Research Article

On Cayley Digraphs That Do Not Have Hamiltonian Paths

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We construct an infinite family $\{\vec{\text{Cay}}(G; a_i; b_i)\}$ of connected, 2-generated Cayley digraphs that do not have hamiltonian paths, such that the orders of the generators a_i and b_i are unbounded. We also prove that if G is any finite group with $||[G, G]|| \leq 3$, then every connected Cayley digraph on G has a hamiltonian path (but the conclusion does not always hold when $||[G, G]|| = 4$ or 5).

1. Introduction

Definition 1. For a subset S of a finite group G , the *Cayley digraph* $\vec{\text{Cay}}(G; S)$ is the directed graph whose vertices are the elements of G and with a directed edge $g \rightarrow gs$ for every $g \in G$ and $s \in S$. The corresponding *Cayley graph* is the underlying undirected graph that is obtained by removing the orientations from all the directed edges.

It has been conjectured that every (nontrivial) connected Cayley graph has a hamiltonian cycle. (See the bibliography of [1] for some of the literature on this problem.) This conjecture does not extend to the directed case, because there are many examples of connected Cayley digraphs that do not have hamiltonian cycles. In fact, infinitely many Cayley digraphs do not even have a hamiltonian path.

Proposition 2 (attributed to Milnor [2, p. 201]). *Assume the finite group G is generated by two elements a and b , such that $a^2 = b^3 = e$. If $|G| \geq 9|ab^2|$, then the Cayley digraph $\vec{\text{Cay}}(G; a, b)$ does not have a hamiltonian path.*

The examples in the above proposition are very constrained, because the order of one generator must be exactly 2 and the order of the other generator must be exactly 3. In this note, we provide an infinite family of examples in which the orders of the generators are not restricted in this way. In fact, a and b can both be of arbitrarily large order.

Theorem 3. *For any $n \in \mathbb{N}$, there is a connected Cayley digraph $\vec{\text{Cay}}(G; a, b)$, such that*

- (1) $\vec{\text{Cay}}(G; a, b)$ does not have a hamiltonian path,
- (2) a and b both have order greater than n .

Furthermore, if p is any prime number such that $p > 3$ and $p \equiv 3 \pmod{4}$, then we may construct the example so that the commutator subgroup of G has order p . More precisely, $G = \mathbb{Z}_m \rtimes \mathbb{Z}_p$ is a semidirect product of two cyclic groups, so G is metacyclic.

Remark 4. Here are some related open questions and other comments.

- (1) The above results show that connected Cayley digraphs on solvable groups do not always have hamiltonian paths. On the other hand, it is an open question whether connected Cayley digraphs on nilpotent groups always have hamiltonian paths. (See [3] for recent results on the nilpotent case.)
- (2) The above results always produce a digraph with an even number of vertices. Do there exist infinitely many connected Cayley digraphs of odd order that do not have hamiltonian paths?
- (3) We conjecture that the assumption “ $p \equiv 3 \pmod{4}$ ” can be eliminated from the statement of Theorem 3. On the other hand, it is necessary to require that $p > 3$ (see Corollary 16).

- (4) If G is abelian, then it is easy to show that every connected Cayley digraph on G has a hamiltonian path. However, some abelian Cayley digraphs do not have a hamiltonian cycle. See Section 5 for more discussion of this.
- (5) The proof of Theorem 3 appears in Section 3, after some preliminaries in Section 2.

2. Preliminaries

We recall some standard notation, terminology, and basic facts.

Notation. Let G be a group, and let H be a subgroup of G . (All groups in this paper are assumed to be finite.)

- (i) e is the identity element of G ;
- (ii) $x^g = g^{-1}xg$, for $x, g \in G$;
- (iii) we write $H \trianglelefteq G$ to say that H is a normal subgroup of G ;
- (iv) $H^G = \langle h^g \mid h \in H, g \in G \rangle$ is the normal closure of H in G , so $H^G \trianglelefteq G$.

Definition 5. Let S be a subset of the group G .

