Research Article

On a Property of a Three-Dimensional Matrix

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Let S_n be the symmetrical group acting on the set $\{1, 2, \dots, n\}$ and $x, y \in S_n$. Consider the set $W = \{(i, x(i), y(i)) \mid 1 \leq i \leq n, |i - x(i)| > 1 \vee |i - y(i)| > 1 \vee |x(i) - y(i)| > 1\}$. The main result of this paper is the following theorem. If the number of W set entries is more than $[n/3]$, then there exist entries $(i_1, x(i_1), y(i_1)), (i_2, x(i_2), y(i_2)), (i_3, x(i_3), y(i_3)) \in W$ such that $|i_1 - x(i_2)| \leq 1$, $|i_1 - y(i_3)| \leq 1$, and $|x(i_2) - y(i_3)| \leq 1$. The application of this theorem to the three-dimensional assignment problem is considered.

1. Introduction

Let R_n^3 be the set of $n \times n \times n$ -matrices over the field of real numbers.

Three-dimensional matrix not only is an interesting mathematical object [1–3], but also has applications in many fields, such as theoretical physics [4] and operational research [5, 6].

Let S_n be the symmetrical group acting on the set $\{1, 2, \dots, n\}$, $x, y \in S_n$, $\|a_{ijk}\| \in R_n^3$ and

$$G = \{a_{ijk} \mid |i - j| \leq 1 \wedge |i - k| \leq 1 \wedge |j - k| \leq 1\},$$

$$H = \{a_{ijk} \mid |i - j| > 1 \vee |i - k| > 1 \vee |j - k| > 1\}, \quad (1)$$

$$T = \{a_{ix(i)y(i)} \mid 1 \leq i \leq n\}.$$

The main result of this paper is the following theorem.

Theorem 1. If $\|a_{ijk}\| \in R_n^3$, $x, y \in S_n$ and the number of $T \cap H$ set entries is more than $[n/3]$, then there exist entries $a_{i_1 j_1 k_1}$, $a_{i_2 j_2 k_2}$, and $a_{i_3 j_3 k_3} \in T \cap H$ such that the entry $a_{i_1 j_2 k_3} \in G$.

We give another formulation of Theorem 1. Consider the set

$$W = \{(i, x(i), y(i)) \mid 1 \leq i \leq n, |i - x(i)| > 1 \vee |i - y(i)| > 1 \vee |x(i) - y(i)| > 1\}. \quad (2)$$

Theorem 2. If the number of W set entries is more than $[n/3]$, then there exist entries $(i_1, x(i_1), y(i_1)), (i_2, x(i_2), y(i_2)),$

$(i_3, x(i_3), y(i_3)) \in W$ such that $|i_1 - x(i_2)| \leq 1$, $|i_1 - y(i_3)| \leq 1$, and $|x(i_2) - y(i_3)| \leq 1$.

2. Proof of Theorem 1

We prove the theorem by contradiction. The set of matrix $\|a_{ijk}\| \in R_n^3$ entries with one index fixed and the two others having values from 1 to n will be called a layer. We denote a layer by L_s^r , where r indicates the location of a fixed index and s indicates its value. For example, $L_s^2 = \{a_{isk} \mid i, k = 1, \dots, n\}$. Furthermore, entries from G will be called basic, entries from H will be called nonbasic; T will be termed a trajectory; the layer containing a basic trajectory entry will be termed a basic layer and the layer containing a nonbasic trajectory entry will be termed a nonbasic layer.

If L_s^1 is a nonbasic layer, then one layer in the pair L_{s-1}^2, L_{s-1}^3 is basic.

Suppose to the contrary that layers L_s^1, L_{s-1}^2 , and L_{s-1}^3 are nonbasic. Let $a_{s j_1 k_1}$, $a_{i_2, s-1, k_2}$, and $a_{i_3, j_3, s-1}$ be the trajectory entries of $L_s^1, L_{s-1}^2, L_{s-1}^3$ layers. Replace $a_{s j_1 k_1}$, $a_{i_2, s-1, k_2}$, and $a_{i_3, j_3, s-1}$ with $a_{s, s-1, s-1}$, $a_{i_2 j_1 k_1}$, and $a_{i_3 j_3 k_2}$. The nonbasic trajectory entries $a_{s j_1 k_1}$, $a_{i_2, s-1, k_2}$, and $a_{i_3, j_3, s-1}$ are replaced with entries $a_{s, s-1, s-1}$, $a_{i_2 j_1 k_1}$, and $a_{i_3 j_3 k_2}$, among which there is a basic one.

Similar assertions may be proved for the L_s^1 layer and the following layer pairs: L_{s-1}^2, L_s^3 ; L_s^2, L_{s-1}^3 ; L_s^2, L_s^3 ; L_s^2, L_{s+1}^3 ; L_{s+1}^2, L_s^3 , and L_{s+1}^2, L_{s+1}^3 .

