Research Article

A Polynomial Representation and a Unique Code of a Simple Undirected Graph

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We introduce a representation of simple undirected graphs in terms of polynomials and obtain a unique code for a simple undirected graph.

1. Introduction

Let M be the set of all positive integers greater than 1. Let $n \in M$ and $V(n)$ be the set of all divisors of n , greater than 1. Define a simple undirected graph $G(n) = (V, E)$ with the vertex set $V = V(n)$ and any two distinct vertices $a, b \in V$ are adjacent if and only if $\gcd(a, b) > 1$. From an observation in [1], it follows that any simple undirected graph is isomorphic to an induced subgraph of $G(n)$ for some $n \in M$. For the sake of completeness and further use of the construction we provide a sketch of the proof below. Throughout the paper by a *graph*, we mean a simple undirected graph.

Theorem 1. *Let G be a graph. Then G is isomorphic to an induced subgraph of $G(n)$ for some $n \in M$.*

Proof. Let $G = (V, E)$ be a graph. Let $\{C_1, C_2, \dots, C_k\}$ be the set of all maximal cliques of G . For $i = 1, 2, \dots, k$, let p_i be the i th prime. For each $v \in V$, define $s_1(v) = \prod \{p_i \mid v \in C_i\}$. Now in order to make the values of $s_1(v)$ distinct for distinct vertices, we modify $s_1(v)$ by using different powers of primes p_i , if required. For each $v \in V$, let $s(v)$ be the modified value of $s_1(v)$. Let n be the least common multiple of $\{s(v) \mid v \in V\}$. Now it is clear that for any $u, v \in V$,

$$u \text{ is adjacent to } v \text{ in } G \iff u, v \in C_i \\ \text{for some } i \in \{1, 2, \dots, k\}$$

$$\iff p_i \text{ is a factor of both } s(u) \text{ and } s(v)$$

$$\text{for some } i \in \{1, 2, \dots, k\}$$

$$\iff \gcd(s(u), s(v)) > 1 \iff s(u) \text{ is adjacent to } s(v) \text{ in } G(n).$$

(1)

Thus, G is isomorphic to the subgraph of $G(n)$ induced by the set $\{s(v) \mid v \in V\}$ of vertices of $G(n)$. $\square$

Example 2. Consider the graph $G = (V, E)$ in Figure 1. The maximal cliques of G are $C_1 = \{v_1, v_2, v_3, v_4\}$, $C_2 = \{v_2, v_3, v_4, v_5, v_6\}$, $C_3 = \{v_7, v_8, v_{10}\}$, $C_4 = \{v_9, v_{10}\}$, and $C_5 = \{v_{10}, v_{11}\}$. We assign the i th prime to the clique C_i for $i = 1, 2, \dots, 5$. Then for each $v \in V$, we compute $s_1(v)$ and $s(v)$ as in the proof of Theorem 1 (Table 1).

So G is isomorphic to the subgraph of $G(n)$ induced by the set $\{2, 3, 5, 6, 7, 9, 11, 12, 18, 25, 385\}$ of vertices of $G(n)$, where $n = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 = 69300$.

2. Main Results

It is important to note that instead of taking all maximal cliques of G in Theorem 1, it is sufficient to consider a set of cliques of G which covers both vertices and edges of G .

TABLE 1

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$s_1(v)$	2	$2 \cdot 3$	$2 \cdot 3$	$2 \cdot 3$	3	3	5	5	7	$5 \cdot 7 \cdot 11$	11
$s_1(v)$	2	6	6	6	3	3	5	5	7	385	11
$s(v)$	2	$2 \cdot 3$	$2^2 \cdot 3$	$2 \cdot 3^2$	3	3^2	5	5^2	7	385	11
$s(v)$	2	6	12	18	3	9	5	25	7	385	11

FIGURE 1: The graph G in Example 2.

Definition 3. Let $G = (V, E)$ be a graph. A set S of cliques of G is called a *total clique covering* of G if S covers both V and E . The minimum size of a total clique covering of G is called the *total clique covering number* of G . We denote it by $\theta_t(G)$.

On the other hand, given a finite sequence of positive integers, one can construct a simple undirected graph as follows.