- (i) $H = \langle SS^{-1} \rangle$ is the *arc-forcing subgroup*, where $SS^{-1} = \{st^{-1} \mid s, t \in S\}$.
- (ii) For any $a \in S$, $a^{-1}H$ is called the *terminal coset*. (This is independent of the choice of a .)
- (iii) Any left coset of H that is not the terminal coset is called a *regular coset*.
- (iv) For $g \in G$ and $s_1, \dots, s_n \in S$, we use $[g](s_i)_{i=1}^n$ to denote the walk in $\vec{\text{Cay}}(G; S)$ that visits (in order) the vertices

$$g, gs_1, gs_1s_2, \dots, gs_1s_2 \cdots s_n. \quad (1)$$

We usually omit the prefix $[g]$ when $g = e$. Also, we often abuse notation when sequences are to be concatenated. For example,

$$\begin{aligned} & (a^4, (s_i)_{i=1}^3, t_j)_{j=1}^2 \\ &= (a, a, a, a, s_1, s_2, s_3, t_1, a, a, a, a, s_1, s_2, s_3, t_2). \end{aligned} \quad (2)$$

Remark 6. Here are two observations about the arc-forcing subgroup.

- (1) It is important to note that $\langle SS^{-1} \rangle \subseteq \langle Sg \rangle$, for every $g \in G$. Furthermore, we have $\langle SS^{-1} \rangle = \langle Sa^{-1} \rangle$, for every $a \in S$.
- (2) It is sometimes more convenient to define the arc-forcing subgroup to be $\langle S^{-1}S \rangle$, instead of $\langle SS^{-1} \rangle$ (e.g., this is the convention used in [3, p. 42]). The difference is minor, because the two subgroups are conjugate: for any $a \in S$, we have

$$\langle S^{-1}S \rangle = \langle a^{-1}S \rangle = \langle Sa^{-1} \rangle^a = \langle SS^{-1} \rangle^a. \quad (3)$$

Definition 7. Suppose L is a hamiltonian path in a Cayley digraph $\vec{\text{Cay}}(G; S)$ and $s \in S$.

- (i) A vertex $g \in G$ travels by s if L contains the directed edge $g \rightarrow gs$.
- (ii) A subset X of G travels by s if every element of X travels by s .

Lemma 8 (see Housman [4, p. 82]). *Suppose L is a hamiltonian path in $\vec{\text{Cay}}(G; a, b)$, with initial vertex e , and let $H = \langle ab^{-1} \rangle$ be the arc-forcing subgroup. Then,*

- (1) the terminal vertex of L belongs to the terminal coset $a^{-1}H$,
- (2) each regular coset either travels by a or travels by b .

3. Proof of Theorem 3

Let

- (i) α be an even number that is relatively prime to $(p-1)/2$, with $\alpha > n$;
- (ii) β a multiple of $(p-1)/2$ that is relatively prime to α , with $\beta > n$;
- (iii) $\bar{a}$ a generator of $\mathbb{Z}_\alpha$;
- (iv) $\bar{b}$ a generator of $\mathbb{Z}_\beta$;
- (v) z a generator of $\mathbb{Z}_p$;
- (vi) r a primitive root modulo p ;
- (vii) $G = (\mathbb{Z}_\alpha \times \mathbb{Z}_\beta) \rtimes \mathbb{Z}_p$, where $z^{\bar{a}} = z^{-1}$ and $z^{\bar{b}} = z^{r^2}$;
- (viii) $a = \bar{a}z$, so $|a| = \alpha$, and a inverts $\mathbb{Z}_p$;
- (ix) $b = \bar{b}z$, so $|b| = \beta$, and b acts on $\mathbb{Z}_p$ via an automorphism of order $(p-1)/2$;
- (x) $H = \langle ab^{-1} \rangle = \langle \bar{a} \bar{b}^{-1} \rangle = \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$.

Suppose L is a hamiltonian path in $\vec{\text{Cay}}(G; a, b)$. This will lead to a contradiction.

It is well known (and easy to see) that Cayley digraphs are vertex-transitive, so there is no harm in assuming that the initial vertex of L is e . Note that

- (i) the terminal coset is $a^{-1}H = z^{-1}H$;
- (ii) since $p \equiv 3 \pmod{4}$, we have $\mathbb{Z}_p^\times = \langle -1, r^2 \rangle$.

Case 1. Assume at most one regular coset travels by a in L . Choose $z' \in \mathbb{Z}_p$, such that $z'H$ is a regular coset, and assume it is the coset that travels by a , if such exists.

For $g \in G$, let

$$\mathcal{B}_g = \{gb^kH \mid k \in \mathbb{Z}\}. \quad (4)$$

Letting $p' = (p-1)/2$, we have

$$\begin{aligned} & (r^2)^{p'-1} + (r^2)^{p'-2} + \cdots + (r^2)^1 + 1 \\ &= \frac{(r^2)^{p'} - 1}{r^2 - 1} = \frac{r^{p-1} - 1}{r^2 - 1} \equiv 0 \pmod{p}, \end{aligned} \quad (5)$$

so

$$\begin{aligned} b^{(p-1)/2} &= (\bar{b}z)^{p'} \\ &= \bar{b}^{p'} z^{(r^2)^{p'-1} + (r^2)^{p'-2} + \dots + (r^2)^1 + 1} \\ &= \bar{b}^{p'} \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta = H. \end{aligned} \quad (6)$$

Therefore $\#\mathcal{B}_e \leq (p-1)/2 \leq p-2$, so we can choose two cosets $z^i H$ and $z^j H$ that do not belong to $\mathcal{B}_e$.