Three nonbasic layers may not be consecutive.

Suppose that layers L_{s-1}^1 , L_s^1 , and L_{s+1}^1 are nonbasic. One of the layers L_s^2 or L_s^3 is basic. In these layers, the basic entries may be $a_{s-1,s,s-1}$, $a_{s-1,s,s}$, $a_{s,s,s-1}$, $a_{s,s,s}$, $a_{s,s,s+1}$, $a_{s+1,s,s}$, and $a_{s+1,s,s+1}$ for the layer L_s^2 , and $a_{s-1,s-1,s}$, $a_{s-1,s,s}$, $a_{s,s-1,s}$, $a_{s,s,s}$, $a_{s+1,s,s}$, $a_{s+1,s+1,s}$ for the layer L_s^3 . However, this contradicts the assumption that the layers L_{s-1}^1 , L_s^1 , and L_{s+1}^1 are nonbasic. The fact that the two first and the two last layers may not be nonbasic is proved in a similar way.

All assertions given below represent conditions that prevent replacing nonbasic trajectory entries with entries that include a basic one.

Consider the sequence of layers $L_1^1, L_2^1, \dots, L_{s-1}^1, L_s^1, L_{s+1}^1, \dots, L_{n-1}^1$, and L_n^1 . Given below are possible arrangements of layers in this sequence. A basic layer L_s^1 is denoted by 1; a nonbasic layer L_s^1 is denoted by 0.

Consider

- (1) ...110011...; 110011...; ...110011; 110011.
- (2) ...1101011...; 01111010...; ...01011110; ...001111010...; ...010111100...; ...0101111010...
- (3) 0111100...; ...00111100...; ...0011110.
- (4) 0111...; ...1110; 1011...; ...11101; 011110.

We prove the first arrangement of item 1. Suppose that the two nonbasic layers are followed only by two basic layers, that is, ...00110... Let there be layers ... L_s^1 , L_{s+1}^1 , L_{s+2}^1 , L_{s+3}^1 , L_{s+4}^1 ...

Consider layer pairs L_{s+1}^2, L_{s+1}^3 ; L_{s+1}^2, L_{s+2}^3 ; L_{s+2}^2, L_{s+2}^3 ; L_{s+2}^2, L_{s+3}^3 ; and L_{s+2}^3 . At least one layer of each pair is basic. These four pairs of layers contain at least two basic entries, and these entries are located in layers L_{s+2}^1 and L_{s+3}^1 . The pair L_{s+3}^2, L_{s+3}^3 includes a basic layer. The first coordinate of the basic layer entry may be $s+2$, $s+3$, or $s+4$. But the L_{s+2}^1 and L_{s+3}^1 layers are occupied, while the L_{s+4}^1 layer is nonbasic. There is a contradiction.

The other arrangements are proved in a similar way.

Thus, nonbasic layers may not be arranged closer than those in the above variants. But these variants do not allow for the composition of a combination containing more than $\lfloor n/3 \rfloor$ nonbasic layers. Hence, it follows that if a trajectory includes more than $\lfloor n/3 \rfloor$ nonbasic entries, then one of the variants is violated and nonbasic trajectory entries can be replaced by a set of entries containing a basic entry.

3. Application to the Three-Dimensional Assignment Problem

The three-dimensional assignment problem (AP3) is an important combinatorial optimization problem. It is sufficient to note that the particular case of AP3, the 3-dimensional matching problem, is one of the six main NP-hard problems [7]. The formal AP3 statement is as follows: for a matrix $\|a_{ijk}\| \in R_n^3$, find permutations $x, y \in S_n$ such that $\sum_{i=1}^n a_{ix(i)y(i)}$ is maximized.

In this paper, the AP3 is considered for a special class of $n \times n \times n$ -matrices $U(n)$. A matrix

$$\begin{aligned} \|a_{ijk}\| &\in U(n), \quad \text{if } a_{ijk} \geq 0, \\ \min_{a_{ijk} \in G} a_{ijk} &\geq C \cdot \max_{a_{ijk} \in H} a_{ijk}, \quad C \geq 3. \end{aligned} \quad (3)$$

One of AP3 interpretations is the following. There are n employees and two job sets of n jobs each. If the i th employee performs the j th job of the first set and the k th job of the second set, then the effect equals a_{ijk} . It is required to distribute the jobs among the employees in such a way so as to maximize the total effect.

Let us describe the situation that will lead to the AP3 for matrices from $U(n)$. As a rule, the employees are ordered by qualification, while the jobs are ordered by complexity. A higher effect is reached when a more qualified employee performs a more complex job, and we may arrive at the AP3 for matrices from $U(n)$.