Definition 4. Let $\sigma = (a_1, a_2, \dots, a_m)$ be a finite sequence of positive integers. Then corresponding to this sequence, define a graph $G[\sigma] = (V, E)$, where $V = \{v_1, v_2, \dots, v_m\}$, v_i corresponds to a_i (a_i is called the *label* of v_i) for $i = 1, 2, \dots, m$, and $v_i v_j \in E$ if and only if $i \neq j$ and $\gcd(a_i, a_j) > 1$. The graph $G[\sigma]$ is said to be *realized* by the sequence σ .

Now by Theorem 1, every graph can be realized by some sequence of positive integers. Also in Definition 4, it is sufficient to take the entries a_i of σ square-free if $a_i > 1$. Thus, for convenience, we specify the sequence in the following manner.

Definition 5. Let $G = (V, E)$ be a graph. Let $|V| > 1$, and let the graph G_1 be obtained from G by deleting isolated vertices of G , if there are any. Let σ_1 be a finite nondecreasing sequence of square-free positive integers greater than 1 such that $G_1 \cong G[\sigma_1]$. If G has s isolated vertices, then we prefix s number of 1's in σ_1 to obtain the sequence σ . For $s = 0$, $\sigma = \sigma_1$. If G is a graph with a single vertex, then $\sigma = (1)$. Then $G \cong G[\sigma]$ and the sequence σ is called a *coding sequence* of G . The entries of σ , which are greater than 1, are called *nontrivial*. Let $\lambda(\sigma)$ be the least common multiple of all nontrivial entries of σ .

Lemma 6. Let $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a coding sequence of a graph $G = (V, E)$. Let $V = \{v_1, v_2, \dots, v_m\}$, where v_i corresponds to λ_i for all $i = 1, 2, \dots, m$ (one writes $\lambda_i = \mu(v_i)$). Let $\{p_1, p_2, \dots, p_k\}$ be the set of all distinct prime factors of $\lambda(\sigma)$.

Suppose G has $s \geq 0$ isolated vertices. Define $S_j = \{v_j\}$ for $j = 1, 2, \dots, s$ and for each $j = 1, 2, \dots, k$, define $S_{s+j} = \{v_i \in V \mid p_j \text{ divides } \mu(v_i)\}$. Then $S = \{S_j \mid j = 1, 2, \dots, s+k\}$ is a total clique covering of G such that $v \in S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_r}$ ($i_t > s$, $t = 1, 2, \dots, r$) and $v \notin S_j$ for all $j \in \{s+1, s+2, \dots, s+k\} \setminus \{i_1, i_2, \dots, i_r\}$ if and only if $\mu(v) = p_{i_1} p_{i_2} \dots p_{i_r}$ and for each $j = 1, 2, \dots, s$, $v \in S_j$ if and only if $\mu(v) = 1$.

Proof. We first note that S_j is a clique containing only the vertex v_j for $j = 1, 2, \dots, s$. Now for each $j \in \{1, 2, \dots, k\}$, p_j divides λ_i for some $i \in \{s+1, s+2, \dots, m\}$ as p_j divides $\lambda(\sigma)$. Thus, $v_i \in S_{s+j}$. So $S_j \neq \emptyset$, for all $j = 1, 2, \dots, s+k$. Also for any two vertices $v_r, v_t \in S_{s+j}$, p_j divides both λ_r and λ_t . So v_r and v_t are adjacent in G . Thus, each S_j is a clique of G . Again since $\{p_1, p_2, \dots, p_k\}$ is the set of all prime factors of $\lambda(\sigma)$, each vertex $v \in V$ belongs to S_j for some $j \in \{1, 2, \dots, s+k\}$. So S covers V . Moreover, two vertices v_i and v_j are adjacent if and only if $\gcd(\lambda_i, \lambda_j) > 1$. So λ_i and λ_j must have at least one common prime factor, say, p_r and hence v_i and v_j both lie in S_{s+r} . So S also covers E and hence S is a total clique covering of G . Now $v \in S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_r}$ ($i_t > s$, $t = 1, 2, \dots, r$) and $v \notin S_j$ for all $j \in \{s+1, s+2, \dots, s+k\} \setminus \{i_1, i_2, \dots, i_r\}$ if and only if $p_{i_1} p_{i_2} \dots p_{i_r}$ divides $\mu(v)$ but p_j does not divide $\mu(v)$ for all $j \in \{s+1, s+2, \dots, s+k\} \setminus \{i_1, i_2, \dots, i_r\}$. Now since nontrivial entries of σ are square-free integers, $\mu(v)$ is a product of distinct primes and hence $\mu(v) = p_{i_1} p_{i_2} \dots p_{i_r}$. By Definition 5, for each $j = 1, 2, \dots, s$, $v \in S_j$ if and only if $\mu(v) = 1$. $\square$

