

Research Article

A Note on Generalized Hardy-Sobolev Inequalities

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We are concerned with finding a class of weight functions g so that the following generalized Hardy-Sobolev inequality holds: $\int_{\Omega} gu^2 \leq C \int_{\Omega} |\nabla u|^2$, $u \in H_0^1(\Omega)$, for some $C > 0$, where Ω is a bounded domain in \mathbb{R}^2 . By making use of Muckenhoupt condition for the one-dimensional weighted Hardy inequalities, we identify a rearrangement invariant Banach function space so that the previous integral inequality holds for all weight functions in it. For weights in a subspace of this space, we show that the best constant in the previous inequality is attained. Our method gives an alternate way of proving the Moser-Trudinger embedding and its refinement due to Hansson.

1. Introduction

In this paper, we are interested to find the best suitable function space for the weights g so that the following generalized Hardy-Sobolev inequality holds:

$$\int_{\Omega} gu^2 \leq C \int_{\Omega} |\nabla u|^2, \quad u \in H_0^1(\Omega), \quad (1)$$

for some $C > 0$, where Ω is a bounded domain in \mathbb{R}^N . We say that g is admissible, if the previous inequality holds. We are also interested to find a class of admissible functions that ensures the best constant in (1) which is attained for some $u \in H_0^1(\Omega)$.

Let us first recall the classical Hardy inequality:

$$\int_0^\infty \left[\frac{1}{x} \int_0^x f(s) ds \right]^2 dx \leq C_2 \int_0^\infty f(x)^2 dx, \quad f \geq 0. \quad (2)$$

By taking $f = u'$ with $u(0) = 0$ in (2), we get

$$\int_0^\infty \frac{1}{x^2} u(x)^2 dx \leq C_2 \int_0^\infty |u'(x)|^2 dx. \quad (3)$$

The higher dimensional analogue of the previous inequality is referred to as the Hardy-Sobolev inequality in the literature:

$$\int_{\Omega} \frac{1}{|x|^2} u(x)^2 dx \leq C_2 \int_{\Omega} |\nabla u(x)|^2 dx, \quad u \in H_0^1(\Omega), \quad (4)$$

where $\Omega \subset \mathbb{R}^N$ is a domain containing the origin with $N \geq 3$. Clearly (4) does not hold when $N = 1$ or 2 , since $1/|x|^2$ is not locally integrable for Ω that contains the origin.

For $N \geq 3$, the Hardy-Sobolev inequality is generalized mainly in two directions, namely, the generalized Hardy-Sobolev inequalities and the improved Hardy-Sobolev inequalities. The first one refers to the inequalities of the form (1) for more general weights instead of the homogeneous weight $1/|x|^2$. The second one relies on the fact that the best constant in (4) is not attained in $H_0^1(\Omega)$ and hence one anticipates to improve (4) by adding nonnegative terms in the left-hand side. The first major improvement in the Hardy-Sobolev inequality is obtained by Brézis and Vázquez in [1] who have proved the following inequality:

$$\begin{aligned} & \left(\frac{N-2}{2} \right)^2 \int_{\Omega} \frac{1}{|x|^2} u(x)^2 dx + \lambda_{\Omega} \int_{\Omega} u(x)^2 dx \\ & \leq \int_{\Omega} |\nabla u(x)|^2 dx, \quad u \in H_0^1(\Omega). \end{aligned} \quad (5)$$

Motivated by the previous inequality, several improved Hardy-Sobolev inequalities have been proved, for example see [2–5].

For $N = 2$, as we pointed out before, the Hardy potential $1/|x|^2$ is not admissible for any domain in \mathbb{R}^2 that contains the origin. In [6], Leray showed that $1/|x|^2 |\log(x)|^2$ is the right admissible function (analogous to Hardy potential) for

$\Omega = B(0; 1) \subset \mathbb{R}^2$. In this paper, we focus on finding a large class of admissible functions including that of Leray's function or its improvements (by adding nonnegative terms) for the generalized Hardy-Sobolev inequalities in bounded domains of \mathbb{R}^2 .

The most general sufficient condition (for any dimension) for the generalized Hardy-Sobolev inequalities is given by Mažja [7], in terms of the capacity. We recall that for a compact set $E \subset \Omega$, the relative capacity of E with respect to Ω is defined as

$$\text{cap}(E; \Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^2 : u \in \mathcal{C}_c^\infty(\Omega), u|_E \geq 1 \right\}. \quad (6)$$

First, let us see that Mažja's capacity condition is very much intrinsic on (1). Let g be a positive weight satisfying (1), then for any compact subset Ω ,

$$\begin{aligned} \int_E g(x) dx &\leq \int_{\Omega} gu^2 \\ &\leq C \int_{\Omega} |\nabla u|^2, \quad u \in \mathcal{C}_c^\infty(\Omega), u|_E \geq 1. \end{aligned} \quad (7)$$

By taking the infimum, we get $\int_E g(x) dx \leq C \text{cap}(E; \Omega)$. Therefore

$$\sup_{E \subset \Omega} \frac{\int_E g(x) dx}{\text{cap}(E; \Omega)} \leq C. \quad (8)$$

Mažja proved that the previous condition is indeed sufficient for (1) (Theorem 1/2.4.1, page 128 of [7]). Since Mažja's condition is necessary and sufficient, all the improved Hardy-Sobolev inequalities follow directly from Mažja's result. However, verifying Mažja's capacity condition for a general weight function is not an easy task. Thus it is of interest to find certain verifiable conditions for the generalized Hardy-Sobolev inequalities by other means.

One such verifiable condition for $N \geq 3$ is obtained by Visciglia in [8]. He proved that (1) holds for weights in the Lorentz space $L(N/2, \infty)$. The embedding of $H_0^1(\Omega)$ into the Lorentz space $L(2^*, 2)$ played a key role in his result. The case $N = 2$ is more subtle, for example, (1) does not hold when $\Omega = \mathbb{R}^2$ and $\int_{\mathbb{R}^2} g > 0$, see [9]. In this paper we obtain a verifiable condition for admissible functions for bounded domains in \mathbb{R}^2 , using one-dimensional weighted Hardy inequalities and certain rearrangement inequalities.

