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Research Article

On Generalized Derivations in Nearrings

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The aim of this paper is to investigate some results of nearrings satisfying certain identities involving generalized derivations. Furthermore, we give some examples to demonstrate the restrictions imposed on the hypothesis of various results which are not superfluous.

1. Introduction

The study of derivations in rings goes back to 1957 when Posner [1] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Many results in this vein were obtained by a number of authors [2–18] in several ways. In view of [19], the concept of generalized derivation is introduced by Hvala [20]. Familiar examples of generalized derivations are derivations and generalized inner derivations, and later includes left multiplier, that is, an additive mapping $F : R \to R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + D(x)$, where c is fixed element of R and D a derivation of R, is a generalized derivation; and if R has 1, all generalized derivations have this form.

Throughout the paper, N will denote a zero symmetric right abelian nearring with multiplicative center $Z(N)$. For all $x, y \in N$, as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie and Jordan products, respectively. A nonempty subset U of N will be called a semigroup right ideal (resp., semigroup left ideal) if $UN \subset U$ ($NU \subset U$). Finally, U is called a semigroup ideal if it is a right as well as a left semigroup ideal. A nearring N is called prime, if $aNb = \{0\} \Rightarrow a = 0$ or $b = 0$ for all $a, b \in N$. We refer to Pilz [21] for the basics definitions and properties of nearring.

As noted in [22], an additive mapping $D : N \to N$ is called a derivation of N if $D(xy) = xD(y) + D(x)y$ holds for all $x, y \in N$. An additive mapping $F : N \to N$ is said to be a right generalized derivation associated with D if

$$F(xy) = F(x)y + xD(y) \quad \forall x, y \in N, \tag{1}$$

and is said to be a left generalized derivation associated with D if

$$F(xy) = xF(y) + D(x)y \quad \forall x, y \in N. \tag{2}$$

F is said to be a generalized derivation associated with D if it is a right as well as a left generalized derivation associated with D.

The purpose of this note is to prove some results which are of independent interest and related to generalized derivations on nearrings.

2. Ideals and Generalized Derivations in Nearrings

Over the last several years, many authors [7, 19, 20, 23] studied the commutativity in prime and semiprime rings admitting derivations and generalized derivations. On other hand, there are several results asserting that prime nearrings with certain constrained derivations have ring-like behavior. It is natural to look for comparable results on nearrings, and this has been done [22, 24–26]. In this section, we investigate some results of nearrings satisfying certain identities involving generalized derivation.

In order to prove our theorems, we will make extensive use of the following lemma.

Lemma A. *If N is prime and F a generalized derivation on N associated with D of N, then*

$$a(bF(c)+D(b)c)=abF(c)+aD(b)c,\quad \forall a,b,c\in N. \tag{3}$$

Proof. Clearly $F(a(bc)) = aF(bc) + D(a)bc = a(bF(c) + D(b)c) + D(a)bc$, and, also, we obtain $F((ab)c) = abF(c) + D(ab)c = abF(c) + aD(b)c + D(a)bc$. Comparing these two expressions for $f(abc)$ gives the desired conclusion. □

Lemma B (see [25, Lemma 1.5]). *Let N be a prime nearring and let $U\neq\{0\}$ be a semigroup ideal of N. If $U\subset Z(N)$, then N is commutative.*

Lemma C (see [25, Lemma 1.4]). *Let N be a prime nearring and let $U\neq\{0\}$ be a semigroup ideal of N. If x is an element of N such that $xU=0$ or $Ux=0$, then $x=0$.*

Lemma D. *Let N be a prime nearring and $U\neq\{0\}$ a semigroup ideal of N. If D is a derivation on N such that $D(U)=0$, then $D=0$.*

Proof. From hypothesis, we get

$$\begin{aligned}0&=D(ux)=uD(x)+D(u)(x)\\&=uD(x),\quad\forall u\in U,\ x\in N.\end{aligned} \tag{4}$$

That is,

$$UD(x)=0,\quad\forall x\in N. \tag{5}$$

Thus, we then concluded the required result by Lemma C. □

Theorem 1. *Let N be a noncommutative prime nearring, U a nonzero semigroup ideal of N, and $F\neq 0$ a generalized derivation associated with D of N such that $F[x,y]-[x,y]=0$, for all $x,y\in U$. Then F is trivial.*

