal{K}$-structure is said to be an $\mathcal{S}$-structure and M^{2n+s} an $\mathcal{S}$-manifold. Finally, if $d\eta^i = 0$ for all $i \in \{1, \ldots, s\}$, then the $\mathcal{K}$-structure is called a $\mathcal{C}$-structure and M^{2n+s} is said to be a $\mathcal{C}$-manifold. Obviously, if $s = 1$, a $\mathcal{K}$-manifold M^{2n+1} is a quasi Sasakian manifold, a $\mathcal{C}$-manifold is a cosymplectic manifold, and an $\mathcal{S}$-manifold is a Sasakian manifold.

The purpose of the present paper is to study Ricci tensor, sectional curvature, and scalar curvature of submanifolds of a generalized $f.p.k.$-space form. In Section 2, we state definitions of $f.p.k.$-space form, its curvature tensor, ξ_α-horizontal CR-submanifold, and ξ_α-vertical CR-submanifold. Section 3 is devoted to the study sectional curvature of submanifold of an $f.p.k.$-space form. Finally, in Section 4, we investigate Ricci tensor and scalar curvature of submanifold of an $f.p.k.$-space form and obtain upper bound for scalar curvature.

2. Preliminaries

We recall that the Levi-Civita connection ∇ of a metric $f.p.k.$-manifold satisfies the following formula [10, 11]:

$$\begin{aligned} 2g((\nabla_X \varphi) Y, Z) &= 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) \\ &\quad + g(N(Y, Z), \varphi X) + N_j^{(2)}(Y, Z)\eta^j X \\ &\quad + d\eta^j(\varphi Y, X)\eta^j(Z) - 2d\eta^j(\varphi Z, X)\eta^j(Y), \end{aligned} \tag{1}$$

where $N_j^{(2)}$ is given by $N_j^{(2)}(X, Y) = 2d\eta^j(\varphi X, Y) - 2d\eta^j(\varphi Y, X)$.

Furthermore, for $\mathcal{S}$-manifolds we have $\nabla_X \xi_j = -\varphi X$, $j = 1, \ldots, s$, [10]. Putting $\overline{\xi} = \sum_{j=1}^s \xi_j$, $\overline{\eta} = \sum_{j=1}^s \eta_j$ is its dual form with respect to g and

$$(\nabla_X \varphi) Y = g(\varphi X, \varphi Y)\overline{\xi} + \overline{\eta}(Y)\varphi^2 X. \tag{2}$$

We remark that (2) together with $\pounds_{\xi_i} g = 0$ and $\pounds_{\xi_i} \eta^j = 0$, $i, j \in \{1, \ldots, s\}$, characterizes the $\mathcal{S}$-manifolds among the metric $f.p.k.$-manifolds.

A metric $f.p.k.$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ has pointwise constant (p.c.) φ-sectional curvature if at any $p \in M^{2n+s}$, $c(p) = R_p(X, \varphi X, X, \varphi X)$ does not depend on the φ-section spanned by $\{X, \varphi X\}$, for any unit $X \in D_p$. Several results involving the pointwise constancy of the φ-sectional curvatures of an almost contact metric manifold (i.e., for $s = 1$) are recently obtained in [17–19]. We refer to [20] for a systematic exposition of the classical curvature results on contact metric manifolds.

We recall some known results.

Proposition 1 (see [6]). *A Sasaki manifold $(M^{2n+s}, \varphi, \xi, \eta, g)$ has p.c. φ-sectional curvature c if and only if its curvature tensor field verifies*

$$\begin{aligned} R(X, Y, Z) &= \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + \frac{1}{4}(c-1)\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\ &\qquad + 2g(X, \varphi Y)\varphi Z + \eta(X)\eta(Z)Y \\ &\qquad - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &\qquad - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{3}$$

for any X, Y, Z tangent to M^{2n+1}.

A Sasaki manifold M^{2n+1} with constant φ-sectional curvature $c \in \mathbb{R}$ is called a Sasakian space form and denoted by $M^{2n+1}(c)$. It is well known that, if $n \geq 2$, a Sasaki manifold M^{2n+1} with p.c. φ-sectional curvature c is a Sasakian space form. As examples of Sasakian space forms, we mention $\mathbb{R}^{2n+1}$ and S^{2n+1}, with standard Sasakian structures [14].

Definition 2 (see [10]). An almost contact metric manifold $(M^{2n+s}, \varphi, \xi, \eta, g)$ is a generalized Sasakian space form, denoted by $(M^{2n+s}, f_1, f_2, f_3)$, if it admits three smooth functions f_1, f_2, f_3 such that its curvature tensor field verifies that, for any $X, Y, Z \in TM$

$$\begin{aligned} R(X, Y, Z) &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\ &\qquad + 2g(X, \varphi Y)\varphi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\qquad + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{4}$$

Remark 3. Any generalized Sasakian space form has p.c. φ-sectional curvature $c = f_1 + 3f_2$. Obviously, a Sasaki manifold of p.c. φ-sectional curvature c satisfies (4) with $f_1 = (1/4)(c+3)$ and $f_2 = f_3 = (1/4)(c-1)$. A cosymplectic manifold with p.c. φ-sectional curvature c satisfies (4) with $f_1 = f_2 = f_3 = (1/4)c$.