Recall that, by definition, $z^i H$ is not the terminal coset $z^{-1} H$, so $z^i z$ is a nontrivial element of $\mathbb{Z}_p$. Then, since $\mathbb{Z}_p^\times = \langle -1, r^2 \rangle$, we can choose some $h \in \langle \bar{a}, \bar{b} \rangle = H$, such that $(z^{j-i})^h = z^i z$. Now, since

$$\begin{aligned} z^i H, z^j H &\notin \mathcal{B}_e, \\ z^{-1} h^{-1} z^{j-i} &\in z^{-1} (z^{j-i})^h H = z^{-1} (z^i z) H = z^i H, \end{aligned} \quad (7)$$

we may multiply on the left by $g = z^{-1} h^{-1} z^{-i}$ to see that

$$z^{-1} H, z^i H \notin \mathcal{B}_g. \quad (8)$$

Therefore, no element of $\mathcal{B}_g$ is either the terminal coset or the regular coset that travels by a . This means that every coset in $\mathcal{B}_g$ travels by b , so L contains the cycle $[g](b^\beta)$, which contradicts the fact that L is a (hamiltonian) path.

Case 2. Assume at least two regular cosets travel by a in L . Let $z^i H$ and $z^j H$ be two regular cosets that both travel by a . Since $\mathbb{Z}_p^\times = \langle -1, r^2 \rangle$, we can choose some $h \in \langle \bar{a}, \bar{b} \rangle = H$, such that $(z^{-1})^h = z^{j-i}$.

Note that $z^i h^{-1} a^k$ travels by a , for every $k \in \mathbb{Z}$.

(i) If $k = 2\ell$ is even, then

$$\begin{aligned} a^k &= (\bar{a}z)^{2\ell} = (\bar{a}z\bar{a}z)^\ell \\ &= (\bar{a}^2 z^2 z)^\ell = (\bar{a}^2 z^{-1} z)^\ell = \bar{a}^{2\ell} \in H, \end{aligned} \quad (9)$$

so $z^i h^{-1} a^k \in z^i H$ travels by a .

(ii) If $k = 2\ell + 1$ is odd, then

$$a^k = (\bar{a}z)^{2\ell+1} = (\bar{a}z)^{2\ell} (\bar{a}z) = \bar{a}^{2\ell} (\bar{a}z) = \bar{a}^k z, \quad (10)$$

so

$$\begin{aligned} z^i h^{-1} a^k &= z^i h^{-1} (\bar{a}^k z) = z^i h^{-1} z^{-1} \bar{a}^k \\ &= z^i (z^{-1})^h h^{-1} \bar{a}^k \in z^i (z^{j-i}) H = z^j H \end{aligned} \quad (11)$$

travels by a .

Therefore L contains the cycle $[z^i h^{-1}](a^k)$, which contradicts the fact that L is a (hamiltonian) path.

4. Cyclic Commutator Subgroups of Very Small Order

It is known that if $|[G, G]| = 2$, then every connected Cayley digraph on G has a hamiltonian path. (Namely, we have $[G, G] \subseteq Z(G)$, so G is nilpotent, and the conclusion, therefore, follows from Theorem 14(2) below.) In this section, we prove the same conclusion when $|[G, G]| = 3$. We also provide counterexamples to show that the conclusion is not always true when $|[G, G]| = 4$ or $|[G, G]| = 5$.

We begin with several lemmas. The first three each provide a way to convert a hamiltonian path in a Cayley digraph on an appropriate subgroup of G to a hamiltonian path in a Cayley digraph on all of G .

Lemma 9. *Assume*

- (i) G is a finite group, such that $[G, G] \cong \mathbb{Z}_{p^k}$, where p is prime and $k \in \mathbb{N}$;
- (ii) S is a generating set for G ;
- (iii) $a, b \in S$, such that $\langle [a, b] \rangle = [G, G]$;
- (iv) $N = \langle a, b \rangle$.

If $\vec{\text{Cay}}(N; a, b)$ has a hamiltonian path, then $\vec{\text{Cay}}(G; S)$ has a hamiltonian path.