A general one dimensional weighted Hardy inequality has the following form:

$$\begin{aligned} &\left(\int_a^b \left[\int_0^x f(s) ds \right]^q u(x) dx \right)^{1/q} \\ &\leq C_{p,q} \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}, \quad f \geq 0. \end{aligned} \quad (9)$$

For an excellent review on the weighted Hardy inequalities, we refer to [10] by Maligranda et al. Many necessary and sufficient conditions on u, v, p, q for holding (9) are available in the literature, see [11–13]. Here we make use of the so

called *Muckenhoupt condition* [13] for obtaining a class of weight functions satisfying (1). For a measurable function g , we denote its decreasing rearrangement by g^* and the maximal function of g^* is denoted by g^{**} , that is, $g^{**}(t) = (1/t) \int_0^t g^*(s) ds$. Now we define

$$\mathcal{M} \log L$$

$$:= \left\{ g \text{ measurable} : \sup_{0 < t < |\Omega|} t \log \left(\frac{|\Omega|}{t} \right) g^{**}(t) < \infty \right\}. \quad (10)$$

Indeed, $\mathcal{M} \log L$ is a rearrangement invariant Banach function space with the norm

$$\|g\|_{\mathcal{M} \log L} = \sup_{0 < t < |\Omega|} t \log \left(\frac{|\Omega|}{t} \right) g^{**}(t), \quad (11)$$

see [14]. More details of the space $\mathcal{M} \log L$ are given in Section 3. Now we state our main theorem.

Theorem 1. *Let Ω be a bounded domain in \mathbb{R}^2 and let $g \in \mathcal{M} \log L$. Then g is admissible and*

$$\int_{\Omega} gu^2 \leq \frac{\|g\|_{\mathcal{M} \log L}}{\pi} \int_{\Omega} |\nabla u|^2, \quad \forall u \in H_0^1(\Omega). \quad (12)$$

Having obtained $\mathcal{M} \log L$, a class of admissible functions, one would like to know for which among them the best constant in (1) is attained in $H_0^1(\Omega)$. Many authors have addressed this question when $N \geq 3$, see [8, 15] and the references therein. For $N = 2$, Mažja has a sufficient condition (see 2.4.2 of [7]) in terms of capacity. Here we consider the weights in a subclass of $\mathcal{M} \log L$ so that the best constant in (1) is attained in $H_0^1(\Omega)$. For a bounded domain $\Omega \subset \mathbb{R}^2$ (analogous to space $\mathcal{F}_{N/2}$ in [15]) we define

$$\mathcal{F}_1 := \overline{\mathcal{C}_c^\infty(\Omega)} \text{ in } \mathcal{M} \log L. \quad (13)$$

We show that the best constant in (1) is attained in $H_0^1(\Omega)$, when $g^+ \in \mathcal{F}_1$, where g^+ is the positive part of g . More precisely, we have the following theorem.

Theorem 2. *Let $g \in \mathcal{M} \log L$ and $g^+ \in \mathcal{F}_1 \setminus \{0\}$. Define*

$$\lambda_1(g) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} gu^2} : u \in H_0^1(\Omega), \int_{\Omega} gu^2 > 0 \right\}. \quad (14)$$

Then $\lambda_1(g)$ is attained for some $u \in H_0^1(\Omega)$.

The Moser-Trudinger embedding of $H_0^1(\Omega)$ into the Orlicz space $L_A(\Omega)$, the Orlicz space generated by the Orlicz function $A(t) = e^{t^2} - 1$, can be used to show that $L \log L$ functions are admissible. An embedding of $H_0^1(\Omega)$, finer than that of Moser-Trudinger, is established independently by Hansson [16], Brézis, and Wainger [17]. As anticipated, this embedding gives a bigger class of admissible functions

than $L \log L$. In [18], the authors used this finer embedding and showed that the Lorentz-Zygmund space $L^{1,\infty}(\log L)^2$ is admissible. We are not going to use any embeddings for proving that $\mathcal{M} \log L$ functions are admissible, instead we use some rearrangement inequalities and one dimensional weighted Hardy inequalities. We would like to stress that the admissibility of $\mathcal{M} \log L$ functions can be used to give alternate proofs for the Moser-Trudinger embedding and its refinement due to Hansson.

The rest of the paper is organised as follows. In Section 2, we recall definition and properties of decreasing rearrangement. Further, we state some classical inequalities that will be used in the subsequent sections. We discuss the properties of the space $\mathcal{M} \log L$ and give examples of function spaces contained in $\mathcal{M} \log L$ in Section 3. In Section 4, we give a proof for Theorem 1. The last section contains a proof of Theorem 2.

2. Preliminaries

In this section, we recall the definition and some of the properties of symmetrization and certain inequalities concerning symmetrization that we use in the subsequent sections. For further details on symmetrization, we refer to the books [19–21].

Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $E_f(s) = \{x : |f(x)| > s\}$, $s > 0$.

Then the *distribution function* α_f of f is defined as

$$\alpha_f(s) := |E_f(s)|, \quad \text{for } s > 0, \quad (15)$$

where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$. Now we define the *decreasing rearrangement* f^* as

$$f^*(t) := \inf \{s > 0 : \alpha_f(s) \leq t\}, \quad t > 0. \quad (16)$$

The *Schwarz symmetrization* of f is given by

$$f^*(x) = f^*(\omega_N|x|^N), \quad \forall x \in \Omega^*, \quad (17)$$

where ω_N is the measure of unit ball in \mathbb{R}^N and Ω^* is the open ball, centered at the origin with same measure as Ω .

Next we give some inequalities concerning the distribution function and rearrangement of a function.

Proposition 3. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let f and g be measurable functions on Ω . Then,*

- (a) $f^*(\alpha_f(s)) \leq s$, $\alpha_f(f^*(t)) \leq t$;
- (b) $t < \alpha_f(s)$ if and only if $s < f^*(t)$.

The map $f \rightarrow f^*$ is not subadditive. However, we obtain a subadditive function from f^* , namely, the maximal function f^{**} of f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau, \quad t > 0. \quad (18)$$

The subadditivity of f^{**} with respect to f helps us to define norms in certain function spaces.

Finally, in the following proposition we state two important inequalities concerning Schwarz symmetrization (decreasing rearrangement).