Proposition 4 (see [10, 21]). *An S-manifold M^{2n+s} has p.c. φ-sectional curvature c if and only if its curvature tensor field verifies*

$$
\begin{aligned}
R&(X,Y,Z)\\
&=\frac{1}{4}(c+3s)\left\{g(\varphi X,\varphi Z)\varphi^2 Y\right.\\
&\qquad\left.-g(\varphi Y,\varphi Z)\varphi^2 X\right\}\\
&+\frac{1}{4}(c-s)\left\{g(Z,\varphi Y)\varphi X-g(Z,\varphi X)\varphi Y\right.\\
&\qquad\left.+2g(X,\varphi Y)\varphi Z\right\}\\
&+\left\{\overline{\eta}(X)\overline{\eta}(Z)\varphi^2 Y-\overline{\eta}(Y)\overline{\eta}(Z)\varphi^2 X\right.\\
&\quad\left.+g(\varphi Y,\varphi Z)\overline{\eta}(Y)\overline{\xi}-g(\varphi X,\varphi Z)\overline{\eta}(X)\overline{\xi}\right\}.
\end{aligned}
\tag{5}
$$

for any X, Y, Z tangent to M^{2n+1}.

An S-manifold M^{2n+s} with constant φ-sectional curvature $c \in \mathbb{R}$ is called an S-space-form and denoted by $M^{2n+s}(c)$. Moreover, it is also well known that if $n \geq 2$, then an S-manifold with p.c. φ-sectional curvature c is an S-space form. We remark that for $s = 1$ (5) reduces to (3).

Definition 5. In [22], Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold M is called trans-Sasakian manifold if there exist two functions α and β on M such that [22–24]

$$
\begin{aligned}
(\nabla_X\varphi)(Y)&=\alpha\{g(X,Y)\xi-\eta(Y)X\}\\
&\quad+\beta\{g(\varphi X,Y)\xi-\eta(Y)\varphi X\},
\end{aligned}
\tag{6}
$$

for vector fields X, Y on M. From (6) it is easy to see that

$$\nabla_X\xi=-\alpha\varphi X+\beta(X-\eta(X))\xi. \tag{7}$$

In particular, if $\beta = 0$, then M is said to be an α-Sasakian manifold. Sasakian manifolds appear as examples of α-Sasakian manifolds with $\alpha = 1$.

On the other hand, if $\alpha = 0$, then M is said to be a β-Kenmotsu manifold. Kenmotsu manifolds, defined in [25], are particular examples with $\beta = 1$.

Another important kind of trans-Sasakian manifolds is that of cosymplectic manifolds obtained for $\alpha = \beta = 0$.

Proposition 6 (see [25]). *An almost contact metric manifold is said to be an almost $C(\alpha)$-manifold if its Riemannian curvature tensor verifies*

$$
\begin{aligned}
R(X,Y,Z,W)&=R(X,Y,\varphi Z,\varphi W)\\
&\quad+\alpha\{g(X,W)g(Y,Z)\\
&\qquad-g(X,Z)g(Y,W)\\
&\qquad+g(X,\varphi Z)g(Y,\varphi W)\\
&\qquad-g(X,\varphi W)g(Y,\varphi Z)\},
\end{aligned}
\tag{8}
$$

for vector fields X, Y, Z, and W on M, where α is a real number. Moreover, if such a manifold has constant φ-sectional curvature equal to c, then its curvature tensor is given by

$$
\begin{aligned}
R&(X,Y)Z\\
&=\frac{1}{4}\left(c+3\alpha^2\right)\{g(Y,Z)X-g(X,Z)Y\}\\
&\quad+\frac{1}{4}\left(c-\alpha^2\right)\{g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X\\
&\qquad+2g(X,\varphi Y)\varphi Z\\
&\qquad+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\
&\qquad+g(X,Z)\eta(Y)\xi\\
&\qquad-g(Y,Z)\eta(X)\xi\},
\end{aligned}
\tag{9}
$$

and so, it is a generalized Sasakian space form with $f_1 = (1/4)(c+3\alpha^2)$ and $f_2 = f_3 = (1/4)(c-\alpha^2)$.

Let $\mathcal{F}$ denote any set of smooth function F_{ij} on M^{2n+s} such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, 2, \ldots, s\}$.

Definition 7 (see [7]). A generalized $f.p.k.$-space form, denoted by $M^{2n+s}(F_1, F_2, \mathcal{F})$, is a metric $f.p.k.$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ which admits smooth functions F_1, F_2, and $\mathcal{F}$ such that its curvature tensor field verifies

$$
\begin{aligned}
R&(X,Y,Z)\\
&=F_1\left\{g(\varphi X,\varphi Z)\varphi^2 Y-g(\varphi Y,\varphi Z)\varphi^2 X\right\}\\
&\quad+F_2\{g(Z,\varphi Y)\varphi X-g(Z,\varphi X)\varphi Y\\
&\qquad+2g(X,\varphi Y)\varphi Z\}\\
&\quad+\sum_{i,j=1}^{s}F_{ij}\left\{\eta^i(X)\eta^j(Z)\varphi^2 Y\right.\\
&\qquad-\eta^i(Y)\eta^j(Z)\varphi^2 X\\
&\qquad+\eta^i(X)\xi_j g(\varphi Y,\varphi Z)\\
&\qquad\left.-\eta^i(Y)\xi_j g(\varphi X,\varphi Z)\right\}.
\end{aligned}
\tag{10}
$$

For $s = 1$, we obtain a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1$, $f_2 = F_2$ and $f_3 = F_1 - F_{11}$. In particular, if the given structure is either Sasakian, Kenmotsu, or possibly cosymplectic, then (10) holds with $F_{11} = 1$, $F_1 = (1/4)(c+3)$, $F_2 = (1/4)(c-1)$, and $f_3 = F_1 - F_{11} = (1/4)(c-1) = f_2$ in the first case, $F_{11} = -1$, $F_1 = (1/4)(c-3)$, $F_2 = (1/4)(c+1)$ and $f_3 = F_1 - F_{11} = (1/4)(c+1) = f_2$ in the second case, and $F_{11} = 0$, $F_1 = (1/4)c$, $F_2 = (1/4)c$, and $f_3 = (1/4)c$ in the last case.

