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Research Article

New Application of (G'/G)-Expansion Method for Generalized (2+1)-Dimensional Nonlinear Evolution Equations

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We established (G'/G)-expansion method for (2+1)-dimensional nonlinear evolution equations. This method was used to construct travelling wave solutions of (2+1)-dimensional nonlinear evolution equations. (2+1)-Dimensional breaking soliton equation, (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation, and (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation are chosen to illustrate the effectiveness of the method.

1. Introduction

In this work, we will study the generalized (2+1)-dimensional nonlinear evolution equations

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \tag{1}$$

where a and b are parameters, for example, namely, the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation for which $a = 4$ and $b = 2$, [1]:

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0, \tag{2}$$

and the (2+1)-dimensional breaking soliton equation for which $a = -4$ and $b = -2$, [2]:

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0, \tag{3}$$

and the (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation for which $a = 4$ and $b = 4$, [3]:

$$u_{xt} + 4u_x u_{xy} + 4u_{xx}u_y + u_{xxxy} = 0. \tag{4}$$

In this paper, we solve (1) by the (G'/G)-expansion method and obtain some exact and new solutions for (2), (3), and (4).

2. The (G'/G)-Expansion Method

In this section we describe the (G'/G)-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x, t is given by

$$P(u, u_t, u_x, u_{tx}, u_{xx}, \ldots) = 0, \tag{5}$$

where $u = u(x,t)$ and P is a polynomial of u and its derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'/G)-expansion method.

Firstly, suppose that

$$u(x,t) = u(\xi), \quad \xi = x + wt. \tag{6}$$

The traveling wave variable (6) permits reducing (5) to an ODE for $u = u(\xi)$

$$P(u, u', u'', u''', \ldots) = 0. \tag{7}$$

Secondly, suppose that the solution of (7) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{8}$$

where $G = G(\xi)$ satisfies the second-order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{9}$$

$\alpha_m, \ldots, \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (8) is also a polynomial in (G'/G), the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7).

Thirdly, substituting (8) into (7) and using the second-order linear ODE (9), collecting all terms with the same order of (G'/G) together, the left-hand side of (7) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m, \ldots, \lambda$ and μ.

Fourthly, assuming that the constants $\alpha_m, \ldots, \lambda$ and μ can be obtained by solving the algebraic equations in *Thirdly*. Since the general solutions of the second-order LODE (9) have been well known for us, then substituting $\alpha_m, \ldots, w$ and the general solutions of (9) into (8) we have traveling wave solutions of the nonlinear evolution equation (5) (for more details see [4–7]).

3. New Application of (G'/G)-Expansion Method

In this section we apply the (G'/G)-expansion method to the generalized (2+1)-dimensional nonlinear evolution equations

$$u_{xt} + a u_x u_{xy} + b u_{xx} u_y + u_{xxxy} = 0. \tag{10}$$

We introduce a transformation

$$u(x, y, t) = u(\xi), \quad \xi = x + y - ct. \tag{11}$$

By substituting (11) into (10), we have

$$-cu'' + a u' u'' + b u' u'' + u^{(4)} = 0, \tag{12}$$

or equivalently

$$-cu'' + (a+b) u' u'' + u^{(4)} = 0, \tag{13}$$

where prime denotes the differential with respect to ξ. Integrating (13) with respect to ξ and taking the integration constant as zero yields

$$-cu' + \frac{(a+b)}{2}(u')^2 + u^{(3)} = g, \tag{14}$$

in which (14) is obtained by integrating and neglecting the constant of integration and where prime denotes the derivative with respect to the same variable ξ and g is the integration constant that can be determined later.

Suppose that the solutions of the O.D.E (14) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{15}$$

where a_i are constants, $G = G(\xi)$ satisfies the second-order LODE in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{16}$$

where λ and μ are constants.