Proposition 4. *Let Ω be a domain in \mathbb{R}^N with $N \geq 2$. Let f and g be two measurable functions on Ω and let $u \in H_0^1(\Omega)$. Then one has the following inequalities.*

- (a) *The Hardy-Littlewood inequality:*

$$\begin{aligned} \int_{\Omega} |f(x)g(x)| dx &\leq \int_{\Omega^*} f^*(x)g^*(x) dx \\ &= \int_0^{|\Omega|} f^*(t)g^*(t) dt. \end{aligned} \quad (19)$$

- (b) *The Polya-Szegö inequality:*

$$\begin{aligned} N^2 \omega_N^{2/N} \int_0^{|\Omega|} t^{2-2/N} [u^{*'}(t)]^2 dt &= \int_{\Omega^*} |\nabla u^*(x)|^2 dx \\ &\leq \int_{\Omega} |\nabla u(x)|^2 dx. \end{aligned} \quad (20)$$

Next we state a necessary and sufficient condition for one dimensional weighted Hardy inequality due to Muckenhoupt (see 4.17, [10]).

Proposition 5. *Let u and v be nonnegative measurable functions such that $v > 0$. Let $1 < p < \infty$ and let p' be the conjugate exponent of p . Then for any $a > 0$,*

$$\int_0^a \left| \int_t^a f(s) ds \right|^p u(t) dt \leq C \int_0^a |f(s)|^p v(s) ds, \quad (21)$$

holds for all measurable function f if and only if

$$A = \sup_{0 < t < a} \left(\int_0^t u(s) ds \right)^{1/p} \left(\int_t^a v(s)^{1-p'} ds \right)^{1/p'} < \infty. \quad (22)$$

Moreover, if C_b is the best constant in (21), then

$$A \leq C_b \leq p^{1/p} [p']^{1/p'} A. \quad (23)$$

3. A Space for Admissible Functions

In this section, we define the function space $\mathcal{M} \log L$ and give its relation with certain classical function spaces. For a bounded domain $\Omega \subset \mathbb{R}^2$, we define

$$\mathcal{M} \log L$$

$$:= \left\{ u \text{ measurable} : \sup_{0 < t < |\Omega|} u^{**}(t) t \log \left(\frac{|\Omega|}{t} \right) < \infty \right\}. \quad (24)$$

One can verify that $\mathcal{M} \log L$ is a Banach function space with the norm

$$\|u\|_{\mathcal{M} \log L} := \sup_{0 < t < |\Omega|} u^{**}(t) t \log \left(\frac{|\Omega|}{t} \right). \quad (25)$$

The nomenclature refers to the fact that $\mathcal{M}\log L$ is the maximal rearrangement invariant Banach function space (Lorentz M -space) on Ω with the fundamental function $t\log(|\Omega|/t)$. Next we give some examples of function spaces in $\mathcal{M}\log L$.

Recall, for a bounded domain $\Omega \subset \mathbb{R}^2$, $L\log L$ is the Orlicz space generated by the Orlicz function $t\log^+(t)$, that is,

$$L\log L := \left\{ u \text{ measurable} : \int_{\Omega} u(x) \log^+(u(x)) dx < \infty \right\}. \quad (26)$$

A similar analysis as in Lemma 6.2 of [14] gives

$$\begin{aligned} L\log L \\ = \left\{ u \text{ measurable} : \int_0^{|\Omega|} u^*(t) \log\left(\frac{|\Omega|}{t}\right) dt < \infty \right\}. \end{aligned} \quad (27)$$

Using this equivalent definition, we show that $L\log L$ is contained in $\mathcal{M}\log L$.

Proposition 6. Let Ω be a bounded domain in \mathbb{R}^2 . Then $L\log L \subset \mathcal{M}\log L$.

Proof. Let $f \in L\log L$, then using the definition of f^{**} and the monotonicity of $\log(|\Omega|/t)$ we obtain

$$\begin{aligned} f^{**}(t) t \log\left(\frac{|\Omega|}{t}\right) &= \log\left(\frac{|\Omega|}{t}\right) \int_0^t f^*(s) ds \\ &\leq \int_0^t \log\left(\frac{|\Omega|}{s}\right) f^*(s) ds. \end{aligned} \quad (28)$$

Now by taking the supremum over $t \in (0, |\Omega|)$ in the previous inequality, we obtain the desired fact. \square

This inclusion is strict as seen in the following example.

Example 7. Let $\Omega = B(0; 1) \subset \mathbb{R}^2$. For δ small, let

$$g(x) = \frac{\chi_{B(0; \delta)}(x)}{|x|^2 (\log(x))^2 (\log|\log(x)|)^{\beta}}. \quad (29)$$

Then for $0 < \beta \leq 1$, g does not belong to $L\log L$ but belongs to $\mathcal{M}\log L$.

For a bounded domain $\Omega \subset \mathbb{R}^2$, the Zygmund space $L^{1,\infty}(\log L)^2$ is defined as

$$L^{1,\infty}(\log L)^2$$

$$:= \left\{ f \text{ measurable} : \sup_{0 < t \leq |\Omega|} t \left[\log\left(\frac{e|\Omega|}{t}\right) \right]^2 f^*(t) < \infty \right\}. \quad (30)$$

From the following proposition we see that $L^{1,\infty}(\log L)^2$ is contained in $\mathcal{M}\log L$.

Proposition 8. Let Ω be a bounded domain in \mathbb{R}^2 . Then for a measurable function f , the following inequality holds:

$$\begin{aligned} f^{**}(t) t \log\left(\frac{|\Omega|}{t}\right) \\ \leq \sup_{0 < s \leq t} f^*(s) s \left[\log\left(\frac{|\Omega|}{s}\right) \right]^2, \quad t \in (0, |\Omega|). \end{aligned} \quad (31)$$

Proof. Let $t \in (0, |\Omega|)$. Then

$$\begin{aligned} f^{**}(t) t \log\left(\frac{|\Omega|}{t}\right) &= \log\left(\frac{|\Omega|}{t}\right) \int_0^t f^*(s) ds \\ &= \log\left(\frac{|\Omega|}{t}\right) \int_0^t f^*(s) s \\ &\quad \times \left[\log\left(\frac{|\Omega|}{s}\right) \right]^2 \frac{1}{s[\log(|\Omega|/s)]^2} ds \\ &\leq \sup_{0 < s \leq t} f^*(s) s \left[\log\left(\frac{|\Omega|}{s}\right) \right]^2, \end{aligned} \quad (32)$$

and the last inequality follows as $\log(|\Omega|/t) \int_0^t (1/(s[\log(|\Omega|/s)]^2)) ds = 1$. \square

Remark 9. In [18], using Hansson's embedding, the authors showed that $L^{1,\infty}(\log L)^2$ functions are admissible. Since $L^{1,\infty}(\log L)^2 \subset \mathcal{M}\log L$, their result also follows from Theorem 1 without using Hansson's embedding. Since $\mathcal{M}\log L$ functions are admissible, Theorem 2.5 of [22] shows that the spaces $L^{1,\infty}(\log L)^2$ and $\mathcal{M}\log L$ are equivalent. Thus, Theorem 1 indeed follows from Hansson's embedding as in [18]. However, our proof for Theorem 1 relies only on certain classical rearrangement inequalities and the Muckenhoupt condition for 1-dimensional weighted Hardy inequalities.