Definition 8. Let M be an m-dimensional submanifold immersed in $\overline{M}$. M is said to be an invariant submanifold if $\xi_\alpha \in TM$ for any $1 \leq \alpha \leq s$ and $\varphi X \in TM$ for any $X \in TM$. On the other hand, it is said to be an anti-invariant submanifold if $\varphi X \in T^\perp M$ for any $X \in TM$.

An m-dimensional Riemannian submanifold M of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}$ is called a CR-submanifold if ξ_α's are tangent to M (so, $\dim M \geq s$) and there exist two differentiable distributions D and $D^\perp$ on M satisfying

(i) $TM = D \oplus D^\perp$ (direct sum),

(ii) the distribution D is invariant under φ, that is, $\varphi D_x = D_x$ for any $x \in M$,

(iii) the distribution $D^\perp$ is anti-invariant under φ, that is, $\varphi D_x^\perp \subseteq T_x^\perp M$ for any $x \in M$.

We denote by $2p$ and q the real dimensions of D_x and $D_x^\perp$, respectively, for any $x \in M$. Then, if $p = 0$, we have an anti-invariant submanifold tangent to $\xi_1, \ldots, \xi_s$, and if $q = 0$, we have an invariant submanifold.

As an example, it is easy to prove that each hypersurface of $\overline{M}$ which is tangent to $\xi_1, \ldots, \xi_s$ inherits the structure of CR-submanifold of $\overline{M}$. Also, pseudoumbilical, totally contact umbilical, totally contact geodesic, totally umbilical, and totally geodesic hypersurfaces of a generalized S-space form are also generalized S-space forms, and, moreover, the bundle space of a principal toroidal bundle over a Kählerian manifold and the warped product of R times a generalized S-space form are generalized S-space forms, too [26].

Definition 9. The φ-sectional curvature H of M determined by a unit vector $X \in D$ orthogonal to ξ_α's is the sectional curvature of the plane section spanned by X and φX. Also, we denote by $\mathrm{Ric}(X, Y)$ (and $K(X, Y)$) Ricci tensor (and sectional curvature) determined by (orthonormal) vector fields $\{X, Y\}$, respectively.

Definition 10. A CR-submanifold M of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}$ is said to be D-totally geodesic (resp., $D^\perp$-totally geodesic) if $h(X, Y) = 0$ for any $X, Y \in D$ (resp., $X, Y \in D^\perp$), and it is said to be $(D, D^\perp)$-mixed totally geodesic if $h(X, Y) = 0$ for any $X \in D$, $Y \in D^\perp$.

Also, CR-submanifold M is said to be minimal if $\mu = 0$, where μ is the mean curvature vector, defined by $\mu = (1/(2n+s))\mathrm{trace}(h)$.

Definition 11. Let M be a CR-submanifold with horizontal distribution D and vertical distribution $D^\perp$. The pair $(D, D^\perp)$ is called ξ_α-horizontal if $\xi_\alpha^x \in D_x$ for any $x \in M$, and in a similar way the pair $(D, D^\perp)$ is called ξ_α-vertical if $\xi_\alpha^x \in D_x^\perp$ for any $x \in M$.

Definition 12. Let $\dim M = n = 2p + s + q$ and $\{e_1, e_2, \ldots, e_{2p}, \xi_1, \xi_2, \ldots, \xi_s, e_1', e_2', \ldots, e_q'\}$ be a local field of orthonormal frames on TM such that in case when M is ξ_α-horizontal, $\{e_1, e_2, \ldots, e_{2p}, \xi_1, \xi_2, \ldots, \xi_s\}$ is a local frame field on D and $\{e_1', e_2', \ldots, e_q'\}$ is a local frame field on $D^\perp$.

Let M be an ξ_α-horizontal CR-submanifold of $\overline{M}$. The mean curvature vector field μ of M in $\overline{M}$ is defined by

$$\mu = \frac{1}{2p+s+q}\left\{\sum_{i=1}^{2p+s} h(e_i, e_i) + \sum_{i=1}^{q} h(e_i', e_i')\right\}. \tag{11}$$

If $\mu = 0$, then M is said to be minimal. Now, we will define

$$\mu_D = \frac{1}{2p+s}\sum_{i=1}^{2p+s} h(e_i, e_i),$$
$$\mu_{D^\perp} = \frac{1}{q}\sum_{i=1}^{q} h(e_i', e_i'). \tag{12}$$

If $\mu_D = 0$, then the CR-submanifold M is said to be D-minimal, and if $\mu_{D^\perp} = 0$, then it is said to be $D^\perp$-minimal. Similar definitions can be given for ξ_α-vertical CR-submanifolds.

We denote by P and Q the projection morphisms of TM on D and $D^\perp$, respectively. We call D (resp., $D^\perp$) the horizontal (resp., vertical) distribution. Then for any vector field X tangent to M, we have:

$$X = PX + QX, \tag{13}$$

where PX and QX belong to the distribution D (horizontal part) and $D^\perp$ (vertical part), respectively. Also, for a vector field N normal to M, we put:

$$\varphi N = tN + fN, \tag{14}$$

where tN and fN denote the horizontal and normal component of φN, respectively.

Definition 13. Let M be a CR-submanifold of an ambient manifold $\overline{M}$, with horizontal distribution D. Then D is called involutive (or integrable) if $[X, Y] \in D$ for any $X, Y \in D$ where $[X, Y]$ is Lie bracket of X, Y. Also, M is a foliate if D is involutive (or integrable).