Proof. Since $[G, G] \subseteq N$, we know that G/N is an abelian group, so there is a hamiltonian path $(s_i)_{i=1}^m$ in $\vec{\text{Cay}}(G/N; S)$ (see Proposition 19 below). Also, by assumption, there is a hamiltonian path $(t_j)_{j=1}^n$ in $\vec{\text{Cay}}(N; a, b)$. Then

$$\left(\left((t_j)_{j=1}^n, s_i \right)_{i=1}^m, (t_j)_{j=1}^n \right) \quad (12)$$

is a hamiltonian path in $\vec{\text{Cay}}(G; S)$. $\square$

Definition 10. If K is a subgroup of G , then $K \setminus \vec{\text{Cay}}(G; S)$ denotes the digraph whose vertices are the right cosets of K in G and with a directed edge $Kg \rightarrow Kgs$ for each $g \in G$ and $s \in S$. Note that $K \setminus \vec{\text{Cay}}(G; S) = \vec{\text{Cay}}(G/K; S)$ if $K \trianglelefteq G$.

Lemma 11 (“Skewed-Generators Argument,” cf. [3, Lem. 2.6], [5, Lem. 5.1]). *Assume*

- (i) S is a generating set for the group G ;
- (ii) K is a subgroup of G , such that every connected Cayley digraph on K has a hamiltonian path;
- (iii) $(s_i)_{i=1}^n$ is a hamiltonian cycle in $K \setminus \vec{\text{Cay}}(G; S)$;
- (iv) $\langle Ss_2s_3 \cdots s_n \rangle = K$.

Then $\vec{\text{Cay}}(G; S)$ has a hamiltonian path.

Proof. Since $\langle Ss_2s_3 \cdots s_n \rangle = K$, we know that $\vec{\text{Cay}}(K; Ss_2s_3 \cdots s_n)$ is connected, so, by assumption, it has a hamiltonian path $(t_j s_2 s_3 \cdots s_n)_{j=1}^m$. Then

$$\left((t_j, (s_i)_{i=2}^n)_{j=1}^{m-1}, t_m, (s_i)_{i=2}^{n-1} \right) \quad (13)$$

is a hamiltonian path in $\vec{\text{Cay}}(G; S)$. $\square$

Lemma 12. Assume

- (i) S is a generating set of G , with arc-forcing subgroup $H = \langle SS^{-1} \rangle$;
- (ii) there is a hamiltonian path in every connected Cayley digraph on H^G ;
- (iii) either $H = H^G$, or H is contained in a unique maximal subgroup of H^G .

Then $\vec{\text{Cay}}(G; S)$ has a hamiltonian path.

Proof. It suffices to show that

$$\begin{aligned} &\text{there exists a hamiltonian cycle } (s_i)_{i=1}^n \text{ in } \vec{\text{Cay}}\left(\frac{G}{H^G}; S\right), \\ &\text{such that } H^G = \langle Ss_2 \cdots s_n \rangle, \end{aligned} \quad (*)$$

for then Lemma 11 provides the desired hamiltonian path in $\vec{\text{Cay}}(G; S)$.

If $H^G = H$, then every hamiltonian cycle in $\vec{\text{Cay}}(G/H^G; S)$ satisfies $(*)$ (see Remark 6 (1)). Thus, we may assume $H^G \neq H$, so, by assumption, H is contained in a unique maximal subgroup M of H^G . Since H^G is generated by conjugates of $S^{-1}S$ (see Remark 6 (2)), there exist $a, b, c \in S$, such that $(a^{-1}b)^c \notin M$.

We may also assume $H^G \neq G$ (since, by assumption, every Cayley digraph on H^G has a hamiltonian path), so, letting $n = |G : H^G| \geq 2$, we have the two hamiltonian cycles (a^{n-1}, c) and (a^{n-2}, b, c) in $\vec{\text{Cay}}(G/H^G; S)$. Since

$$(a^{n-1}c)^{-1}(a^{n-2}bc) = (a^{-1}b)^c \notin M, \quad (14)$$

the two products $a^{n-1}c$ and $a^{n-2}bc$ cannot both belong to M . Hence, either (a^{n-1}, c) or (a^{n-2}, b, c) is a hamiltonian cycle $(s_i)_{i=1}^n$ in $\vec{\text{Cay}}(G/H^G; S)$, such that $s_1s_2 \cdots s_n \notin M$. Since M is the unique maximal subgroup of H^G that contains H , this implies

$$H^G = \langle H, s_1s_2 \cdots s_n \rangle = \langle Ss_2s_3 \cdots s_n \rangle, \quad (15)$$

as desired. $\square$

The final hypothesis of the preceding lemma is automatically satisfied when $[G, G]$ is cyclic of prime-power order.