Proof. From hypothesis, we have

$$F[x,y]=[x,y],\quad\forall x,y\in U. \tag{6}$$

Replacing y with yx in (6) and using it, we get

$$xyD(x)=yxD(x),\quad\forall x,y\in U. \tag{7}$$

Again replacing y with nz in (7) and using it, we obtain

$$[x,n]zD(x)=\{0\},\quad\forall x,z\in U,\ n\in N; \tag{8}$$

that is,

$$[x,n]UD(x)=\{0\},\quad\forall x\in U,\ n\in N. \tag{9}$$

It follows from Lemma C that either $[x,n]=0$ or $D(x)=0$, for all $x\in U$, $n\in N$. Therefore, in view of hypothesis and from Lemma B, we are forced to consider later case $D(U)=0$ and so $D=0$ by Lemma D. Hence, our hypothesis becomes

$$\begin{aligned}F(xy)-F(yx)&=xy-yx,\\(F(x)y-xy)&=(F(y)x-yx),\\(F(x)-x)y&=(F(y)-y)x.\end{aligned} \tag{10}$$

Let $\mathbb{B}(x)=F(x)-x$, for all $x\in U$, and so $\mathbb{B}(xy)=\mathbb{B}(x)y$, for all $x,y\in U$. Then, the last equality can be rewritten as

$$\mathbb{B}(x)y=\mathbb{B}(y)x,\quad\forall x,y\in U. \tag{11}$$

Taking zn' instead of x in (11) and using Lemma A, we find

$$\mathbb{B}(z)[y,n']=0,\quad\forall y,z\in U,\ n'\in N. \tag{12}$$

Replacing z with yn in the last equality, we obtain

$$\mathbb{B}(y)n[y,n]=0,\quad\forall y\in U,\ n,n'\in N. \tag{13}$$

It implies that

$$\mathbb{B}(y)N[y,n']=\{0\},\quad\forall y\in U,\ n\in N. \tag{14}$$

Thus, we then concluded, by the primeness of N, that either $\mathbb{B}(y)=0$ or $[y,n]$ for all $y\in U$, $n\in N$, that is, $U\subset Z(N)$. If $\mathbb{B}(y)\neq 0$, then we conclude that N is commutative by Lemma B, which is a contradiction. Hence, this completes the proof. □

A slight modification in the proof of Theorem yields the following.

Theorem 2. *Let N be a noncommutative prime nearring, U a nonzero semigroup ideal of N, and N admit a generalized derivation F associated with D such that $F[x,y]+[x,y]=0$, for all $x,y\in U$. Then F is trivial.*

As a consequence of above theorems, we obtain the following remarks.

Remark 3. Suppose N is a prime nearring and U a nonzero semigroup ideal of N. If N admits a generalized derivation F associated with D such that $F[x,y]+[x,y]=0$, for all $x,y\in U$ or $F[x,y]-[x,y]=0$, for all $x,y\in U$, then N is commutative or F is trivial.

Remark 4. Suppose N is a prime nearring. If N admits a generalized derivation F associated with $D\neq 0$ of N such that $F[x,y]-[x,y]=0$, for all $x,y\in N$ or $F[x,y]+[x,y]=0$, for all $x,y\in N$, then N is commutative.

Proof. If $F=0$, then we have the desired conclusion. Now, we consider $F\neq 0$, and following the same technique as in the proof of Theorem 1, we reach $[x,y]D(x)=0$, for all $x,y\in N$. Taking yx instead of y in the last relation, we obtain $[x,y]ND(x)=\{0\}$, for all $x,y\in N$. Since N is prime, we obtain the required result by hypothesis.

Similar argument can be adapted in the case $F[x,y]+[x,y]=0$ for all $x,y\in N$, and we can omit the similar proof. □

Here, we try to construct an example to demonstrates that the above result do not hold for arbitrary rings.

Example 5. Let $N = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) : a, b, c \in R\}$, where R is a commutative ring and $U = \{\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) : a, b \in R\}$. We define a map $F : N \rightarrow N$ by $F\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ a+b & 0 \end{smallmatrix}\right)$; then it is easy to check that F is a generalized derivation associated with D, where $D : N \rightarrow N$ define as $D\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$ on N. However, F satisfies the properties of Theorems 1 and 2 and Remarks 3 and 4, but neither N is commutative nor F is trivial.