Definition 7. Let $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a coding sequence of a graph G . Then the total clique covering S of G as defined in Lemma 6 is called the *total clique covering of G corresponding to the sequence σ* and is denoted by S_σ .

Lemma 8. Let $G = (V, E)$ be a graph, and let S be a total clique covering of G . Then there exists a coding sequence σ of G such that $S = S_\sigma$.

Proof. Let $S = \{C_1, C_2, \dots, C_{s+k}\}$, where $|C_i| = 1$ for $i = 1, 2, \dots, s$, $s \geq 0$. Let p_i denote the i th prime. We define a map $f : V \rightarrow \mathbb{N}$ by $f(v) = 1$ if $v \in C_1 \cup C_2 \cup \dots \cup C_s$ and $f(v) = p_{i_1} p_{i_2} \dots p_{i_r}$ if $v \in C_{s+i_1} \cap C_{s+i_2} \cap \dots \cap C_{s+i_r}$ (where $1 \leq r \leq k$) and $v \notin C_j$ for all $j \in \{s+1, s+2, \dots, s+k\} \setminus \{s+i_1, s+i_2, \dots, s+i_r\}$. Now S being a total clique covering, every vertex is assigned a label in this way. Also note that two vertices u, v are adjacent in G if and only if they lie in some $C_{s+j} \in S$ and consequently, if and only if $f(u)$ and $f(v)$ have a common prime factor p_j . Let us arrange the vertices of G according to the nondecreasing order of $\{f(v) \mid v \in V\}$ and define the sequence σ by $\sigma = (f(v_1), f(v_2), \dots, f(v_m))$, where $f(v_1) \leq f(v_2) \leq \dots \leq f(v_m)$. Then clearly σ is a coding sequence of G and $S = S_\sigma$. $\square$

Theorem 9. Let G be a graph with $s \geq 0$ isolated vertices, and let k be the minimum number of prime factors of $\lambda(\sigma)$ among all coding sequences σ of G . Then $\theta_t(G) = k + s$.

Proof. Let σ be a coding sequence of G such that the number of prime factors of $\lambda(\sigma)$ is k . Then by Lemma 6, $k + s = |S_\sigma| \geq \theta_t(G)$. Again let S be a total clique covering of G such that $|S| = \theta_t(G)$. Then by Lemma 8, there is a coding sequence σ of G such that $S = S_\sigma$ and it follows from the proof of Lemma 8 that there are $|S| - s$ prime factors of $\lambda(\sigma)$. Thus, $k \leq |S| - s = \theta_t(G) - s$. Therefore, $\theta_t(G) = k + s$. $\square$

Definition 10. Let $G = (V, E)$ be a graph with $|V| = m$ and $s \geq 0$ isolated vertices. Let $k = \theta_t(G) - s$. Let $S = \{C_1, C_2, \dots, C_{k+s}\}$ be a total clique covering of G such that $|S| = \theta_t(G)$ and $|C_i| = 1$ for $i = 1, 2, \dots, s$. Now as in the proof of Lemma 8, there are $k!$ ways of assigning the first k primes to the cliques $C_{s+1}, C_{s+2}, \dots, C_{s+k}$ to obtain at most $k!$ different coding sequences. Let $\sigma[S]$ be the least among them in the lexicographic ordering in $\mathbb{R}^m$. Then $\sigma[S]$ is called the *coding sequence of G with respect to S* . Let $\{S_1, S_2, \dots, S_r\}$ be the set of all total clique coverings of G such that $|S_i| = \theta_t(G)$ for all $i = 1, 2, \dots, r$. Let $\sigma(G)$ be the least element of $\{\sigma[S_i] \mid i = 1, 2, \dots, r\}$ in the lexicographic ordering in $\mathbb{R}^m$. Then $\sigma(G)$ is called the *code* of the graph G .