In the next proposition, we show that the weights considered in [2] for the improved Hardy-Sobolev inequalities are in $L^{1,\infty}(\log L)^2$ and hence belong to $\mathcal{M}\log L$ as well.

Proposition 10. Let Ω be a bounded domain in \mathbb{R}^2 and let $R \geq \sup_{x \in \Omega} |x| e$. Then for $\gamma \geq 2$,

$$\frac{1}{|x|^2 [\log(R/|x|)]^{\gamma}} \in L^{1,\infty}(\log L)^2. \quad (33)$$

Proof. Let $g(x) = \{|x|^2 [\log(R/|x|)]^{\gamma}\}^{-1}$, $x \in B(0; R)$, and $f(x) = g(x)\chi_{\Omega}(x)$. Note that $f(x) \leq g(x)$, $x \in B(0; R)$, and hence

$$f^*(t) \leq g^*(t), \quad t \in (0, R). \quad (34)$$

A straightforward calculation gives

$$\begin{aligned} g^*(t) &= \pi \left\{ t \left[\log \left(\sqrt{\frac{R^2 \pi}{t}} \right) \right]^\gamma \right\}^{-1} \\ &= 2^\gamma \pi \left\{ t \left[\log \left(\frac{R^2 \pi}{t} \right) \right]^\gamma \right\}^{-1} \\ &\leq 2^\gamma \pi \left\{ t \left[\log \left(\frac{e|\Omega|}{t} \right) \right]^\gamma \right\}^{-1}, \end{aligned} \quad (35)$$

where the last inequality follows since $R \geq \sup_{x \in \Omega} |x| e$. Therefore

$$t \left[\log \left(\frac{e|\Omega|}{t} \right) \right]^2 g^*(t) \leq 2^\gamma \pi \left[\log \left(\frac{e|\Omega|}{t} \right) \right]^{2-\gamma}. \quad (36)$$

Hence the proof. \square

Next we verify that weights in $\mathcal{M} \log L$ satisfy Mažja's capacity condition.

Proposition 11. *Let Ω be a bounded domain in \mathbb{R}^2 , $g \in \mathcal{M} \log L$, and $g \geq 0$. Then g satisfies Mažja's capacity condition, that is,*

$$\sup_{K \subset \subset \Omega} \frac{\int_K g dx}{\text{cap}(K, \Omega)} < \infty. \quad (37)$$

Proof. Using Polya-Szegö inequality, we can easily verify that $\text{cap}(K^*, \Omega^*) \leq \text{cap}(K, \Omega)$. Further, $\text{cap}(K^*, \Omega^*) = (\pi/2) \log(|\Omega|/|K|)^{-1}$ (see, e.g., page 106 of [7]). Now for a compact set K ,

$$\begin{aligned} \frac{\int_K g(x) dx}{\text{cap}(K, \Omega)} &\leq \frac{2}{\pi} \log \left(\frac{|\Omega|}{|K|} \right) \int_{K^*} g^*(x) dx \\ &\leq \frac{2}{\pi} \log \left(\frac{|\Omega|}{|K|} \right) \int_0^{|K|} g^*(s) ds. \end{aligned} \quad (38)$$

Therefore,

$$\begin{aligned} \sup_{K \subset \subset \Omega} \frac{\int_K g(x) dx}{\text{cap}(K, \Omega)} &\leq \frac{2}{\pi} \sup_{0 < t < |\Omega|} \log \left(\frac{|\Omega|}{t} \right) \int_0^t g^*(s) ds \\ &= \frac{2}{\pi} \|g\|_{\mathcal{M} \log L} < \infty. \end{aligned} \quad (39)$$

\square

In the next section, we see that the space $\mathcal{M} \log L$ almost characterizes the radial weights satisfying Mažja's condition (Theorem 13).

4. The Generalized Hardy-Sobolev Inequalities

In this section, we give a proof for Theorem 1. Further, when $\Omega = B(0; r)$ for some $r > 0$, we show that all radial and radially decreasing admissible weights necessarily lie in $\mathcal{M} \log L$. First we have the following theorem on one dimensional weighted Hardy inequalities.

Theorem 12. *Let Ω be a bounded domain in \mathbb{R}^2 and let $g \in \mathcal{M} \log L$. Then*

$$\begin{aligned} &\int_0^{|\Omega|} \left(\int_t^{|\Omega|} f(s) ds \right)^2 g^*(t) dt \\ &\leq 4\|g\|_{\mathcal{M} \log L} \int_0^{|\Omega|} s [f(s)]^2 ds, \quad f \geq 0. \end{aligned} \quad (40)$$

The previous inequality is known for more general weights (even for measures), see [11–13]. Note that when $g \in \mathcal{M} \log L$, g^* satisfies Muckenhoupt condition (22) and hence the inequality follows from Proposition 5. We prove Theorem 12 by adapting the proof of Theorem 4, chapter 4 of [10].

Proof. For the simplicity, we let $a = |\Omega|$ and $\|g\| = \|g\|_{\mathcal{M} \log L}$. Since $g \in \mathcal{M} \log L$, we have

$$\log \left(\frac{a}{t} \right) \left(\int_0^t g^*(\tau) d\tau \right) \leq \|g\|, \quad \forall t \in (0, a). \quad (41)$$

Now using the Hölder inequality we obtain

$$\begin{aligned} &\left(\int_t^a f(s) ds \right)^2 \\ &= \left(\int_t^a f(s) \sqrt{s} \left[\log \left(\frac{a}{s} \right) \right]^{1/4} \frac{1}{\sqrt{s} [\log(a/s)]^{1/4}} ds \right)^2 \\ &\leq \int_t^a [f(s)]^2 s \left[\log \left(\frac{a}{s} \right) \right]^{1/2} ds \\ &\quad \times \int_t^a \frac{1}{s [\log(a/s)]^{1/2}} ds \\ &= 2 \left[\log \left(\frac{a}{t} \right) \right]^{1/2} \int_t^a [f(s)]^2 s \left[\log \left(\frac{a}{s} \right) \right]^{1/2} ds. \end{aligned} \quad (42)$$