Let M be an m-dimensional submanifold immersed in a generalized $f.p.k.$-space form $\overline{M}^{2n+s}$. The Gauss-Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad X, Y \in TM,$$
$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N; \quad X \in TM, N \in T^\perp M, \tag{15}$$

where $\nabla^\perp$ is the connection in the normal bundle, h is the second fundamental form of M and A_N the Weingarten endomorphism associated with N. Then A_N and h are related by:

$$g(A_N X, Y) = g(h(X, Y), N). \tag{16}$$

We denote by $\overline{R}$ and R the curvature tensor fields associated with $\overline{\nabla}$ and ∇, respectively. The Gauss equation is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \tag{17}$$

where $X, Y, Z,$ and W belong to TM.

3. Sectional Curvature of Submanifolds

Let M be a submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$. Then from the equation of Gauss, we have

$$\begin{aligned}
&R(X,Y,Z,W) \\
&= \overline{R}(X,Y,Z,W) - g(h(X,Z), h(Y,W)) \\
&\quad + g(h(X,W), h(Y,Z)) \\
&= F_1\{g(\varphi X, \varphi Z)\, g(\varphi^2 Y, W) \\
&\qquad - g(\varphi Y, \varphi Z)\, g(\varphi^2 X, W)\} \\
&\quad + F_2\{g(Z, \varphi Y)\, g(\varphi X, W) \\
&\qquad - g(Z, \varphi X)\, g(\varphi Y, W) \\
&\qquad + 2g(X, \varphi Y)\, g(\varphi Z, W)\} \\
&\quad + \sum_{i,j=1}^{s} F_{ij}\{\eta^i(X)\eta^j(Z)\, g(\varphi^2 Y, W) \\
&\qquad - \eta^i(Y)\eta^j(Z)\, g(\varphi^2 X, W) \\
&\qquad + \eta^i(X)\eta^j(W)\, g(\varphi Y, \varphi Z) \\
&\qquad - \eta^i(Y)\eta^j(W)\, g(\varphi X, \varphi Z)\} \\
&\quad - g(h(X,Z), h(Y,W)) \\
&\quad + g(h(X,W), h(Y,Z)),
\end{aligned} \tag{18}$$

for any $X, Y, Z,$ and W tangent to M.

Let $K_M(X,Y)$ be the sectional curvature determined by orthonormal vectors X and Y. Then from (18), we have

$$\begin{aligned}
&K_M(X,Y) \\
&= g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + 3F_2 g^2(X, \varphi Y) \\
&\quad + F_1\left\{\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad \left. - \left(\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right)^2\right\} \\
&\quad + \sum_{i,j=1}^{s} F_{ij}\left\{\eta^i(X)\eta^j(X)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad + \eta^i(Y)\eta^j(Y)\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right) \\
&\qquad \left. + 2\eta^i(X)\eta^j(Y)\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right\}.
\end{aligned} \tag{19}$$

Thus we have the following theorem.

Theorem 14. *Let M be a submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$. Then the sectional curvature of M determined by orthonormal tangent vectors $\{X, Y\}$ is given by*

$$\begin{aligned}
&K_M(X,Y) \\
&= g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + 3F_2 g^2(X, \varphi Y) \\
&\quad + F_1\left\{\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad \left. - \left(\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right)^2\right\} \\
&\quad + \sum_{i,j=1}^{s} F_{ij}\left\{\eta^i(X)\eta^j(X)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad + \eta^i(Y)\eta^j(Y)\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right) \\
&\qquad \left. + 2\eta^i(X)\eta^j(Y)\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right\}.
\end{aligned} \tag{20}$$

From this we have the following corollaries for the sectional curvature of submanifold determined by orthonormal tangent vectors $\{X, Y\}$.

Corollary 15. *The sectional curvature of a submanifold of an S-space form $\overline{M}^{2n+s}(c)$ is given by*

$$\begin{aligned}
&K_M(X,Y) \\
&= g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 \\
&\quad + \frac{3}{4}(c-s)\, g^2(X, \varphi Y) + \frac{1}{4}(c+3s) \\
&\quad \times \left\{\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad \left. - \left(\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right)^2\right\} \\
&\quad + \sum_{i,j=1}^{s}\left\{\eta^i(X)\eta^j(X)\left(1 - \sum_{k=1}^{s}\eta^k(Y)^2\right) \right. \\
&\qquad + \eta^i(Y)\eta^j(Y)\left(1 - \sum_{k=1}^{s}\eta^k(X)^2\right) \\
&\qquad \left. + 2\eta^i(X)\eta^j(Y)\sum_{k=1}^{s}\eta^k(X)\eta^k(Y)\right\}.
\end{aligned} \tag{21}$$

Proof. We will get the result by using $F_{ij} = 1$, for all $1 \le i, j \le s$; $F_1 = (1/4)(c + 3s)$; $F_2 = (1/4)(c - s)$ in (20). □

Corollary 16. *The sectional curvature of a submanifold of a generalized Sasakian space form $\overline{M}(c)$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + f_1 + 3f_2 g^2(X,\varphi Y) - f_3\left(\eta^2(X) + \eta^2(Y)\right). \quad (22)$$

Proof. We will get the result by using $s = 1$, $F_1 = f_1$, $F_2 = f_2$, and $F_{11} = f_1 - f_3$ in (20). □

Corollary 17. *The sectional curvature of a submanifold of a Sasakian space form $\overline{M}(c)$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c+3) + \frac{1}{4}(c-1) \times \left(3g^2(X,\varphi Y) - \eta^2(X) - \eta^2(Y)\right). \quad (23)$$

Proof. We get the result by using $f_1 = (1/4)(c + 3)$, $f_2 = f_3 = (1/4)(c - 1)$ in (22). □

Corollary 18. *The sectional curvature of a submanifold of a Kenmotsu space form $\overline{M}(c)$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c-3) + \frac{1}{4}(c+1) \times \left(3g^2(X,\varphi Y) - \eta^2(X) - \eta^2(Y)\right). \quad (24)$$