For example, $(2, 3, 3, 5, 5, 6, 6, 7, 11, 385)$ and $(2, 3, 3, 5, 5, 7, 10, 10, 10, 11, 231)$ are two coding sequences of the graph G in Example 2 with respect to the total clique covering $S = \{C_1, C_2, \dots, C_5\}$. It is easy to see that $\theta_t(G) = 5$ and S is the only total clique covering with 5 cliques. The code of G is $(2, 2, 3, 3, 5, 7, 10, 10, 10, 11, 231)$. Note that for any graph G , we have $G \cong G[\sigma(G)]$.

Theorem 11. Let G_1 and G_2 be two graphs. Then $G_1 \cong G_2$ if and only if $\sigma(G_1) = \sigma(G_2)$.

Proof. The proof follows from Definition 10. $\square$

Definition 12. Let $G = (V, E)$ be a graph with $s \geq 0$ isolated vertices. Let $S = \{C_1, C_2, \dots, C_{s+k}\}$ be a total clique covering of G , where $|C_i| = 1$ for $i = 1, 2, \dots, s$. If $v \notin C_1 \cup C_2 \cup \dots \cup C_s$, let $m(v) = \prod \{x_j \mid v \in C_{s+j}\}$ be a monomial in the polynomial semiring $\mathbb{Z}_0^+[x_1, x_2, \dots, x_k]$ of k indeterminates over the semiring of nonnegative integers with usual addition and multiplication. For any $v \in C_i$, $i = 1, 2, \dots, s$, define $m(v) = 1$. Now we define

$$f(G, S) = f(x_1, x_2, \dots, x_k) = \sum_{v \in V} m(v). \quad (2)$$

Then $f(G, S)$ is said to be a *polynomial representation of G with respect to S* .

Consider the graph G in Example 2. Then $f(G, S) = x_1 + 2x_2 + 2x_3 + x_4 + x_5 + 3x_1x_2 + x_3x_4x_5$ is a polynomial representation of G with respect to $S = \{C_1, C_2, C_3, C_4, C_5\}$. Now let $G = (V, E)$ be a graph, and let $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a coding sequence of G . From the construction of S_σ (cf. Lemma 6, Definition 7) and by Definition 12, it follows that $f(G, S_\sigma)$ can also be obtained from σ by replacing primes p_i by x_i ($i < j \Leftrightarrow p_i < p_j$) and commas by the addition symbol. It is important

to note that the constant term of $f(G, S)$ in Definition 12 is the number of isolated vertices of G .

Definition 13. Let G be a graph and $\sigma(G)$ the code of G . Then the polynomial representation of G corresponding to $\sigma(G)$, that is, $f(G, S_{\sigma(G)})$, is called the *normal polynomial representation* or the *canonical polynomial representation* of G and is denoted by $F(G)$.

The normal polynomial representation of the graph G in Example 2 is given by

$$F(G) = 2x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_1x_3 + x_2x_4x_5. \quad (3)$$

The following interesting observations are immediate from Definition 13.

Observation 1. A graph G is disconnected if and only if $F(G) = f_1 + f_2$, where f_1 and f_2 are polynomials with no common variables between them. The same is true for $f(G, S)$ for any total clique covering S of G .

Observation 2. A graph G is bipartite if and only if $F(G) = f_1 + f_2$, where monomials belonging to the same f_i have no common variables, for $i = 1, 2$. The same is true for $f(G, S)$ for any total clique covering S of G .

We now proceed to obtain a formula for $F(G(n))$.

Theorem 14. Let $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \in M$, where the p_i 's are distinct primes. Then $\theta_t(G(n)) = k$ and there is only one total clique covering of $G(n)$ with precisely k cliques.