Therefore,

$$\begin{aligned} &\int_0^a \left(\int_t^a f(s) ds \right)^2 g^*(t) dt \\ &\leq 2 \int_0^a \left[\int_t^a [f(s)]^2 s \left[\log \left(\frac{a}{s} \right) \right]^{1/2} ds \right] \\ &\quad \times \left[\log \left(\frac{a}{t} \right) \right]^{1/2} g^*(t) dt \\ &= 2 \int_0^a \left[\int_0^s \left[\log \left(\frac{a}{t} \right) \right]^{1/2} g^*(t) dt \right] [f(s)]^2 \\ &\quad \times s \left[\log \left(\frac{a}{s} \right) \right]^{1/2} ds, \end{aligned} \quad (43)$$

The equality in the last step follows from Fubini's theorem on the interchange of the order of integration. Next we estimate

the innermost integral on the right-hand side of the previous inequality using (41):

$$\begin{aligned} & \int_0^s \left[\log \left(\frac{a}{t} \right) \right]^{1/2} g^*(t) dt \\ & \leq \sqrt{\|g\|} \int_0^s \left(\int_0^t g^*(\tau) d\tau \right)^{(-1)/2} g^*(t) dt \\ & = 2\sqrt{\|g\|} \int_0^s \frac{d}{dt} \left(\int_0^t g^*(\tau) d\tau \right)^{1/2} dt \\ & = 2\sqrt{\|g\|} \left(\int_0^s g^*(\tau) d\tau \right)^{1/2} \\ & \leq \frac{2\|g\|}{[\log(a/s)]^{1/2}}. \end{aligned} \quad (44)$$

(using (41)).

Now by substituting back into (43), we get the result. \square

Proof of Theorem 1. Let $g \in \mathcal{M} \log L$. From the inequalities given in Proposition 4, it is clear that the inequality

$$\int_{\Omega} g u^2 \leq C \int_{\Omega} |\nabla u|^2, \quad \forall u \in H_0^1(\Omega) \quad (45)$$

holds, if

$$\int_0^{|\Omega|} g^*(t) u^*(t)^2 dt \leq 4\pi C \int_0^{|\Omega|} t [u^*'(t)]^2 dt, \quad \forall u \in H_0^1(\Omega). \quad (46)$$

Since $u^*(|\Omega|) = 0$, we can rewrite the previous inequality as

$$\int_0^{|\Omega|} \left(\int_t^{|\Omega|} [-u^*'(s)] ds \right)^2 g^*(t) dt \leq 4\pi C \int_0^{|\Omega|} t [-u^*'(t)]^2 dt. \quad (47)$$

Now by Theorem 12, we see that the previous inequality holds with $C = \|g\|_{\mathcal{M} \log L}/\pi$. \square

In the following theorem, we show that our condition is almost necessary for the generalized Hardy-Sobolev inequality.

Theorem 13. Let $\Omega = B(0; R) \subset \mathbb{R}^2$ and let $g \in L^1(\Omega)$ be such that g is positive, radial, and radially decreasing. If g is admissible, then $g \in \mathcal{M} \log L$.

Proof. Let g be admissible and let $C > 0$ be such that

$$\int_{\Omega} g u^2 \leq C \int_{\Omega} |\nabla u|^2, \quad \forall u \in H_0^1(\Omega). \quad (48)$$

We use certain test functions in $H_0^1(\Omega)$ to estimate $\|g\|_{\mathcal{M} \log L}$. For $r > 0$, let

$$u_r(x) = \begin{cases} \log \left(\frac{R}{r} \right), & |x| \leq r, \\ \log \left(\frac{R}{|x|} \right), & |x| \geq r. \end{cases} \quad (49)$$

Clearly $u_r \in H_0^1(\Omega)$ and

$$\nabla u_r(x) = \begin{cases} 0, & |x| \leq r, \\ -\frac{x}{|x|^2}, & |x| \geq r. \end{cases} \quad (50)$$

Therefore

$$\int_{\Omega} |\nabla u_r|^2 = 2\pi \int_r^R \frac{1}{t^2} t dt = 2\pi \log \left(\frac{R}{r} \right). \quad (51)$$

Further, since g is radial and radially decreasing, we get

$$\begin{aligned} \int_{\Omega} g u_r^2 & \geq \left[\log \left(\frac{R}{r} \right) \right]^2 \int_{B(0;r)} g(x) dx \\ & = \left[\log \left(\frac{R}{r} \right) \right]^2 \int_0^{\pi r^2} g^*(t) dt. \end{aligned} \quad (52)$$

Now (48), (51), and (52) yield

$$\int_0^{\pi r^2} g^*(s) ds \leq 2\pi C \frac{1}{\log(R/r)}. \quad (53)$$

Thus by substituting $t = \pi r^2$ in the previous inequality and noting that $2\log(R/r) = \log(|\Omega|/\pi r^2)$, we get

$$t \log \left(\frac{|\Omega|}{t} \right) g^{**}(t) \leq 4\pi C, \quad \forall t \in (0, |\Omega|). \quad (54)$$

Hence $g \in \mathcal{M} \log L$ and $\|g\|_{\mathcal{M} \log L} \leq 4\pi C$. \square

Next we see how one can obtain Moser-Trudinger embedding and Hansson's embedding using Theorem 1.

Remark 14 (an alternate proof for some classical embeddings). From Theorem 1, for each $g \in \mathcal{M} \log L$, we have the generalized Hardy-Sobolev inequality

$$\int_{\Omega} g u^2 \leq \frac{\|g\|_{\mathcal{M} \log L}}{\pi} \int_{\Omega} |\nabla u|^2, \quad \forall u \in H_0^1(\Omega). \quad (55)$$

(i) Moser-Trudinger embedding: since $L \log L \subset \mathcal{M} \log L$, there exists $C_1 > 0$ such that

$$\|g\|_{\mathcal{M} \log L} \leq C_1 \pi \|g\|_{L \log L}. \quad (56)$$

Thus from (55) we have

$$\int_{\Omega} g u^2 \leq C_1 \|g\|_{L \log L} \int_{\Omega} |\nabla u|^2, \quad \forall u \in H_0^1(\Omega). \quad (57)$$