Proof. We get the result by using $f_1 = (1/4)(c - 3)$ and $f_2 = f_3 = (1/4)(c + 1)$ in (22). □

Corollary 19. *The sectional curvature of a submanifold of a cosymplectic space form $\overline{M}(c)$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}c\left(1 + 3g^2(X,\varphi Y) - \eta^2(X) - \eta^2(Y)\right). \quad (25)$$

Proof. By taking $f_1 = f_2 = f_3 = (1/4)c$ in (22), we obtain the above. □

Corollary 20. *The sectional curvature of a submanifold of an almost $C(\alpha)$-manifold $\overline{M}(c)$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}\left(c + 3\alpha^2\right) + \frac{1}{4}\left(c - \alpha^2\right)\left(3g^2(X,\varphi Y) - \eta^2(X) - \eta^2(Y)\right). \quad (26)$$

Proof. By getting $f_1 = (1/4)(c + 3\alpha^2)$, $f_2 = f_3 = (1/4)(c - \alpha^2)$ in (22), we obtain (26). □

Proposition 21. *If M is a ξ_α-horizontal CR-submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + F_1. \quad (27)$$

Proof. From (20) and by replacing $\eta^k(X) = 0 = \eta^k(Y)$; $1 \le k \le s$ and $g(X,\varphi Y) = 0$ we get the result immediately. □

Corollary 22. *If M is a ξ_α-horizontal CR-submanifold of an S-space form $\overline{M}^{2n+s}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c+3s). \quad (28)$$

Corollary 23. *If M is a ξ-horizontal CR-submanifold of a generalized Sasakian space form $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + f_1. \quad (29)$$

Corollary 24. *If M is a ξ-horizontal CR-submanifold of a Sasakian space form $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c+3). \quad (30)$$

Corollary 25. *If M is a ξ-horizontal CR-submanifold of a Kenmotsu space form $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c-3). \quad (31)$$

Corollary 26. *If M is a ξ-horizontal CR-submanifold of a cosymplectic space form $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}c. \quad (32)$$

Corollary 27. *If M is a ξ-horizontal CR-submanifold of a $C(\alpha)$-manifold $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D^\perp$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}\left(c + 3\alpha^2\right). \quad (33)$$

Proposition 28. *If M is a ξ_α-vertical CR-submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}$, then the sectional curvature of M determined by $X, Y \in D$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + F_1 + 3F_2 g^2(X,\varphi Y). \tag{34}$$

Corollary 29. *If M is a ξ_α-vertical CR-submanifold of an S-space form $\overline{M}^{2n+s}(c)$, then the sectional curvature of M determined by $X, Y \in D$ is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c+3s) + \frac{3}{4}(c-s) g^2(X,\varphi Y). \tag{35}$$

Corollary 30. *If M is a ξ-vertical CR-submanifold of space form $\overline{M}(c)$, then the sectional curvature of M determined by $X, Y \in D$,*

(i) *where $\overline{M}(c)$ is a generalized Sasakian space form, is given by:*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + f_1 + 3f_2 g^2(X,\varphi Y), \tag{36}$$

(ii) *where $\overline{M}(c)$ is a Sasakian space form, is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c+3) + \frac{3}{4}(c-1) g^2(X,\varphi Y), \tag{37}$$

(iii) *where $\overline{M}(c)$ is a Kenmotsu space form, is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}(c-3) + \frac{3}{4}(c+1) g^2(X,\varphi Y), \tag{38}$$

(iv) *where $\overline{M}(c)$ is a cosymplectic space form, is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}c\left(1 + 3g^2(X,\varphi Y)\right), \tag{39}$$

(v) *where $\overline{M}(c)$ is a $C(\alpha)$-manifold, is given by*

$$K_M(X,Y) = g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 + \frac{1}{4}\left(c+3\alpha^2\right) + \frac{3}{4}\left(c-\alpha^2\right) g^2(X,\varphi Y). \tag{40}$$

Proposition 31. *The φ-sectional curvature of a CR-submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$, determined by $X \in TM$, is given by*

$$H(X) = g\left(h(X,X), h(\varphi X,\varphi X)\right) - \left\|h(X,\varphi X)\right\|^2 + F_1 + 3F_2. \tag{41}$$

Proof. By using $\eta^k(X) = 0$, for all $1 \le k \le s$ in (20), we will get the result. □

Proposition 32. *The φ-sectional curvature of a CR-submanifold of a generalized Sasakian space form $\overline{M}(c)$, determined by $X \in TM$ is given by*

$$H(X) = g\left(h(X,X), h(\varphi X,\varphi X)\right) - \left\|h(X,\varphi X)\right\|^2 + f_1 + 3f_2. \tag{42}$$

Corollary 33. *The φ-sectional curvature of a CR-submanifold of either an S-space form, a Sasakian space form, a Kenmotsu space form, a cosymplectic space form, or an almost $C(\alpha)$-manifold $\overline{M}^{2n+s}$, determined by $X \in TM$, is given by:*

$$H(X) = g\left(h(X,X), h(\varphi X,\varphi X)\right) - \left\|h(X,\varphi X)\right\|^2 + c. \tag{43}$$

We recall the following Lemma [27].

Lemma 34. *Let M be a foliate ξ_α-horizontal CR-submanifold of a S-space form $\overline{M}^{2n+s}(c)$; then*

$$h(\varphi X,\varphi Y) = -h(X,Y); \quad X, Y \in D. \tag{44}$$

Proposition 35. *If M is a foliate ξ_α-horizontal CR-submanifold of a S-space form $\overline{M}^{2n+s}(c)$; then*

$$H(X) \le c; \quad X \in D, \tag{45}$$

and the equality holds if and only if M is D-totally geodesic.