Proof. If n is prime, then the result is obvious. Suppose n is not a prime number. Now the vertices of $G(n)$ correspond to all the divisors of n , greater than 1. So there are k vertices labeled p_i , $i = 1, 2, \dots, k$. Clearly, no two of them can lie in the same clique as $\gcd(p_i, p_j) = 1$ for all $i \neq j$. So in any total clique covering of $G(n)$, these k vertices will be in k different cliques. In other words, any total clique covering has at least k cliques. Now let $C_i = \{v \in V(G(n)) \mid p_i \text{ divides } v\}$. Then $S = \{C_i \mid i = 1, 2, \dots, k\}$ is easily seen to be a total clique covering of $G(n)$. So we have a total clique covering consisting of precisely k cliques. Hence, $\theta_t(G(n)) = k$.

Now we show that there is only one total clique covering of $G(n)$ containing precisely k cliques. Suppose there is another total clique covering $D = \{D_1, D_2, \dots, D_k\}$ of G with exactly k cliques. Here also, the vertices labeled p_i , $i = 1, 2, \dots, k$, are in distinct cliques. Without loss of generality, let $p_i \in D_i$ for $i = 1, 2, \dots, k$. Now for any i , p_i is not adjacent to those vertices which are not in C_i . So $D_i \subseteq C_i$ for all $i = 1, 2, \dots, k$. Now suppose that for some j , there is a vertex v in C_j which is not in D_j . Now D being a total clique covering, v has to lie in at least one D_l , where $l \neq j$. Since a total clique covering covers all the edges, the edge between v and p_j will be covered. $v \notin D_j$ implies that v and p_j both lie in some D_l , where $l \neq j$. However, this contradicts the fact that p_j is not adjacent to p_l . So $D_i = C_i$ for all $i = 1, 2, \dots, k$ and hence $D = S$. $\square$

Let $\alpha(G)$ denote the maximum size of an independent set in a graph G . In general, $\theta_t(G) \geq \alpha(G)$ for any graph G . The proof of the following proposition is similar to that of Theorem 14 and so it is omitted.

Proposition 15. *Let G be a graph with $\alpha(G) = k$. If $\{v_1, v_2, \dots, v_k\}$ is an independent set and $S = \{S_1, S_2, \dots, S_k\}$ is a total clique covering of G such that each v_i lies only in S_i among cliques in S for each $i = 1, 2, \dots, k$, then $\theta_t(G) = \alpha(G) = k$, and S is the only total clique covering containing exactly k cliques.*

Now we provide a formula for $F(G(n))$.

Theorem 16. *Let $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \in M$, where the p_i 's are distinct primes and $r_1 \geq r_2 \geq \dots \geq r_k \geq 1, k \geq 1$. Then $F(G(n))$ contains all the monomials $x_{i_1} x_{i_2} \dots x_{i_s}$, where $\{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, k\}$, $s \geq 1$ with the coefficient $r_{i_1} r_{i_2} \dots r_{i_s}$, that is,*

$$F(G(n)) = \sum_{\substack{i_1 < i_2 < \dots < i_s \\ \emptyset \neq \{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, k\}}} r_{i_1} r_{i_2} \dots r_{i_s} x_{i_1} x_{i_2} \dots x_{i_s}, \quad (4)$$

unless n is prime. If n is prime, then $F(G(n)) = 1$.