The particular cases of AP3 [5, 8–11], the nonpolynomial exact algorithms for the AP3 [6, 12], and heuristics for the AP3 [13, 14] were considered.

The NP-hard particular cases of the traveling salesman problem, with sets of matrices whose structures are similar to those of matrices from $U(n)$, have been considered previously [15, 16].

Theorem 3. *The AP3 for matrices from $U(n)$ is NP-hard.*

Proof. Let $\|b_{ijk}\| \in R_n^3$, $b_{ijk} \geq 0$ and numbers A, B are such that $A = 3 \cdot \max b_{ijk}$, $B = nA$.

Consider $4n \times 4n \times 4n$ -matrix $\|a_{ijk}\|$ with nonnegative entries

$$\begin{aligned} a_{4s-3,4s-2,4s-2} &= B & s &= 1, \dots, n; \\ a_{4s-2,4s-1,4s-1} &= B & s &= 1, \dots, n; \\ a_{4s,4s,4s} &= B & s &= 1, \dots, n; \\ a_{ruv} &= A & r, u, v &= 1, \dots, n; \quad \text{if } |r-u| \leq 1, \\ |r-v| \leq 1, \quad |u-v| \leq 1, \quad a_{ruv} &\neq B; \\ a_{4i-1,4j-3,4k-3} &= b_{ijk} & i, j, k &= 1, \dots, n; \\ a_{ijk} &= 0 & \text{in other cases.} \end{aligned} \quad (4)$$

The matrix $\|a_{ijk}\| \in U(4n)$.

All entries that are equal to B belong to the optimal solution of AP3 for $\|a_{ijk}\|$ matrix. This signifies that if the optimal solution of AP3 for $\|a_{ijk}\|$ matrix is known, then the optimal solution of AP3 for $\|b_{ijk}\|$ matrix is known as well. Therefore, the AP3 for arbitrary matrices is polynomially reducible to the AP3 for matrices from $U(n)$. $\square$

Corollary 4. *If $\|a_{ijk}\| \in U(n)$, then a set T exists on which the AP3 optimum is reached and the number of $T \cap H$ set entries is no more than $\lfloor n/3 \rfloor$.*

Indeed, if for optimal T of matrix $\|a_{ijk}\| \in U(n)$ the number of $T \cap H$ set entries is more than $\lfloor n/3 \rfloor$, then, according to

Theorem 1, it is possible to make such replacement of entries such that the sum of T set entries will not change and the number of $T \cap H$ set entries is reduced.

Theorem 5. *If $\|a_{ijk}\| \in U(n)$, then the AP3 optimum for a $\|a'_{ijk}\|$ matrix, where $a'_{ijk} = a_{ijk}$ for $a_{ijk} \in G$ and $a'_{ijk} = 0$ for $a_{ijk} \in H$, represents an approximation for the AP3 optimum of the $\|a_{ijk}\|$ matrix with a relative error not exceeding $1/(3C)$.*

Proof. Let $\|a_{ijk}\| \in U(n)$; m_1 is the AP3 optimum for $\|a_{ijk}\|$; $f, g \in S_n$ are such that $m_1 = \sum_{i=1}^n a_{if(i)g(i)}$, and the number of $\{a_{if(i)g(i)} \mid i = 1, \dots, n\} \cap H$ set entries is no more than $\lfloor n/3 \rfloor$; m_2 is the sum of $\{a_{if(i)g(i)} \mid i = 1, \dots, n\} \cap H$ set entries and $m_3 = m_1 - m_2$; m' is the AP3 optimum for $\|a'_{ijk}\|$.

Note that $m_3 = \sum_{i=1}^n a'_{if(i)g(i)}$ and $m' \geq m_3$.

Insofar as $a_{ijk} \geq a'_{ijk}$, $i, j, k = 1, \dots, n$, then $m_1 \geq m'$. Hence, $m_3 \leq m' \leq m_1$ and $(m_1 - m')/m_1 \leq (m_1 - m_3)/m_1 = m_2/m_1$.

Since $m_2 \leq (1/3)n \cdot \max_{a_{ijk} \in H} a_{ijk}$ and $m_1 \geq n \cdot \min_{a_{ijk} \in G} a_{ijk}$, then $m_2/m_1 \leq 1/(3C)$. Hence, $(m_1 - m')/m_1 \leq 1/(3C)$, and the relative error of m' taken as the AP3 approximate solution for the $\|a_{ijk}\|$ matrix is no more than $1/(3C)$. $\square$

Remark 6. An exact $O(n)$ -algorithm of AP3 solution has been constructed [9] for $n \times n \times n$ -matrices $\|a_{ijk}\|$ such that $a_{ijk} \geq 0$ for $a_{ijk} \in G$ and $a_{ijk} = 0$ for $a_{ijk} \in H$ using dynamical programming.

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