Remark 6. In Remark 4, the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a nearring. Then the same proof will lead to the conclusion that N is commutative.

Conclusion of Theorems 1 and 2 is still hold if we replace the product $[x, y]$ by $x \circ y$. In fact, we obtain the following results.

Theorem 7. *Let N be a noncommutative prime nearring, U a nonzero semigroup ideal of N, and N admit a generalized derivation F associated with D such that $F(x \circ y) - x \circ y = 0$, for all $x, y \in U$. Then F is trivial.*

Proof. From hypothesis, we have

$$F(x)y + xD(y) + F(y)x + yD(x) - x \circ y = 0, \quad \forall x, y \in U. \tag{15}$$

Replacing y by yx in (15) and using it, we obtain that

$$xyD(x) = -yxD(x), \quad \forall x, y \in U. \tag{16}$$

Replacing y by nz in the last expression and using it, we reach $[x, n]zD(x) = 0$ for all $x, z \in U$, $n \in N$, that is,

$$[x, n]UD(x) = \{0\}, \quad \forall x \in U, \ n \in N. \tag{17}$$

Equation (17) is the same as (9) in Theorem 1. Thus, by same argument of Theorem 1, we can conclude the result here. □

Proceeding the same line as above with necessary variation, we can prove the following.

Theorem 8. *Let N be a noncommutative prime nearring, U a nonzero semigroup ideal of N, and admit a generalized derivation F associated with D such that $F(x \circ y) + x \circ y = 0$, for all $x, y \in U$. Then F is trivial.*

As a consequence of above theorems, we obtain the following remarks.

Remark 9. Suppose N is a prime nearring and U a nonzero semigroup ideal of N. If N admits a generalized derivation F associated with D such that $F(x \circ y) - (x \circ y) = 0$, for all $x, y \in N$ or $F(x \circ y) + (x \circ y) = 0$, for all $x, y \in N$, then N is commutative or F is trivial.

Remark 10. Suppose N is a prime nearring. If N admits a generalized derivation F associated with $D \neq 0$ such that $F(x \circ y) - (x \circ y) = 0$, for all $x, y \in N$ or $F(x \circ y) + (x \circ y) = 0$, for all $x, y \in N$, then N is commutative.

Proof. For any $x, y \in N$, we have $F(x \circ y) - (x \circ y) = 0$, and following the same technique as in proof of Theorem 7, we reach $(x \circ y)D(x) = 0$, for all $x, y \in N$. Taking yz instead of y in last relation and using it, we obtain $[x, y]zD(x) = 0$, for all $x, y, z \in N$. This implies that $[x, y]ND(x) = \{0\}$, for all $x, y \in N$. Due to hypothesis and primeness of N, we get the required result.

Similar results hold in case $F(x \circ y) + (x \circ y) = 0$, for all $x, y \in N$. □

The following example demonstrates that the above results do not hold for arbitrary rings.

Example 11. Let $N = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) : a, b, c \in R\}$ and $U = \{\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) : b \in R\}$ a nonzero ideal of N, where R is a noncommutative ring with condition $a^2 = 0$, for all $a \in R$. We define a map $F : N \rightarrow N$ by $F\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ a+b & 0 \end{smallmatrix}\right)$. Then, it is easy to check that F is a generalized derivation associated with D, where $D : N \rightarrow N$ is defined as $D\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$ on N. However, F satisfies the properties of Theorems 7 and 8 and Remarks 9 and 10, but neither N is commutative nor F is trivial.

Remark 12. In Remark 10, the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a nearring. Then, the same proof will lead to the conclusion that N is commutative.

In conclusion of our paper, it would be interesting to prove or disprove the following problem.

Problem 13. Let $n > 1$ be a fixed positive number. Suppose N is a prime nearring. Let U be a nonzero ideal of N and let N admit a generalized derivation F associated with a derivation $D \neq 0$.

(i) Does the condition that $F^n[x, y] + [x, y] = 0$, for all $x, y \in N$ or $F^n[x, y] - [x, y] = 0$, for all $x, y \in N$ imply that N is commutative?

(ii) Does the condition that $F^n(x \circ y) + (x \circ y) = 0$, for all $x, y \in N$ or $F^n(x \circ y) - (x \circ y) = 0$, for all $x, y \in N$ imply that N is commutative?

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