Lemma 13. If $[G, G]$ is cyclic of order p^k , where p is prime, and H is any subgroup of G , then either $H = H^G$ or H is contained in a unique maximal subgroup of H^G .

Proof. Note that the normal closure H^G is the (unique) smallest normal subgroup of G that contains H . Therefore, $H^G \subseteq H [G, G]$ (since $H [G, G]$ is normal in G). This implies that if M is any proper subgroup of H^G that contains H , then

$$M = H \cdot (M \cap [G, G]) \subseteq H \cdot (H^G \cap [G, G])^p. \quad (16)$$

Therefore, $H \cdot (H^G \cap [G, G])^p$ is the unique maximal subgroup of H^G that contains M . $\square$

The following known result handles the case where G is nilpotent.

Theorem 14 (see Morris [3]). Assume G is nilpotent, and S generates G . If either

- (1) $\#S \leq 2$ or
- (2) $|[G, G]| = p^k$, where p is prime and $k \in \mathbb{N}$,

then $\vec{\text{Cay}}(G; S)$ has a hamiltonian path.

We now state the main result of this section.

Theorem 15. Suppose

- (i) $[G, G]$ is cyclic of prime-power order,
- (ii) every element of G either centralizes $[G, G]$ or inverts it.

Then every connected Cayley digraph on G has a hamiltonian path.

Proof. Let S be a generating set for G . Write $[G, G] = \mathbb{Z}_{p^k}$ for some p and k . Since every minimal generating set of $\mathbb{Z}_{p^k}$ has only one element, there exist $a, b \in S$, such that $\langle [a, b] \rangle = [G, G]$. Then, by Lemma 9, we may assume $S = \{a, b\}$.

Let $H = \langle ba^{-1} \rangle$ be the arc-forcing subgroup. We may assume $H^G = G$, for otherwise we could assume, by induction on $|G|$, that every connected Cayley digraph on H^G has a hamiltonian path, and then Lemma 12 would apply (since Lemma 13 verifies the remaining hypothesis). So

$$H\mathbb{Z}_{p^k} = H [G, G] \supset H^G = G. \quad (17)$$

If a and b both invert $\mathbb{Z}_{p^k}$, then $H = \langle ba^{-1} \rangle$ centralizes $\mathbb{Z}_{p^k} = [G, G]$, so G is nilpotent. Then Theorem 14 applies.

Therefore, we may now assume that a does not invert $\mathbb{Z}_{p^k}$. Then, by assumption, a centralizes $\mathbb{Z}_{p^k}$. Let $n = |G : H|$, and write $a = \bar{a}z$, where $\bar{a} \in H$ and $z \in \mathbb{Z}_{p^k}$. Then $a = \bar{a}z \in Hz$ and $b = (ba^{-1})(\bar{a}z) \in Hz$. Since $\langle a, b \rangle = G$, this implies $H\langle z \rangle = G$. Therefore,

$$[H](a^n) = [H, Hz, Hz^2, \dots, Hz^{n-1}, H] \quad (18)$$

is a hamiltonian cycle in $H \setminus \vec{\text{Cay}}(G; S)$, so Lemma 11 applies. $\square$

Corollary 16. If $|[G, G]| \leq 3$ or $[G, G] \cong \mathbb{Z}_4$, then every connected Cayley digraph on G has a hamiltonian path.

Proof. Theorem 15 applies, because the groups $\{e\}$ and $\mathbb{Z}_2$ have no nontrivial automorphisms, and inversion is the only nontrivial automorphism of $\mathbb{Z}_3$ or $\mathbb{Z}_4$. $\square$

Remark 17 ([6, p. 266]). In the statement of Corollary 16, the assumption that $[G, G] \cong \mathbb{Z}_4$ cannot be replaced with the weaker assumption that $|[G, G]| = 4$. For a counterexample, let $G = A_4 \times \mathbb{Z}_2$. Then $|[G, G]| = 4$, but it can be shown without much difficulty that $\vec{\text{Cay}}(G; a, b)$ does not have a hamiltonian path when $a = ((1\ 2)(3\ 4), 1)$ and $b = ((1\ 2\ 3), 0)$.

Here is a counterexample when $|[G, G]| = 5$.