Corollary 36. *If M is a foliate ξ-horizontal CR-submanifold of a generalized Sasakian space form $\overline{M}(c)$, then*

$$H(X) \le f_1 + 3f_2; \quad X \in D, \tag{46}$$

and the equality holds if and only if M is D-totally geodesic.

Corollary 37. *If M is a foliate ξ-horizontal CR-submanifold of either a Sasakian space form, a Kenmotsu space form, a cosymplectic space form, or an almost $C(\alpha)$-manifold form $\overline{M}(c)$, then*

$$H(X) \le c; \quad X \in D, \tag{47}$$

and the equality holds if and only if M is D-totally geodesic.

Proposition 38. *If M be a $D^\perp$-minimal ξ_α-horizontal CR-submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$; then M is $D^\perp$-totally geodesic iff*

$$K_M(X,Y) = F_1, \tag{48}$$

for any $X, Y \in D^\perp$.

Proof. Let M is $D^\perp$-minimal ξ_α-horizontal CR-submanifold of generalized f.p.k.-space form $\overline{M}^{2n+s}$, then by definition of $D^\perp$-minimal, we have:

$$\sum_{i=1}^{q} h\left(e_i', e_i'\right) = 0, \tag{49}$$

where $\{e_1', e_2', \dots, e_q'\}$ is a local frame field on $D^\perp$. Therefore, for any $X \in D^\perp$

$$h(X, X) = 0. \tag{50}$$

On the other hand, from (20), we have for $X, Y \in D^\perp$

$$K_M(X, Y) = g(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 + F_1. \tag{51}$$

Hence, M is $D^\perp$-totally geodesic if and only if for any $X, Y \in D^\perp$

$$K_M(X, Y) = F_1. \tag{52}$$

□

Proposition 39. *If M is a D-minimal ξ_α-vertical CR-submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$, then M is D-totally geodesic if and only if*

$$K_M(X, Y) = F_1, \tag{53}$$

for any $X, Y \in D$ with $g(X, \varphi Y) = 0$.

Proposition 40. *If M is a ξ_α-horizontal CR-submanifold and $(D, D^\perp)$-mixed totally geodesic of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$, then*

$$\begin{aligned} K_M(X, Y) = {} & g(h(X, X), h(Y, Y)) \\ & + F_1\left(1 - \sum_{k=1}^{s} \eta^k(X)^2\right) \\ & + \sum_{i,j=1}^{s} F_{ij}\eta^i(X)\eta^j(X), \end{aligned} \tag{54}$$

for any $X \in D$ and $Y \in D^\perp$.

Proof. By using $\eta^k(Y) = 0$, for all $1 \le k \le s$ and $g(X, \varphi X) = 0$ and $h(X, Y) = 0$, we arrive at the aforementioned equation, easily. □

Proposition 41. *If M is a ξ_α-vertical CR-submanifold and $(D, D^\perp)$-mixed totally geodesic of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$, then*

$$\begin{aligned} K_M(X, Y) = {} & g(h(X, X), h(Y, Y)) \\ & + F_1\left(1 - \sum_{k=1}^{s} \eta^k(Y)^2\right) \\ & + \sum_{i,j=1}^{s} F_{ij}\eta^i(Y)\eta^j(Y), \end{aligned} \tag{55}$$

for any $X \in D$ and $Y \in D^\perp$.

Proof. By using $\eta^k(X) = 0$, for all $1 \le k \le s$, $g(X, \varphi X) = 0$, $h(X, Y) = 0$ and (20), we arrive at the abovementioned equation, easily. □

4. The Ricci Tensor and Scalar Curvature of a Submanifold

Let M be a submanifold of a generalized f.p.k.-space form $\overline{M}^{2n+s}(F_1, F_2, \mathcal{F})$. Then it is straightforward to calculate the Ricci tensor of M as follows

