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Research Article

New Weighted Norm Inequalities for Pseudodifferential Operators and Their Commutators

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This paper is dedicated to study weighted L^p inequalities for pseudodifferential operators with amplitudes and their commutators by using the new class of weights A_p^∞ and the new BMO function space BMO_∞ which are larger than the Muckenhoupt class of weights A_p and classical BMO space BMO, respectively. The obtained results therefore improve substantially some well-known results.

1. Introduction and the Main Results

For $f \in C_0^\infty(\mathbb{R}^n)$ a pseudodifferential operator given formally by

$$T_a f(x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n} a(x, y, \xi) e^{i\langle x-y,\xi\rangle} f(y)\, dy\, d\xi, \quad (1)$$

where the amplitude a satisfies certain growth conditions. The boundedness of pseudodifferential operators has been studied extensively by many mathematicians; see, for example, [1–7] and the references therein. One of the most interesting problems is studying the weighted norm inequalities for pseudodifferential operators and their commutators with BMO function; see, for example, [5–9].

In this paper we consider the following classes of symbols and amplitudes a (in what follows we set $\langle x\rangle = (1+|x|^2)^{1/2}$).

Definition 1. Let $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m \in \mathbb{R}$, $\rho \in [0,1]$ and $\delta \in [0,1]$.

(a) We say $a \in A^m_{\rho,\delta}$ when for each triple of multi-indices α, β, and γ there exists a constant C such that

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \le C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta+\gamma|}. \quad (2)$$

(b) We say $a \in L^\infty A^m_{\rho,\delta}$ when for each triple of multi-indices α, β, and γ there exists a constant C such that

$$\left\|\partial_\xi^\alpha \partial_y^\beta a(\cdot, y, \xi)\right\|_{L^\infty} \le C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \quad (3)$$

Definition 2. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m \in \mathbb{R}$, $\rho \in [0,1]$ and $\delta \in [0,1]$.

(a) We say $a \in S^m_{\rho,\delta}$ when for each pair of multi-indices α and β there exists a constant C such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\right| \le C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \quad (4)$$

(b) We say $a \in L^\infty S^m_\rho$ when for each multi-indices α there exists a constant C such that

$$\left\|\partial_\xi^\alpha a(\cdot, \xi)\right\|_{L^\infty} \le C\langle\xi\rangle^{m-\rho|\alpha|}. \quad (5)$$

It is easy to see that $S^m_{\rho,\delta} \subset A^m_{\rho,\delta}$, $L^\infty S^m_\rho \subset L^\infty A^m_{\rho,\delta}$, $S^m_{\rho,\delta} \subset L^\infty S^m_\rho$, and $A^m_{\rho,\delta} \subset L^\infty A^m_{\rho,\delta}$. The classes $A^m_{\rho,\delta}$ and $S^m_{\rho,\delta}$ were studied in [3, 8]. For further information about these two classes, we refer the reader to, for example, [3, 10]. The class $L^\infty S^m_\rho$ was introduced by [11], and it is the natural generalization of the class $S^m_{\rho,\delta}$. This class is much rougher than that considered in [6, 7]. The amplitude class $L^\infty A^m_{\rho,\delta}$

in Definition 1 is rough in the x variable, but smooth in the y variable. This is smaller than the class $L^\infty A^m_\rho$ introduced in [5] but includes the class $A^m_{\rho,\delta}$.

The aim of this paper is to study the weighted norm inequalities for pseudodifferential operators T_a and their commutators by using the new BMO functions and the new class of weights. Firstly, we would like to give brief definitions on the new class of weights and the new BMO function space (we refer to Section 2 for details).

The new classes of weights $A^\infty_p = \cup_{\theta>0} A^\theta_p$ for $p \geq 1$, where A^θ_p, $\theta \geq 0$, is the set of those weights satisfying

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-1/(p-1)}\right)^{1/p'} \leq C\,|B|\,(1+r_B)^\theta \tag{6}$$

for all ball $B = B(x_B, r_B)$. We denote that $A^\infty_\infty = \cup_{p\geq 1} A^\infty_p$. It is easy to see that the new class A^∞_p is strictly larger than the Muckenhoupt class A_p. Indeed, for example, the weight $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$ belongs to the class A^∞_p, but it is not in A_p, for $p > 1$, see, for example, [12].

The new BMO space BMO_θ with $\theta \geq 0$ is defined as a set of all locally integrable functions b satisfying

$$\frac{1}{|B|}\int_B |b(y) - b_B|\,dy \leq C(1+r_B)^\theta, \tag{7}$$

where $B = B(x_B, r_B)$ and $b_B = (1/|B|)\int_B b$. A norm for $b \in \mathrm{BMO}_\theta$, denoted by $\|b\|_\theta$, is given by the infimum of the constants satisfying (12). Clearly $\mathrm{BMO}_{\theta_1} \subset \mathrm{BMO}_{\theta_2}$ for $\theta_1 \leq \theta_2$ and $\mathrm{BMO}_0 = \mathrm{BMO}$. We define $\mathrm{BMO}_\infty = \cup_{\theta>0}\mathrm{BMO}_\theta$.