The previous inequality shows that for each $u \in H_0^1(\Omega)$, u^2 is a continuous linear functional on $L \log L$ with $\|u^2\|_{[L \log L]^*} \leq C_1 \|\nabla u\|_2^2$. In other words, $u^2 \in L_{\exp}$ (the dual space of $L \log L$) and

$$\int_{\Omega} e^{(u^2/C_1 \|\nabla u\|_2^2)} - 1 \leq 1. \quad (58)$$

In particular, for each $u \in H_0^1(\Omega)$, $u \in L_{\exp}$ and also

$$\|u\|_{L_{\exp}} \leq \sqrt{C_1} \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega). \quad (59)$$

(ii) Hansson's embedding: since $L^{1,\infty}(\log L)^2 \subset \mathcal{M} \log L$, there exists $C_2 > 0$ such that

$$\|g\|_{\mathcal{M} \log L} \leq C_2 \pi \|g\|_{L^{1,\infty}(\log L)^2}. \quad (60)$$

As before, for each $u \in H_0^1(\Omega)$, $u^2 \in [L^{1,\infty}(\log L)]^* (= L^{\infty,1}(\log L)^{-2})$ with $\|u^2\|_{[L^{1,\infty}(\log L)]^*} \leq C_2 \|\nabla u\|_2^2$, that is,

$$\int_0^{|\Omega|} \frac{[u^*(t)]^2}{t[\log(e|\Omega|/t)]^2} \leq C_2 \|\nabla u\|_2^2, \quad \forall u \in H_0^1(\Omega). \quad (61)$$

Thus for $u \in H_0^1(\Omega)$, $u \in L^{\infty,2}(\log L)^{-1}$ and

$$\|u\|_{L^{\infty,2}(\log L)^{-1}} \leq \sqrt{C_2} \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega). \quad (62)$$

5. The Best Constant in the Hardy-Sobolev Inequalities

In this section, we give a proof of Theorem 2 using a direct variational method. For $g \in \mathcal{M} \log L$, we define the following:

$$\mathcal{M} := \left\{ u \in H_0^1(\Omega) : \int_{\Omega} g u^2 = 1 \right\}, \quad (63)$$

$$G^+(u) := \int_{\Omega} g^+ u^2, \quad u \in H_0^1(\Omega). \quad (64)$$

Recall that

$$\lambda_1(g) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} g u^2} : u \in H_0^1(\Omega), \int_{\Omega} g u^2 > 0 \right\}. \quad (65)$$

It is easy to see that $\lambda_1(g) = \inf\{J(u) : u \in \mathcal{M}\}$, where $J(u) = \int_{\Omega} |\nabla u|^2$. Note that, for an admissible g , $\lambda_1(g) > 0$ and $1/\lambda_1(g)$ is the best constant in (1). Thus the best constant in (1) is attained for some $u \in H_0^1(\Omega)$ if J has a minimizer on \mathcal{M} . We show that, under the assumptions of Theorem 2, J admits a minimizer on \mathcal{M} .

First we prove the following compactness theorem.

Lemma 15. Let Ω be a bounded domain in \mathbb{R}^2 and let $h \in \mathcal{F}_1$. Then the map

$$H(u) = \int_{\Omega} h u^2, \quad u \in H_0^1(\Omega) \quad (66)$$

is compact.

Proof. For $u_n \rightharpoonup u$ we show that $H(u_n) \rightarrow H(u)$. First we estimate the following:

$$\begin{aligned} |H(u_n) - H(u)| &\leq \int_{\Omega} |h| |u_n^2 - u^2| \\ &\leq \int_{\Omega} (\sqrt{|h|} |u_n + u|) (\sqrt{|h|} |u_n - u|) \\ &\leq \left(\int_{\Omega} |h| (u_n + u)^2 \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} |h| (u_n - u)^2 \right)^{1/2}. \end{aligned} \quad (67)$$

Since u_n is bounded, from (55) we get that $(\int_{\Omega} |h| (u_n + u)^2)^{1/2}$ is bounded. Thus $H(u_n) \rightarrow H(u)$ that if $\int_{\Omega} |h| (u_n - u)^2 \rightarrow 0$. Let $m = \sup_n \|\nabla(u_n - u)\|^2$. For a given $\varepsilon > 0$, since $|h| \in \mathcal{F}_1$, we choose $h_{\varepsilon} \in \mathcal{C}_c^{\infty}(\Omega)$ such that

$$\| |h| - h_{\varepsilon} \|_{\mathcal{M} \log L} < \frac{\varepsilon}{m}. \quad (68)$$

Therefore

$$\begin{aligned} \int_{\Omega} (|h| - h_{\varepsilon}) (u_n - u)^2 \\ \leq \| |h| - h_{\varepsilon} \|_{\mathcal{M} \log L} \int_{\Omega} |\nabla(u_n - u)|^2 < \varepsilon. \end{aligned} \quad (69)$$

Now

$$\begin{aligned} \int_{\Omega} |h| (u_n - u)^2 &= \int_{\Omega} h_{\varepsilon} (u_n - u)^2 + \int_{\Omega} (|h| - h_{\varepsilon}) (u_n - u)^2 \\ &\leq \int_{\Omega} h_{\varepsilon} (u_n - u)^2 + \varepsilon. \end{aligned} \quad (70)$$

For sufficiently large n , the first integral can be made smaller than ε since $H_0^1(\Omega)$ is embedded compactly in $L^2(\Omega)$ and $h_{\varepsilon} \in \mathcal{C}_c^{\infty}(\Omega)$. Thus we conclude that $\int_{\Omega} |h| (u_n - u)^2 \rightarrow 0$. \square

Proof of Theorem 2. Since $g \in L_{\text{loc}}^1(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in \mathcal{C}_c^{\infty}(\Omega)$ such that $\int_{\Omega} g \varphi^2 > 0$ (see for e.g., Proposition 4.2 of [23]) and hence the set \mathcal{M} is nonempty. Let $\{u_n\}$ be a minimizing sequence of J on \mathcal{M} , that is,

$$\lim_{n \rightarrow \infty} J(u_n) = \lambda_1(g) = \inf_{u \in \mathcal{M}} J(u). \quad (71)$$

Using the coercivity of J , the reflexivity of $H_0^1(\Omega)$, and the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we can further assume that $\{u_n\}$ converges weakly and almost everywhere to some $u \in H_0^1(\Omega)$. Since $g^+ \in \mathcal{F}_1$, from the previous lemma we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} g^+ u_n^2 = \int_{\Omega} g^+ u^2. \quad (72)$$