Example 18. Let $G = \mathbb{Z}_{12} \ltimes \mathbb{Z}_5 = \langle h \rangle \ltimes \langle z \rangle$, where $z^h = z^3$. Then $|[G, G]| = 5$, and the Cayley digraph $\vec{\text{Cay}}(G; h^2z, h^3z)$ is connected but does not have a hamiltonian path.

Proof. A computer search can confirm the nonexistence of a hamiltonian path very quickly, but, for completeness, we provide a human-readable proof.

Let $a = h^2z = z^4h^2$ and $b = h^3z = z^3h^3$. The argument in Case 2 of the proof of Theorem 3 shows that no more than one regular coset travels by a in any hamiltonian path. On the other hand, since a hamiltonian path cannot contain any cycle of the form $[g](b^4)$, we know that at least $\lfloor (|G| - 1)/4 \rfloor = 14$ vertices must travel by a . Since $|ab^{-1}| = 12 < 14$, this implies that some regular coset travels by a . So exactly one regular coset travels by a in any hamiltonian path.

For $0 \leq i \leq 3$ and $0 \leq m \leq 11$, let $L_{i,m}$ be the spanning subdigraph of $\vec{\text{Cay}}(G; a, b)$ in which

- (i) all vertices have outvalence 1, except $b^{-1}(ab^{-1})^m = z^4h^{9-m}$, which has outvalence 0;
- (ii) the vertices in the regular coset z^iH travel by a ;
- (iii) a vertex $b^{-1}h^{-j} = z^4h^{9-j}$ in the terminal coset travels by a if $0 \leq j < m$;
- (iv) all other vertices travel by b .

An observation of D. Housman [7, Lem. 6.4(b)] tells us that if L is a hamiltonian path from e to $b^{-1}(ab^{-1})^m$, in which z^iH is the regular coset that travels by a , then $L = L_{i,m}$. Thus, from the conclusion of the preceding paragraph, we see that every hamiltonian path (with initial vertex e) must be equal to $L_{i,m}$, for some i and m .

However, $L_{i,m}$ is not a (hamiltonian) path. More precisely, for each possible value of i and m , the following list displays a cycle that is contained in $L_{i,m}$:

- (i) if $i = 0$ and $0 \leq m \leq 8$,

$$z^2h^3 \xrightarrow{b} zh^6 \xrightarrow{b} z^3h^9 \xrightarrow{b} z^4 \xrightarrow{b} z^2h^3; \quad (19)$$

- (ii) if $i = 0$ and $9 \leq m \leq 11$,

$$\begin{aligned} h^2 &\xrightarrow{a} zh^4 \xrightarrow{b} z^4h^7 \xrightarrow{a} zh^9 \xrightarrow{b} z^2 \\ &\xrightarrow{b} h^3 \xrightarrow{a} z^2h^5 \xrightarrow{b} z^3h^8 \xrightarrow{b} zh^{11} \xrightarrow{b} h^2; \end{aligned} \quad (20)$$

- (iii) if $i = 1$ and $0 \leq m \leq 7$,

$$h^4 \xrightarrow{b} z^3h^7 \xrightarrow{b} z^2h^{10} \xrightarrow{b} z^4h \xrightarrow{b} h^4; \quad (21)$$

- (iv) if $i = 1$ and $8 \leq m \leq 11$,

$$\begin{aligned} h &\xrightarrow{b} zh^4 \xrightarrow{a} h^6 \xrightarrow{b} z^2h^9 \xrightarrow{b} z^3 \xrightarrow{b} zh^3 \\ &\xrightarrow{a} z^3h^5 \xrightarrow{b} z^4h^8 \xrightarrow{a} z^3h^{10} \xrightarrow{b} h; \end{aligned} \quad (22)$$

- (v) if $i = 2$ and $0 \leq m \leq 9$,

$$h^5 \xrightarrow{b} zh^8 \xrightarrow{b} z^4h^{11} \xrightarrow{b} z^3h^2 \xrightarrow{b} h^5; \quad (23)$$

- (vi) if $i = 2$ and $10 \leq m \leq 11$,

$$\begin{aligned} z^2h^3 &\xrightarrow{a} z^4h^5 \xrightarrow{a} z^2h^7 \xrightarrow{a} z^4h^9 \\ &\xrightarrow{a} z^2h^{11} \xrightarrow{a} z^4h \xrightarrow{a} z^2h^3; \end{aligned} \quad (24)$$

- (vii) if $i = 3$ and $0 \leq m \leq 10$:

$$h^7 \xrightarrow{b} z^4h^{10} \xrightarrow{b} zh \xrightarrow{b} z^2h^4 \xrightarrow{b} h^7; \quad (25)$$