Our main result is the following theorem.

Theorem 3. *Let $a \in L^\infty A^m_{\rho,\delta}$ with $m < n(\rho - 1)$ or $a \in L^\infty A^0_{1,\delta}$, $\delta \in [0,1]$. If T_a is bounded on L^p for all $1 < p < \infty$, then*

(a) *T_a is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$;*

(b) *for any $b \in \mathrm{BMO}_\infty$, the commutator $[b, T_a]$ bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$.*

In particular, the obtained results in (a) and (b) still hold for $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$.

We would like to specify some applications of Theorem 3.

In [8], the author studied the weighted L^p inequalities of T_a when the symbol a belongs to the class $S^0_{1,\delta} \subset L^\infty A^0_{1,\delta}$ with $\delta \in (0,1)$. It was proved that T_a is bounded on $L^p(w)$ for $1 < p < \infty$, $w \in A_p$. Recently, the author in [9] showed that T_a and its commutator with a BMO function $[b, T_a]$ are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$ by the different approach. Here, by using Theorem 3, we not only reobtain the boundedness of T_a on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$ but also obtain the new result on the boundedness of its commutator with BMO_∞ functions.

Corollary 4. *Let $a \in S^0_{1,\delta} \subset L^\infty A^0_{1,\delta}$, $0 < \delta < 1$. Then we have the following:*

(i) *T_a is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$;*

(ii) *for each $b \in \mathrm{BMO}_\infty$, the commutator $[b, T_a]$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$.*

In particular, the obtained results in (i) and (ii) still hold for $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$.

Now we consider the class $L^\infty S^m_\rho$. If $a \in L^\infty S^m_\rho$ with $\rho \in [0,1]$ and $m < n(\rho - 1)$, then the authors in [5] proved that the pseudodifferential operator T_a and its commutators with BMO functions $[b, T_a]$ are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$; see [5, Theorems 3.3 and 4.5]. So, Theorem 3 leads us to the following result.

Corollary 5. *Let $a \in L^\infty S^m_\rho$ with $\rho \in [0,1]$ and $m < n(\rho - 1)$. Then we have the following:*

(i) *T_a is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$;*

(ii) *for each $b \in \mathrm{BMO}_\infty$, the commutator $[b, T_a]$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$.*

In particular, the obtained results in (i) and (ii) still hold for $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$.

It was proved in [5, Theorem 3.7] that if $a \in L^\infty A^m_\rho$ with $0 \leq \rho \leq 1$ and $m < n(\rho - 1)$, then T_a and $[b, T_a]$ are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$ with $b \in \mathrm{BMO}$. Therefore, in the light of Theorem 3, we have the following:

Corollary 6. *Let $a \in L^\infty A^m_{\rho,\delta}$ with $0 \leq \rho \leq 1$ and $m < n(\rho - 1)$. Then we have the following:*

(i) *T_a is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$;*

(ii) *for each $b \in \mathrm{BMO}_\infty$, the commutator $[b, T_a]$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$.*

In particular, the obtained results in (i) and (ii) still hold for $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$.

For smooth amplitudes, we have the following result.

Corollary 7. *Let $a \in A^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$. Then we have the following:*

(i) *T_a is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$;*

(ii) *for each $b \in \mathrm{BMO}_\infty$, the commutator $[b, T_a]$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$.*

In particular, the obtained results in (i) and (ii) still hold for $w(x) = 1 + |x|^\gamma$ with $\gamma > n(p-1)$.

Proof. The remark in [1, page 11] tells us that T_a is bounded on L^p for $1 < p < \infty$. Thanks to Theorem 3, we conclude that T_a and $[b, T_a]$, $b \in \mathrm{BMO}_\infty$ are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\infty_p$. □

The outline of the paper is as follows. In Section 2, we first recall some definitions of the new class of weights A^∞_p and the new BMO function spaces BMO_∞. Then we also review some basic properties concerning A^∞_p and BMO_∞. Section 3 represents some kernel estimates for the pseudodifferential operator T_a. The proof of the main result will be given in Section 4.