For $u_n \in \mathcal{M}$, we write

$$\int_{\Omega} g^- u_n^2 = \int_{\Omega} g^+ u_n^2 - 1. \quad (73)$$

Now use the Fatou's lemma to obtain

$$\int_{\Omega} g^- u^2 \leq \int_{\Omega} g^+ u^2 - 1. \quad (74)$$

Thus we get $\int_{\Omega} g u^2 \geq 1$. Set $\tilde{u} := u / (\int_{\Omega} g u^2)^{1/2}$. Now the homogeneity and the weak lower semicontinuity of J yields the following:

$$\lambda_1(g) \leq J(\tilde{u}) = \frac{J(u)}{\int_{\Omega} g u^2} \leq J(u) \leq \liminf_n J(u_n) = \lambda_1(g). \quad (75)$$

Thus equality must hold at each step and hence $\int_{\Omega} g u^2 = 1$. This shows that $u \in \mathcal{M}$ and $J(u) = \lambda_1(g)$. \square

Next we give an equivalent definition for the space \mathcal{F}_1 . Recall that \mathcal{F}_1 is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{M}\log L$.

Theorem 16. Let Ω be a bounded domain in \mathbb{R}^2 . Then the following statements are equivalent:

- (i) $g \in \mathcal{F}_1$,
- (ii) $g \in \mathcal{M}\log L$ and $\lim_{t \rightarrow 0} g^{**}(t)t \log(|\Omega|/t) = 0$.

Proof. (i) \Rightarrow (ii) We show that $\lim_{t \rightarrow 0} g^{**}(t)t \log(|\Omega|/t) = 0$. Let $g_n \in \mathcal{C}_c^\infty(\Omega)$ and $g_n \rightarrow g$ in $\mathcal{M}\log L$. Note that for each g_n , $g_n^{**}(t)$ is bounded and hence

$$\lim_{t \rightarrow 0} g_n^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) = 0, \quad \forall n \in \mathbb{N}. \quad (76)$$

Let $\varepsilon > 0$ be given. We show that there exists $\delta > 0$ such that $g^{**}(t)t \log(|\Omega|/t) < \varepsilon$, for all $t \in (0, \delta)$. Since $g_n \rightarrow g$ in $\mathcal{M}\log L$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{0 < t < |\Omega|} (g - g_{n_0})^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) < \varepsilon. \quad (77)$$

Using the subadditivity of the maximal operator $f \rightarrow f^{**}$, we get

$$\begin{aligned} & g^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) \\ & \leq (g - g_{n_0})^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) + g_{n_0}^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) \\ & < \varepsilon + g_{n_0}^{**}(t)t \log\left(\frac{|\Omega|}{t}\right). \end{aligned} \quad (78)$$

Note that from (76), there exists $\delta > 0$ such that $g_{n_0}^{**}(t)t \log(|\Omega|/t) < \varepsilon$, for all $t \in (0, \delta)$. Now (78) yields the result as

$$g^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) < 2\varepsilon, \quad \forall t \in (0, \delta). \quad (79)$$

(ii) \Rightarrow (i). Let $\varepsilon > 0$ be given. For proving $g \in \mathcal{F}_1$, we show that there exists $\tilde{g}_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that $\|g - \tilde{g}_\varepsilon\|_{\mathcal{M}\log L} < 2\varepsilon$. Since $\lim_{t \rightarrow 0} g^{**}(t)t \log(|\Omega|/t) = 0$, we get $\delta > 0$ such that

$$g^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) < \varepsilon, \quad \forall t \in (0, \delta]. \quad (80)$$

Let

$$A_\varepsilon := \{x \in \Omega : |g(x)| < g^*(\delta)\}, \quad g_\varepsilon = g\chi_{A_\varepsilon}. \quad (81)$$

Clearly $g_\varepsilon \in L^2(\Omega)$. Since $L^2(\Omega)$ is continuously embedded in $\mathcal{M}\log L$ and $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, we can choose $\tilde{g}_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ so that

$$\|g_\varepsilon - \tilde{g}_\varepsilon\|_{\mathcal{M}\log L} < \varepsilon. \quad (82)$$

Next we estimate $\|g - g_\varepsilon\|_{\mathcal{M}\log L}$. Let us compute the distribution function α_{g-g_ε} :

$$\alpha_{g-g_\varepsilon}(s) = \alpha_{g\chi_{A_\varepsilon}}(s) = \begin{cases} \alpha_g(s) & s \geq g^*(\delta), \\ \alpha_g(g^*(\delta)) & s < g^*(\delta). \end{cases} \quad (83)$$

For computing the symmetric rearrangement $(g - g_\varepsilon)^*(t)$, we set $\delta_1 = \alpha_{g-g_\varepsilon}(g^*(\delta)) = \alpha_g(g^*(\delta))$. If $\delta_1 = 0$, $\alpha_{g-g_\varepsilon}(s) = 0$, for all $s > 0$, we get the desired result as $\|g - g_\varepsilon\|_{\mathcal{M}\log L} = 0$. Next we calculate $(g - g_\varepsilon)^*(t)$, when $\delta_1 > 0$. For $0 < t < \delta_1$, note that $\alpha_{g-g_\varepsilon}(s) \leq t$ implies that $s \geq g^*(\delta)$ (if $s < g^*(\delta)$, then $\delta_1 = \alpha_{g-g_\varepsilon}(g^*(\delta)) \leq \alpha_{g-g_\varepsilon}(s) \leq t$, a contradiction). Thus for $t < \delta_1$, using (83) we have

$$\begin{aligned} (g - g_\varepsilon)^*(t) &= \inf\{s : \alpha_{g-g_\varepsilon}(s) \leq t\} \\ &= \inf\{s \geq g^*(\delta) : \alpha_{g-g_\varepsilon}(s) \leq t\} \\ &= \inf\{s : \alpha_g(s) \leq t\} = g^*(t). \end{aligned} \quad (84)$$

Now for $t \geq \delta_1$, again using (83), we see that $\alpha_{g-g_\varepsilon}(s) \leq \alpha_g(g^*(\delta)) = \delta_1 \leq t$, for all $s < g^*(\delta)$ and hence $(g - g_\varepsilon)^*(t) = 0$, for all $t \geq \delta_1$. Thus

$$(g - g_\varepsilon)^*(t) = \begin{cases} g^{**}(t) & t < \delta_1, \\ \frac{\delta_1}{t} g^{**}(\delta_1) & t \geq \delta_1. \end{cases} \quad (85)$$

Therefore,

$$\begin{aligned} & (g - g_\varepsilon)^*(t)t \log\left(\frac{|\Omega|}{t}\right) \\ & \leq \begin{cases} g^{**}(t)t \log\left(\frac{|\Omega|}{t}\right) & t < \delta_1, \\ g^{**}(\delta_1)\delta_1 \log\left(\frac{|\Omega|}{\delta_1}\right) & t \geq \delta_1. \end{cases} \end{aligned} \quad (86)$$

Since $\delta_1 \leq \delta$, from (80), we have $\|g - g_\varepsilon\|_{\mathcal{M}\log L} < \varepsilon$. Now (82) yields $\|g - \tilde{g}_\varepsilon\|_{\mathcal{M}\log L} < 2\varepsilon$ and hence the proof is done. \square

Next we give example of functions in \mathcal{F}_1 .