- (viii) if $i = 3$ and $m = 11$,

$$\begin{aligned} z^3h^2 &\xrightarrow{a} z^4h^4 \xrightarrow{a} z^3h^6 \xrightarrow{a} z^4h^8 \\ &\xrightarrow{a} z^3h^{10} \xrightarrow{a} z^4 \xrightarrow{a} z^3h^2. \end{aligned} \quad (26)$$

Since $L_{i,m}$ is never a hamiltonian path, we conclude that $\vec{\text{Cay}}(G; a, b)$ does not have a hamiltonian path. $\square$

5. Nonhamiltonian Cayley Digraphs on Abelian Groups

When G is abelian, it is easy to find a hamiltonian path in $\vec{\text{Cay}}(G; S)$.

Proposition 19 (see [6, Thm. 3.1]). *Every connected Cayley digraph on any abelian group has a hamiltonian path.*

On the other hand, it follows from Lemma 8(2) that sometimes there is no hamiltonian cycle.

Proposition 20 (see Rankin [8, Thm. 4]). *Assume $G = \langle a, b \rangle$ is abelian. Then there is a hamiltonian cycle in $\vec{\text{Cay}}(G; a, b)$ if and only if there exist $k, \ell \geq 0$, such that $\langle a^k b^\ell \rangle = \langle ab^{-1} \rangle$, and $k + \ell = |G : \langle ab^{-1} \rangle|$.*

Example 21. If $\gcd(a, n) > 1$ and $\gcd(a + 1, n) > 1$, then $\vec{\text{Cay}}(\mathbb{Z}_n; a, a + 1)$ does not have a hamiltonian cycle.

All of the non-hamiltonian Cayley digraphs provided by Proposition 20 are 2-generated. However, a few 3-generated examples are also known. Specifically, the following result lists (up to isomorphism) the only known examples of connected, non-hamiltonian Cayley digraphs $\vec{\text{Cay}}(G; S)$, such that $\#S > 2$ (and $e \notin S$).

Theorem 22 (see Locke and Witte [9]). *The following Cayley digraphs do not have hamiltonian cycles:*

- (1) $\vec{\text{Cay}}(\mathbb{Z}_{12k}; 6k, 6k + 2, 6k + 3)$, for any $k \in \mathbb{Z}^+$;
- (2) $\vec{\text{Cay}}(\mathbb{Z}_{2k}; a, b, b + k)$, for $a, b, k \in \mathbb{Z}^+$, such that certain technical conditions (Remark 23) are satisfied.

Remark 23. The precise conditions in (2) are (i) either a or k is odd, (ii) either a is even or b and k are both even, (iii) $\gcd(a - b, k) = 1$, (iv) $\gcd(a, 2k) \neq 1$, and (v) $\gcd(b, k) \neq 1$.

It is interesting to note that, in the examples provided by Theorem 22, the group G is cyclic (either $\mathbb{Z}_{12k}$ or $\mathbb{Z}_{2k}$), and either

- (1) one of the generators has order 2 or
- (2) two of the generators differ by an element of order 2.

S. J. Curran (personal communication) asked whether the constructions could be generalized by allowing G to be an abelian group that is not cyclic. We provide a negative answer for case (2).

Proposition 24. *Let G be an abelian group (written additively), and let $a, b, k \in G$, such that k is an element of order 2. (Also assume $\{a, b, b + k\}$ consists of three distinct, nontrivial elements of G .) If the Cayley digraph $\vec{\text{Cay}}(G; a, b, b + k)$ is connected, but does not have a hamiltonian cycle, then G is cyclic.*

Proof. We prove the contrapositive: assume G is not cyclic, and we will show that the Cayley digraph has a hamiltonian cycle (if it is connected). The argument is a modification of the proof of [9, Thm. 4.1(⇐)].

Construct a subdigraph H_0 of G as in [9, Defn. 4.2], but with G in the place of $\mathbb{Z}_{2k}$, with $|G|$ in the place of $2k$, and with $|a|$ in the place of d . (Case 1 is when $k \notin \langle a \rangle$; Case 2 is when $k \in \langle a \rangle$.) Every vertex of H_0 has both invallence 1 and outvalence 1.