2. Preliminaries

To simplify notation, we will often just use B for $B(x_B, r_B)$ and $|E|$ for the measure of E for any measurable subset $E \subset \mathbb{R}^n$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset \mathbb{R}^n$ we set that

$$S_0(B) = B, \qquad S_j(B) = 2^j B \setminus 2^{j-1} B \quad \text{for } j \in \mathbb{N}. \tag{8}$$

2.1. The New Class of Weights and New BMO *Function Spaces.* Recently, in [12], a new class of weights associated to Schrödinger operators $L := -\Delta + V$, where the potential $V \in RH_{n/2}$, the reverse Hölder class has been introduced. According to [12], the authors defined the new classes of weights $A_p^L = \cup_{\theta \geq 0} A_p^{L,\theta}$ for $p \geq 1$, where $A_p^{L,\theta}$, $\theta \geq 0$, is the set of those weights satisfying

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-1/(p-1)}\right)^{1/p'} \leq C|B|\left(1 + \frac{r}{\rho(x)}\right)^{\theta} \tag{9}$$

for all ball $B = B(x, r)$. We denote that $A_\infty^L = \cup_{p \geq 1} A_p^L$, where the critical radius function $\rho(\cdot)$ is defined by

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V \leq 1\right\}, \quad x \in \mathbb{R}^n. \tag{10}$$

In this paper, we consider the particular case when $\rho(\cdot) \equiv 1$. In this situation the new classes of weights are defined by $A_p^\infty = \cup_{\theta \geq 0} A_p^\theta$ for $p \geq 1$, where $A_p^\theta, \theta \geq 0$ is the set of those weights satisfying

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-1/(p-1)}\right)^{1/p'} \leq C|B|(1 + r_B)^{\theta} \tag{11}$$

for all ball $B = B(x_B, r_B)$. We denote that $A_\infty^\infty = \cup_{p \geq 1} A_p^\infty$.

It is easy to see that the new class A_p^∞ is larger than the Muckenhoupt class A_p. The following properties hold for the new classes A_p^∞; see [12, Proposition 5].

Proposition 8. *The following statements hold:*

(i) $A_p^\infty \subset A_q^\infty$ *for* $1 \leq p \leq q < \infty$,

(ii) *if* $w \in A_p^\infty$ *with* $p > 1$, *then there exists* $\epsilon > 0$ *such that* $w \in A_{p-\epsilon}^\infty$. *Consequently,* $A_p^\infty = \cup_{q<p} A_q^\infty$.

Similarly, by adapting the ideas to [13], the new BMO space BMO_θ with $\theta \geq 0$ is defined as a set of all locally integrable functions b satisfying

$$\frac{1}{|B|} \int_B |b(y) - b_B|\, dy \leq C(1 + r_B)^{\theta}, \tag{12}$$

where $B = B(x_B, r_B)$ and $b_B = (1/|B|)\int_B b$. A norm for $b \in \mathrm{BMO}_\theta$, denoted by $\|b\|_\theta$, is given by the infimum of the constants satisfying (12). Clearly $\mathrm{BMO}_{\theta_1} \subset \mathrm{BMO}_{\theta_2}$ for $\theta_1 \leq \theta_2$ and $\mathrm{BMO}_0 = \mathrm{BMO}$. We define $\mathrm{BMO}_\infty = \cup_{\theta>0} \mathrm{BMO}_\theta$.

The following result can be considered to be a variant of John-Nirenberg inequality for the spaces BMO_θ.

Proposition 9. *Let* $\theta > 0$, $s \geq 1$. *If* $b \in \mathrm{BMO}_\theta$, *then for all balls* B

(i)

$$\left(\frac{1}{|B|} \int_B |b(y) - b_B|^s dx\right)^{1/s} \lesssim \|b\|_\theta (1 + r_B)^{\theta}; \tag{13}$$

(ii)

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B|^s dx\right)^{1/s} \lesssim \|b\|_\theta k\left(1 + 2^k r_B\right)^{\theta} \tag{14}$$

for all $k \in \mathbb{N}$.

The proof is similar (even easier) to [13, Lemma 1 and Proposition 3] and hence we omit details.

2.2. Weighted Estimates for Some Localized Operators. A ball of the form $B(x_B, r_B)$ is called *a critical ball* if $r_B = 1$. We have the following result.

Proposition 10. *There exists a sequence of points* x_j, $j \geq 1$ *in* $\mathbb{R}^n$ *so that the family of critical balls* $\{Q_j\}_j$ *where* $Q_j := B(x_j, 1)$, $j \geq 1$ *satisfies the following:*

(i) $\cup_j Q_j = \mathbb{R}^n$,

(ii) *there exists a constant* C *such that for any* $\sigma > 1$, $\sum_j \chi_{\sigma Q_j} \leq C\sigma^n$.

Note that the more general version of Proposition 10 is obtained by [14]. However, in our particular situation, for convenience, we would like to give a simple proof of this proposition.