Example 17. We have already seen that $L\log L \subset \mathcal{M}\log L$ (Proposition 6). In fact, $L\log L \subset \mathcal{F}_1$ since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L\log L$ and the inclusion of $L\log L$ in $\mathcal{M}\log L$ is continuous. Also, using Theorem 16, one can verify that $1/|x|^2[\log(R/|x|)]^\gamma \in \mathcal{F}_1$, for $\gamma > 2$.

Remark 18 (an open problem). Whether all the admissible functions are in $\mathcal{M}\log L$ or not, by setting $\lambda_1(g^*) = \inf\{\int_\Omega |\nabla u^*|^2 / \int_\Omega g^* u^{*2} : u \in H_0^1(\Omega), \int_\Omega gu^2 > 0\}$, the problem can be rephrased as whether $\lambda_1(g) > 0$ implies $\lambda_1(g^*) > 0$ or not.

Remark 19. For $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ (not necessarily bounded), using the Muckenhoupt condition (22) and the

inequalities given in Proposition 4, one can show that (as in Theorem 1) g is admissible if g satisfies

$$\sup_{0 < t < |\Omega|} t^{2/N} g^{**}(t) < \infty. \quad (87)$$

The previous condition shows that one can obtain Visciglia's result [8] without using the Lorentz-Sobolev embedding. In fact, one can give an alternate proof for the Lorentz Sobolev embedding of $\mathcal{D}_0^{1,2}(\Omega)$ into the Lorentz space $L(2^*, 2)$ using similar arguments as in Remark 14.

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References

- [1] H. Brezis and J. L. Vázquez, "Blow-up solutions of some nonlinear elliptic problems," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 10, no. 2, pp. 443–469, 1997.
- [2] Adimurthi, N. Chaudhuri, and M. Ramaswamy, "An improved Hardy-Sobolev inequality and its application," *Proceedings of the American Mathematical Society*, vol. 130, no. 2, pp. 489–505, 2002.
- [3] N. Chaudhuri and M. Ramaswamy, "Existence of positive solutions of some semilinear elliptic equations with singular coefficients," *Proceedings of the Royal Society of Edinburgh A*, vol. 131, no. 6, pp. 1275–1295, 2001.
- [4] S. Filippas and A. Tertikas, "Optimizing improved Hardy inequalities," *Journal of Functional Analysis*, vol. 192, no. 1, pp. 186–233, 2002.
- [5] N. Ghoussoub and A. Moradifam, "On the best possible remaining term in the Hardy inequality," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 105, no. 37, pp. 13746–13751, 2008.
- [6] J. Leray, "Etude de diverses équations intégrales non linéaires et de quelques problèmes qui posent l'hydrodynamique," *Journal de Mathématiques Pures et Appliquées*, vol. 12, no. 9, pp. 1–82, 1933.
- [7] V. G. Maz'ja, *Sobolev Spaces*, Springer Series in Soviet Mathematics, Springer, Berlin, Germany, 1985, Translated from the Russian by T. O. Shaposhnikova.
- [8] N. Visciglia, "A note about the generalized Hardy-Sobolev inequality with potential in $L^{p/d}(\mathbb{R}^n)$," *Calculus of Variations and Partial Differential Equations*, vol. 24, no. 2, pp. 167–184, 2005.
- [9] C. Cosner, K. J. Brown, and J. Fleckinger, "Principal eigenvalues for problems with indefinite weight function on \mathbb{R}^n ," *Proceedings of the American Mathematical Society*, vol. 109, no. 1, pp. 147–155, 1990.
- [10] L. Maligranda, L.-E. Persson, and A. Kufner, *The Hardy Inequality: About its History and Some Related Results*, 2007.
- [11] G. Talenti, "Osservazioni sopra una classe di diseguaglianze," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 39, pp. 171–185, 1969.
- [12] G. Tomaselli, "A class of inequalities," *Bollettino Della Unione Matematica Italiana*, vol. 2, no. 4, pp. 622–631, 1969.
- [13] B. Muckenhoupt, "Hardy's inequality with weights," *Studia Mathematica*, vol. 44, pp. 31–38, 1972, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.
- [14] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1988.
- [15] T. V. Anoop, M. Lucia, and M. Ramaswamy, "Eigenvalue problems with weights in Lorentz spaces," *Calculus of Variations and Partial Differential Equations*, vol. 36, no. 3, pp. 355–376, 2009.
- [16] K. Hansson, "Imbedding theorems of Sobolev type in potential theory," *Mathematica Scandinavica*, vol. 45, no. 1, pp. 77–102, 1979.
- [17] H. Brézis and S. Wainger, "A note on limiting cases of Sobolev embeddings and convolution inequalities," *Communications in Partial Differential Equations*, vol. 5, no. 7, pp. 773–789, 1980.
- [18] M. Krbec and T. Schott, "Superposition of imbeddings and Fefferman's inequality," *Bollettino della Unione Matematica Italiana B*, vol. 2, no. 3, pp. 629–637, 1999.
- [19] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 2004.
- [20] S. Kesavan, *Symmetrization & Applications*, vol. 3 of *Series in Analysis*, World Scientific Publishing, Hackensack, NJ, USA, 2006.
- [21] M. Loss and E. H. Lieb, *Analysis*, vol. 14 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2nd edition, 2001.
- [22] D. E. Edmunds and H. Triebel, "Sharp Sobolev embeddings and related Hardy inequalities: the critical case," *Mathematische Nachrichten*, vol. 207, pp. 79–92, 1999.
- [23] B. Kawohl, M. Lucia, and S. Prashanth, "Simplicity of the principal eigenvalue for indefinite quasilinear problems," *Advances in Differential Equations*, vol. 12, no. 4, pp. 407–434, 2007.