Proof. If n is prime, then $G(n) \cong K_1$ and so $F(G(n)) = 1$. Suppose n is not prime. Now by Theorem 14, $\theta_t(G(n)) = k$ and there is only one total clique covering, say, S of $G(n)$ with precisely k cliques. Let $S = \{C_1, C_2, \dots, C_k\}$, where $C_i = \{v \in V(G(n)) \mid p_i \text{ divides } v\}$ for $i = 1, 2, \dots, k$. So $\sigma(G(n))$ is $\sigma[S]$. First, let $k > 1$. Now $\sigma[S]$ is the least (in the lexicographic ordering) among the coding sequences of G obtained from S . By Definition 10, any such coding sequence involves the primes $\{t_1, t_2, \dots, t_k\}$, where t_i is the i th prime. Now for any p_i , there are r_i elements (namely, $p_i, p_i^2, \dots, p_i^{r_i}$) which belong to only the clique C_i . So considering the labellings and the fact that $r_1 \geq r_2 \geq \dots \geq r_k \geq 1$, it is easy to see that the coding sequence will be the least in the lexicographic ordering if we assign t_i to the clique C_i . So the vertex corresponding to a number $p_{i_1}^{q_{i_1}} p_{i_2}^{q_{i_2}} \dots p_{i_s}^{q_{i_s}}$ ($1 \leq q_{i_j} \leq r_{i_j}$ for $j = 1, 2, \dots, s$ where $1 \leq s \leq k$) is assigned the label $t_{i_1} t_{i_2} \dots t_{i_s}$. In other words, to find out $\sigma(G(n))$, we first consider the set of all divisors $\{m_1, m_2, \dots, m_r\}$ (say) of n , greater than 1, where $r = (r_1 + 1)(r_2 + 1) \dots (r_k + 1) - 1$. Then in each m_j , we first replace p_i by t_i , then make the resultant entries square-free and arrange them in nondecreasing order. This gives us $\sigma(G(n)) = (\lambda_1, \lambda_2, \dots, \lambda_r)$, where each λ_i is a product of primes of the form $t_{i_1} t_{i_2} \dots t_{i_s}$, ($1 \leq s \leq k$) and this particular number repeats, say, ν times in the sequence $\sigma(G(n))$, where ν is the number of divisors of n (greater than 1) which are of the form $p_{i_1}^{q_{i_1}} p_{i_2}^{q_{i_2}} \dots p_{i_s}^{q_{i_s}}$, $1 \leq q_{i_j} \leq r_{i_j}$ for $j = 1, 2, \dots, s$; that is, $\nu = \prod_{j=1}^s r_{i_j}$.

Thus, by Definition 13 we have $F(G(n)) = \sum_{i_1 < i_2 < \dots < i_s, \emptyset \neq \{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, k\}} r_{i_1} r_{i_2} \dots r_{i_s} x_{i_1} x_{i_2} \dots x_{i_s}$.

Finally, if $k = 1$, $r_1 > 1$, then $G(n) \cong K_{r_1}$. So $F(G(n)) = r_1 x_1$, which satisfies the aforementioned formula. This completes the proof. $\square$

For example, $F(G(p^2qr)) = F(G(60)) = 2x_1 + x_2 + x_3 + 2x_1x_2 + 2x_1x_3 + x_2x_3 + 2x_1x_2x_3$, where p, q , and r are distinct primes. Further, one may easily verify codes and normal polynomial representations for the following special classes of graphs:

- (i) $\sigma(K_1) = (1)$, $F(K_1) = 1$, $\sigma(K_n) = (2, 2, \dots, 2)$ (n times), $F(K_n) = nx_1$ for $n \geq 2$,
- (ii) $\sigma(P_n) = (p_1, p_2, p_1p_3, p_2p_4, \dots, p_{n-3}p_{n-1}, p_{n-2}p_{n-1})$, $F(P_n) = x_1 + x_2 + \sum_{i=1}^{n-3} x_i x_{i+2} + x_{n-2} x_{n-1}$ for $n \geq 3$,
- (iii) $\sigma(C_n) = (p_1p_2, p_1p_3, p_2p_4, \dots, p_{n-2}p_n, p_{n-1}p_n)$, $F(C_n) = x_1x_2 + x_1x_3 + \sum_{i=1}^{n-2} x_i x_{i+2} + x_{n-1} x_n$ for $n \geq 4$,

where p_i is the i th prime for $i = 1, 2, \dots, n$ and K_n , P_n , and C_n are, respectively, the complete graph, the path, and the cycle with n vertices.

3. Conclusion

There are some representations of simple undirected graphs in terms of adjacency matrices, adjacency lists, and unordered pairs which are not unique for isomorphic graphs. There are some other instances for unique representations [2, 3]. In the nauty algorithm, McKay [2] defined a *canonical isomorph* [4] which is a graph rather than a sequence of integers as the code of a graph introduced here. The importance of the code $\sigma(G)$ is its uniqueness and its simple form. It is the same for any set of isomorphic graphs. The determination of $\sigma(G)$ is not always easy, but once it is obtained for a graph, it becomes the characteristic of the graph. The authors believe that further study of the code and the normal polynomial representation of a simple undirected graph will be helpful in further research on graph theory. The purpose of this paper is to communicate these interesting observations to all graph theorists.

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