$$\begin{aligned} \operatorname{Ric}(X, Y) = {} & \sum_{k=1}^{n} g(R(e_k, X)Y, e_k) \\ = {} & \sum_{k=1}^{n} F_1\{g(\varphi e_k, \varphi Y)g(\varphi^2 X, e_k) \\ & - g(\varphi X, \varphi Y)g(\varphi^2 e_k, e_k)\} \\ & + \sum_{k=1}^{n} F_2\{g(Y, \varphi X)g(\varphi e_k, e_k) \\ & - g(Y, \varphi e_k)g(\varphi X, e_k) \\ & + 2g(e_k, \varphi X)g(\varphi Y, e_k)\} \\ & + \sum_{k=1}^{n}\sum_{i,j=1}^{s} F_{ij}\{\eta^i(e_k)\eta^j(Y)g(\varphi^2 X, e_k) \\ & - \eta^i(X)\eta^j(Y)g(\varphi^2 X, e_k) \\ & + \eta^i(e_k)\eta^j(e_k)g(\varphi X, \varphi Y) \\ & - \eta^i(X)\eta^j(e_k)g(\varphi e_k, \varphi Y)\} \\ & - g(h(e_k, Y), h(X, e_k)) \\ & + g(h(e_k, e_k), h(X, Y)) \\ = {} & F_1\Bigg\{ - g(\varphi X, \varphi Y) - g(\varphi X, \varphi Y) \\ & \times \sum_{k=1}^{n}\left(-1 + \sum_{\alpha=1}^{s}(\eta^\alpha(e_k))^2\right)\Bigg\} \\ & + 3F_2\left\{g(X, Y) - \sum_{\alpha=1}^{s}\eta^\alpha(X)\eta^\alpha(Y)\right\} \\ & + \sum_{i,j=1}^{s} F_{ij}\Bigg\{-\sum_{k=1}^{n}\eta^i(e_k)\eta^j(Y)g(\varphi X, \varphi e_k) \\ & + \eta^i(X)\eta^j(Y)\sum_{k=1}^{n} g(\varphi e_k, \varphi e_k) \\ & + g(\varphi X, \varphi Y)\sum_{k=1}^{n}\eta^i(e_k)\eta^j(e_k) \\ & - \eta^i(X)\sum_{k=1}^{n}\eta^j(e_k)g(\varphi Y, \varphi e_k)\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\left[g\left(h\left(e_k,e_k\right),h\left(X,X\right)\right)\right.\\
&\qquad\left.-g\left(h\left(X,e_k\right),h\left(Y,e_k\right)\right)\right]\\
&=F_1\left(n-s-1\right)\\
&\quad\times\left(g\left(X,Y\right)-\sum_{\alpha=1}^{s}\eta^{\alpha}\left(X\right)\eta^{\alpha}\left(Y\right)\right)\\
&\quad+3F_2\left(g\left(X,Y\right)-\sum_{\alpha=1}^{s}\eta^{\alpha}\left(X\right)\eta^{\alpha}\left(Y\right)\right)\\
&\quad+\sum_{i,j=1}^{s}F_{ij}\left(n-s\right)\eta^{i}\left(X\right)\eta^{j}\left(Y\right)\\
&\quad+\left(\sum_{i=1}^{s}F_{ii}\right)\left(g\left(X,Y\right)-\sum_{\alpha=1}^{s}\eta^{\alpha}\left(X\right)\eta^{\alpha}\left(Y\right)\right)\\
&\quad+\sum_{k=1}^{n}\left[g\left(h\left(X,X\right),h\left(e_k,e_k\right)\right)\right.\\
&\qquad\left.-g\left(h\left(X,e_k\right),h\left(Y,e_k\right)\right)\right].
\end{aligned}
\tag{56}
$$

Also, the scalar curvature ρ of a submanifold M of $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$ is then given by

$$
\begin{aligned}
\rho&=\sum_{t=1}^{n}\operatorname{Ric}\left(e_t,e_t\right)=\left(F_1\left(n-s-1\right)+3F_2\right)\\
&\quad\times\left(\sum_{t=1}^{n}g\left(e_t,e_t\right)-\sum_{t=1}^{n}\sum_{\alpha=1}^{s}\left(\eta^{\alpha}\left(e_t\right)\right)^2\right)\\
&\quad+\sum_{i,j=1}^{s}F_{ij}\left(n-s\right)\sum_{t=1}^{n}\eta^{i}\left(e_t\right)\eta^{j}\left(e_t\right)+\left(\sum_{i=1}^{s}F_{ii}\right)\\
&\quad\times\left(\sum_{t=1}^{n}g\left(e_t,e_t\right)-\sum_{t=1}^{n}\sum_{\alpha=1}^{s}\left(\eta^{\alpha}\left(e_t\right)\right)^2\right)\\
&\quad+\sum_{t=1}^{n}\sum_{k=1}^{n}\left[g\left(h\left(e_t,e_t\right),h\left(e_k,e_k\right)\right)\right.\\
&\qquad\left.-g\left(h\left(e_t,e_k\right),h\left(e_t,e_k\right)\right)\right]\\
&=\left(n-s\right)\left(\left(n-s-1\right)F_1+3F_2\right)\\
&\quad+\left(n-s\right)\sum_{i=1}^{s}F_{ii}+\left(n-s\right)\sum_{i=1}^{s}F_{ii}\\
&\quad+\sum_{i,j=1}^{n}\left[g\left(h\left(e_i,e_i\right),h\left(e_j,e_j\right)\right)\right.\\
&\qquad\left.-g\left(h\left(e_i,e_j\right),h\left(e_i,e_j\right)\right)\right].
\end{aligned}
\tag{57}
$$

Thus, we obtain the following.

Theorem 42. *Let M be a submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$. Then the Ricci tensor and scalar curvature of M (resp.) are given by*

$$
\begin{aligned}
\operatorname{Ric}\left(X,Y\right)&=\left(\left(n-s-1\right)F_1+3F_2+\sum_{i=1}^{s}F_{ii}\right)g\left(\varphi X,\varphi Y\right)\\
&\quad+\left(n-s\right)\sum_{i,j=1}^{s}F_{ij}\eta^{i}\left(X\right)\eta^{j}\left(Y\right)\\
&\quad+\sum_{k=1}^{n}\left[g\left(h\left(X,X\right),h\left(e_k,e_k\right)\right)\right.\\
&\qquad\left.-g\left(h\left(X,e_k\right),h\left(Y,e_k\right)\right)\right],\\
\rho&=\left(n-s\right)\left(\left(n-s-1\right)F_1+3F_2+2\sum_{i=1}^{s}F_{ii}\right)\\
&\quad+\sum_{i,j=1}^{n}\left[g\left(h\left(e_i,e_i\right),h\left(e_j,e_j\right)\right)\right.\\
&\qquad\left.-g\left(h\left(e_i,e_j\right),h\left(e_i,e_j\right)\right)\right].
\end{aligned}
\tag{58}
$$

Theorem 43. *Let M be a minimal CR-submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}$. Then*

$$
\begin{aligned}
&\operatorname{Ric}\left(X,Y\right)\\
&\quad-\left(\left(n-s-1\right)F_1+3F_2+\sum_{i=1}^{s}F_{ii}\right)g\left(\varphi X,\varphi Y\right)\\
&\quad-\left(n-s\right)\sum_{i,j=1}^{s}F_{ij}\eta^{i}\left(X\right)\eta^{j}\left(Y\right)
\end{aligned}
\tag{59}
$$

is negative semidefinite and

$$
\begin{aligned}
&\rho\leq\left(n-s\right)\\
&\quad\times\left(\left(n-s-1\right)F_1+3F_2+2\sum_{i=1}^{s}F_{ii}\right).
\end{aligned}
\tag{60}
$$