By balancing the order of $(u')^2$ and $u^{(3)}$ in (11), we have $2m + 2 = m + 3$, then $m = 1$. So we can write

$$u(\xi) = a_1 \left(\frac{G'}{G}\right) + a_0, \quad a_1 \neq 0, \tag{17}$$

where a_1, a_0 are constants to be determined later. Then it follows

$$u'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu, \tag{18}$$

$$u''(\xi) = 2a_1 \left(\frac{G'}{G}\right)^3 + 3a_1 \lambda \left(\frac{G'}{G}\right)^2 + a_1 \left(\lambda^2 - 2\mu\right) \left(\frac{G'}{G}\right) + a_1 \lambda \mu, \tag{19}$$

$$\begin{aligned} u'''(\xi) = & -6a_1 \left(\frac{G'}{G}\right)^4 - 12a_1 \lambda \left(\frac{G'}{G}\right)^3 \\ & - \left(7a_1 \lambda^2 + 8a_1 \mu\right) \left(\frac{G'}{G}\right)^2 \\ & - \left(a_1 \lambda^3 + 8a_1 \lambda \mu\right) \left(\frac{G'}{G}\right) - a_1 \lambda^2 \mu - 2a_1 \mu^2. \end{aligned} \tag{20}$$

Substituting (18), (19), and (20) into (14) and collecting all the terms with the same power of (G'/G) together, equating each coefficient to zero yields a set of simultaneous algebraic equations as follows:

$$\begin{aligned} & \left(\frac{G'}{G}\right)^4 : a_1 (a+b) - 12 = 0, \\ & \left(\frac{G'}{G}\right)^3 : 2a_1 (a+b) \lambda - 24\lambda = 0, \\ & \left(\frac{G'}{G}\right)^2 : 2c + 2a_1 (a+b) \mu - 14\lambda^2 + a_1 (a+b) \lambda^2 - 16\mu = 0, \\ & \left(\frac{G'}{G}\right)^1 : 2a_1 (a+b) \lambda\mu - 2\lambda^3 + 2c\lambda - 16\lambda\mu = 0, \\ & \left(\frac{G'}{G}\right)^0 : a_1 (a+b) \mu^2 + 2c\mu - 2\lambda^2 \mu - 4\mu^2 - g = 0. \end{aligned} \tag{21}$$

Solving the algebraic equations above by using the Maple, we get

$$a_0 = a_0, \quad a_1 = \frac{12}{a+b}, \quad g = 0, \quad c = \lambda^2 - 4\mu, \tag{22}$$

where c, λ, μ are arbitrary constants.

Substituting (22) into (17) and by using (11), we have

$$u(\xi) = \frac{12}{a+b}\left(\frac{G'}{G}\right) + a_0, \tag{23}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$.

Substituting the general solutions of (16) into (23), we obtain the following.

Case 1. When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{6\sqrt{\lambda^2 - 4\mu}}{a+b} \times \left(\frac{C_1 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh(1/2)\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi} - \lambda\right) + a_0, \tag{24}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{6\sqrt{4\mu - \lambda^2}}{a+b} \times \left(\frac{-C_1 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2 \cos(1/2)\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi} - \lambda\right) + a_0, \tag{25}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 3. When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{12}{a+b}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) + a_0, \tag{26}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then $u_1(\xi)$ becomes

$$u_{11}(\xi) = \frac{6\sqrt{\lambda^2 - 4\mu}}{a+b}\left(\tanh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0 \tag{27}$$

and $u_2(\xi)$ becomes

$$u_{21}(\xi) = -\frac{6\sqrt{4\mu - \lambda^2}}{a+b}\left(\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + \lambda\right) + a_0. \tag{28}$$

When we take $C_1 = 0$ and $C_2 \neq 0$, then $u_1(\xi)$ becomes

$$u_{12}(\xi) = \frac{6\sqrt{\lambda^2 - 4\mu}}{a+b}\left(\coth\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0 \tag{29}$$

and $u_2(\xi)$ becomes

$$u_{22}(\xi) = \frac{6\sqrt{4\mu - \lambda^2}}{\alpha+b}\left(\cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi - \lambda\right) + a_0. \tag{30}$$

In the following Sections 4–6, we will apply solutions $u_i(\xi)$, $i = 1, 2, 3$ to (2+1)-dimensional breaking soliton equation and (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation and (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation.

4. Exact Solution of (2)

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation given in [1]

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0. \tag{31}$$

By using Section 3, we have following exact solution.