The argument in case 3 of the proof of [9, Thm. 4.1(⇐)] shows that the Cayley digraph $\vec{\text{Cay}}(G; a, b, b + k)$ has a hamiltonian cycle if $\langle a - b, k \rangle \neq G$. Therefore, we may assume $\langle a - b, k \rangle = G$. On the other hand, we know $\langle a - b \rangle \neq G$ (because G is not cyclic). Since $|k| = 2$, this implies $G = \langle a - b \rangle \oplus \langle k \rangle$. Since G is not cyclic, this implies that $a - b$ has even order. Also, we may write $a = a' + k'$ and $b = b' + k''$ for some (unique) $a', b' \in \langle a - b \rangle$ and $k', k'' \in \langle k \rangle$. (Since

$a' - b' \in \langle a - b \rangle$, it is easy to see that $k' = k''$, but we do not need this fact.)

Claim. H_0 has an odd number of connected components. Arguing as in the proof of [9, Lem. 4.1] (except that, as before, Case 1 is when $k \notin \langle a \rangle$ and Case 2 is when $k \in \langle a \rangle$), we see that the number of connected components in H_0 is

$$\begin{aligned} |G : \langle a, k \rangle| + |G : \langle b, k \rangle| & \quad \text{if } k \notin \langle a \rangle, \\ |G : \langle b, k \rangle| & \quad \text{if } k \in \langle a \rangle. \end{aligned} \quad (27)$$

Since $\langle a' - b' \rangle = \langle a - b \rangle$, we know that one of a' and b' is an even multiple of $a - b$, and the other is an odd multiple. (Otherwise, the difference would be an even multiple of $a - b$, so it would not generate $\langle a - b \rangle$.) Thus, one of $|G : \langle a, k \rangle|$ and $|G : \langle b, k \rangle|$ is even, and the other is odd. So $|G : \langle a, k \rangle| + |G : \langle b, k \rangle|$ is odd. This establishes the claim if $k \notin \langle a \rangle$.

We may now assume $k \in \langle a \rangle$. This implies that the element a' has odd order (and k' must be nontrivial, but we do not need this fact). This means that a' is an even multiple of $a - b$, so b' must be an odd multiple of $a - b$ (since $\langle a' - b' \rangle = \langle a - b \rangle$). Therefore, $|\langle a - b \rangle : \langle b' \rangle|$ is odd, which means $|G : \langle b, k \rangle|$ is odd. This completes the proof of the claim.

Now, if $|G : \langle b, k \rangle|$ is odd, we can apply a very slight modification of the argument in case 4 of the proof of [9, Thm. 4.1(⇐)]. (Subcase 4.1 is when $k \notin \langle a \rangle$ and subcase 4.2 is when $k \in \langle a \rangle$.) We conclude that $\vec{\text{Cay}}(G; a, b, b + k)$ has a hamiltonian cycle, as desired.

Finally, if $|G : \langle b, k \rangle|$ is even, then more substantial modifications to the argument in [9] are required. For convenience, let $m = |G : \langle a, k \rangle|$. Note that, since $|G : \langle b, k \rangle|$ is even, the proof of the claim shows that m is odd and $k \notin \langle a \rangle$.

Define H'_0 as in subcase 4.1 of [9, Thm. 4.1(⇐)] (with G in the place of $\mathbb{Z}_{2k}$ and replacing $\gcd(b, k)$ with $|G : \langle b, k \rangle|$). Let $H_1 = H'_0$, and inductively construct, for $1 \leq i \leq (m + 1)/2$, an element H_i of $\mathcal{E}$, such that

$$\begin{aligned} \{v \mid z_v = 0, 0 \leq y_v \leq 2i - 2\} \\ \cup \{v \mid z_v = 1, x_v = 0 \text{ or } 1 \pmod{|G : \langle b, k \rangle|}\} \end{aligned} \quad (28)$$

is a component of H_i , and all other components are components of H_0 . The construction of H_i from H_{i-1} is the same as in subcase 4.1, but with $2i$ replaced by $2i - 1$.

We now let $K_1 = H_{(m+1)/2}$ and inductively construct, for $1 \leq i \leq |G : \langle b, k \rangle|/2$, an element K_i of $\mathcal{E}$, such that

$$\begin{aligned} \{v \mid z_v = 0\} \\ \cup \{v \mid z_v = 1, x_v \equiv 0, 1, \dots, \text{ or } 2i - 1 \pmod{|G : \langle b, k \rangle|}\} \end{aligned} \quad (29)$$

is a single component of K_i . Namely, [9, Lem. 4.2] implies there is an element $K_i = K'_{i-1}$, such that $(2i - 2)a$, $(2i - 2)a + k$, and $(2i - 1)a + k$ are all in the same component of K_i . Then, for $i = |G : \langle b, k \rangle|/2$, we see that K_i is a hamiltonian cycle. $\square$

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