Theorem 44. *Let M be a ξ_α-horizontal (resp., ξ_α-vertical) CR-submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$. Then the Ricci tensor of M for any $X, Y \in D^{\perp}$ (resp., $X, Y \in D$) is given by*

$$
\begin{aligned}
\operatorname{Ric}\left(X,Y\right)&=\left(\left(n-s-1\right)F_1+3F_2+\sum_{i=1}^{s}F_{ii}\right)g\left(X,Y\right)\\
&\quad+\sum_{i=1}^{n}\left[g\left(h\left(X,Y\right),h\left(e_i,e_i\right)\right)\right.\\
&\qquad\left.-g\left(h\left(X,e_i\right),h\left(Y,e_i\right)\right)\right].
\end{aligned}
\tag{61}
$$

Table 1

Manifold	Ricci tensor of Minimal ξ_α-horizontal CR-submanifold
$\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$	$((n-s-1)F_1 + 3F_2 + \sum_{i=1}^{s} F_{ii})g(\varphi X, \varphi Y) + (n-s)\sum_{i,j=1}^{s} F_{ij}\eta^i(X)\eta^j(Y)$
$\overline{M}^{2n+s}$	$\frac{1}{4}(3c + s + (n-s-1)(c+3s))g(\varphi X, \varphi Y) + (n-s)\sum_{i,j=1}^{s}\eta^i(X)\eta^j(Y)$
$\overline{M}(f_1, f_2, f_3)$	$((n-1)f_1 + 3f_2 - f_3)g(X,Y) - \eta(X)\eta(Y)(3f_2 + (n-2)f_3)$
$\overline{M}_{\mathrm{Sas}}(c)$	$\frac{1}{4}g(X,Y)(n(c+3) + c - 5) - \frac{1}{4}(c-1)(n+1)\eta(X)\eta(Y)$
$\overline{M}_{\mathrm{Ken}}(c)$	$\frac{1}{4}g(X,Y)(n(c-3) + c + 5) - \frac{1}{4}(c+1)(n+1)\eta(X)\eta(Y)$
$\overline{M}_{\mathrm{cosym}}(c)$	$\frac{1}{4}c(n+1)(g(X,Y) - \eta(X)\eta(Y))$
$\overline{M}_{C(\alpha)}(c)$	$\frac{1}{4}g(X,Y)(n(c+3\alpha^2) + c - 5\alpha^2) - \frac{1}{4}(n+1)(c-\alpha^2)\eta(X)\eta(Y)$

Where $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$, $\overline{M}^{2n+s}$, $\overline{M}(f_1, f_2, f_3)$, $\overline{M}_{\mathrm{Sas}}(c)$, $\overline{M}_{\mathrm{Ken}}(c)$, $\overline{M}_{\mathrm{cosym}}(c)$, and $\overline{M}_{C(\alpha)}(c)$ denote generalized $f.p.k.$-space form, S-space form, generalized Sasakian space form, Sasakian space form, Kenmotsu space form, cosymplectic space form, and $C(\alpha)$-space form, respectively.

Table 2

Manifold	Scalar curvature of Minimal ξ_α-horizontal CR-submanifold
$\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$	$(n-s)((n-s-1)F_1 + 3F_2 + 2\sum_{i=1}^{s} F_{ii})$
$\overline{M}^{2n+s}$	$\frac{1}{4}(n-s)(3c + 5s + (n-c-1)(c+3s))$
$\overline{M}(f_1, f_2, f_3)$	$(n-1)(nf_1 + 3f_2 - 2f_3)$
$\overline{M}_{\mathrm{Sas}}(c)$	$\frac{1}{4}(n-1)(c-1+n(c+3))$
$\overline{M}_{\mathrm{Ken}}(c)$	$\frac{1}{4}(n-1)(c+1+n(c-3))$
$\overline{M}_{\mathrm{cosym}}(c)$	$\frac{1}{4}c(n^2-1)$
$\overline{M}_{C(\alpha)}(c)$	$\frac{1}{4}(n-1)(c-\alpha^2+n(c+3\alpha^2))$

Where $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$, $\overline{M}^{2n+s}$, $\overline{M}(f_1, f_2, f_3)$, $\overline{M}_{\mathrm{Sas}}(c)$, $\overline{M}_{\mathrm{Ken}}(c)$, $\overline{M}_{\mathrm{cosym}}(c)$, and $\overline{M}_{C(\alpha)}(c)$ denote generalized $f.p.k.$-space form, S-space form, generalized Sasakian space form, Sasakian space form, Kenmotsu space form, cosymplectic space form, and $C(\alpha)$-space form, respectively.

Theorem 45. *Let M be minimal ξ_α-horizontal (resp., ξ_α-vertical) CR-submanifold of a generalized $f.p.k.$-space form $\overline{M}^{2n+s}(F_1, F_2, \mathscr{F})$. Then for any $X, Y \in D^\perp$ (resp., $X, Y \in D$)*

$$\operatorname{Ric}(X,Y) - \left((n-s-1)F_1 + 3F_2 + \sum_{i=1}^{s} F_{ii}\right)g(X,Y) \quad (62)$$

is negative semidefinite.

Corollary 46. *One has for Ricci tensor of minimal ξ_α-horizontal CR-submanifold Table 1.*

Corollary 47. *One has for scalar curvature of minimal ξ_α-horizontal CR-submanifold Table 2.*

Remark 48. Similar results can be written for minimal ξ_α-vertical CR-submanifolds, easily.

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