Case 1. When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh(1/2)\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi} - \lambda\right) + a_0, \tag{32}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \sqrt{\lambda^2 - 4\mu} \times \left(\frac{-C_1 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2 \cos(1/2)\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi} - \lambda\right) + a_0, \tag{33}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 3. When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = 2\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) + a_0, \tag{34}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then $u_1(\xi)$ becomes

$$u_{11}(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\tanh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0, \tag{35}$$

and $u_2(\xi)$ becomes

$$u_{21}(\xi) = -\sqrt{\lambda^2 - 4\mu}\left(\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + \lambda\right) + a_0 \quad (36)$$

when we take $C_1 = 0$ and $C_2 \neq 0$, then $u_1(\xi)$ becomes

$$u_{12}(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\coth\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0 \quad (37)$$

and $u_2(\xi)$ becomes

$$u_{22}(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi - \lambda\right) + a_0. \quad (38)$$

5. Exact Solution of (3)

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Breaking soliton equation given in [2]

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0. \quad (39)$$

By using Section 3, we have following exact solution:

Case 1. When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = -\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2\cosh(1/2)\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2\sinh(1/2)\sqrt{\lambda^2-4\mu}\xi} - \lambda\right) + a_0, \quad (40)$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = -\sqrt{\lambda^2 - 4\mu} \times \left(\frac{-C_1 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2\cos(1/2)\sqrt{4\mu-\lambda^2}\xi}{C_1\cos(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2\sin(1/2)\sqrt{4\mu-\lambda^2}\xi} - \lambda\right) + a_0, \quad (41)$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 3. When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = -2\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) + a_0, \quad (42)$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then $u_1(\xi)$ becomes

$$u_{11}(\xi) = -\sqrt{\lambda^2 - 4\mu}\left(\tanh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0, \quad (43)$$

and $u_2(\xi)$ becomes

$$u_{21}(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + \lambda\right) + a_0. \quad (44)$$

When we take $C_1 = 0$ and $C_2 \neq 0$, then $u_1(\xi)$ becomes

$$u_{12}(\xi) = -\sqrt{\lambda^2 - 4\mu}\left(\coth\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0, \quad (45)$$

and $u_2(\xi)$ becomes

$$u_{22}(\xi) = -\sqrt{\lambda^2 - 4\mu}\left(\cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi - \lambda\right) + a_0. \quad (46)$$

6. Exact Solution of (4)

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation given in [3]:

$$u_{xt} + 4u_x u_{xy} + 4u_{xx}u_y + u_{xxxy} = 0. \quad (47)$$

By using Section 3, we have following exact solutions.

Case 1. When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{3\sqrt{\lambda^2 - 4\mu}}{4} \times \left(\frac{C_1 \sinh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2\cosh(1/2)\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh(1/2)\sqrt{\lambda^2-4\mu}\xi + C_2\sinh(1/2)\sqrt{\lambda^2-4\mu}\xi} - \lambda\right) + a_0, \quad (48)$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{3\sqrt{4\mu - \lambda^2}}{4} \times \left(\frac{-C_1 \sin(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2\cos(1/2)\sqrt{4\mu-\lambda^2}\xi}{C_1\cos(1/2)\sqrt{4\mu-\lambda^2}\xi + C_2\sin(1/2)\sqrt{4\mu-\lambda^2}\xi} - \lambda\right) + a_0, \quad (49)$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

Case 3. When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{3}{2}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) + a_0, \tag{50}$$

where $\xi = x + y - (\lambda^2 - 4\mu)t$ and $C_1, C_2, c, \lambda, \mu$ are arbitrary constant.

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then $u_1(\xi)$ becomes

$$u_{11}(\xi) = \frac{3\sqrt{\lambda^2 - 4\mu}}{4}\left(\tanh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0, \tag{51}$$

and $u_2(\xi)$ becomes

$$u_{21}(\xi) = -\frac{3\sqrt{4\mu - \lambda^2}}{4}\left(\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + \lambda\right) + a_0. \tag{52}$$

When we take $C_1 = 0$ and $C_2 \neq 0$, then $u_1(\xi)$ becomes

$$u_{12}(\xi) = \frac{3\sqrt{\lambda^2 - 4\mu}}{4}\left(\coth\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \lambda\right) + a_0, \tag{53}$$

and $u_2(\xi)$ becomes

$$u_{22}(\xi) = \frac{3\sqrt{4\mu - \lambda^2}}{4}\left(\cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi - \lambda\right) + a_0. \tag{54}$$

7. Conclusions

In this paper, by using the (G'/G)-expansion method, we obtained some explicit formulas of solutions for the generalized (2+1)-dimensional nonlinear evolution equations. We chose (2+1)-dimensional breaking soliton equation and (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation and (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation to illustrate the effectiveness of the method. Those solutions were similar to the solutions obtained in other paper. The study reveals the power of the method.

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