Proof. Let us consider the family of balls $\{B(x, 1/5) : x \in \mathbb{R}^n\}$. Using Vitali covering lemma, we can pick the subfamily of balls $\{B_j := B(x_j, 1/5) : j \geq 1\}$ so that $\{Q_j\}_j$ is pairwise disjoint and $\mathbb{R}^n \subset \cup_j Q_j$ where $Q_j = 5B_j = B(x_j, 1)$. This gives (i).

To prove (ii), pick any $x \in \mathbb{R}^n$. Let $\mathbb{J}$ be the set of all indices j so that $x \in \sigma Q_j$. Note that if $x \in \sigma Q_j$, then $\sigma Q_j \subset B(x, 2\sigma)$. Therefore, $B(x_j, 1/5) \subset B(x, 2\sigma)$ for all $j \in \mathbb{J}$. Since $\{B(x_j, 1/5)\}_{j \in \mathbb{J}}$ is pairwise disjoint, $\sum_{j \in \mathbb{J}} |B(x_j, 1/5)| \leq |B(x, 2\sigma)|$. This is equivalent to that $|\mathbb{J}|/5^n \leq C\sigma^n$. Hence, $|\mathbb{J}| \leq C\sigma^n$. This completes our proof. □

We consider the following maximal functions for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$\begin{aligned} M_{\mathrm{loc},\alpha} g(x) &= \sup_{x \in B \in \mathscr{B}_\alpha} \frac{1}{|B|} \int_B |g|, \\ M^\sharp_{\mathrm{loc},\alpha} g(x) &= \sup_{x \in B \in \mathscr{B}_\alpha} \frac{1}{|B|} \int_B |g - g_B|, \end{aligned} \tag{15}$$

where $\mathscr{B}_\alpha = \{B(y, r) : y \in \mathbb{R}^n \text{ and } r \leq \alpha\}$.

Also, given a ball Q, we define the following maximal functions for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in Q$:

$$M_Q g(x) = \sup_{x \in B \in \mathcal{F}(Q)} \frac{1}{|B \cap Q|} \int_{B \cap Q} |g|,$$
$$M_Q^\sharp g(x) = \sup_{x \in B \in \mathcal{F}(Q)} \frac{1}{|B \cap Q|} \int_{B \cap Q} \left|g - g_{B \cap Q}\right|, \tag{16}$$

where $\mathcal{F}(Q) = \{B(y, r) : y \in Q, r > 0\}$.

We have the following lemma.

Lemma 11. *For $1 < p < \infty$, let $\{Q_k\}_k$ be a sequence of balls as in Proposition 10. Then*

$$\int_{\mathbb{R}^n} \left|M_{\mathrm{loc},1/2} g(x)\right|^p w(x)\, dx \lesssim \int_{\mathbb{R}^n} \left|M^\sharp_{\mathrm{loc},4} g(x)\right|^p w(x)\, dx + \sum_k w(Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g|\right)^p \tag{17}$$

for all $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w \in A^\infty_\infty$.

Proof. We adapt the argument in [13, Lemma 2] to our present situation.

By Proposition 10, we have

$$\int_{\mathbb{R}^n} \left|M_{\mathrm{loc},1/2} g(x)\right|^p w(x)\, dx \le C \sum_k \int_{Q_k} \left|M_{\mathrm{loc},1/2} g(x)\right|^p w(x)\, dx. \tag{18}$$

It can be verified that, for $x \in Q_k$, $M_{\mathrm{loc},1/2} g(x) \le M_{2Q_k}(g\chi_{2Q_k})$. Note that since $g\chi_{2Q_k}$ is supported in $2Q_k$, operators M_{2Q_k} and $M^\sharp_{2Q_k}$ are Hardy-Littlewood and sharp maximal functions defined in $2Q_k$ viewed as a space of homogeneous type with the Euclidean metric and the Lebesgues measure restricted to $2Q_k$. Moreover, by definition of A^∞_∞, if $w \in A^\infty_\infty$, then $w \in A_\infty(2Q_k)$, where $A_\infty(2Q_k) = \cup_{p \ge 1} A_p(2Q_k)$, and $A_p(2Q_k)$ is the class of Muckenhoupt weights on the spaces of homogeneous type $2Q_k$. Moreover, due to [12, Lemma 5], $[w]_{A_\infty(2Q_k)} \le C$ for all $k \ge 1$. Therefore, using Proposition 3.4 in [15] gives

$$\begin{aligned} \int_{\mathbb{R}^n} \left|M_{\mathrm{loc},1/2} g(x)\right|^p w(x)\, dx &\le C \sum_k \int_{Q_k} \left|M_{\mathrm{loc},1/2} g(x)\right|^p w(x)\, dx \\ &\le C \sum_k \int_{Q_k} \left|M_{2Q_k}\left(g\chi_{2Q_k}\right)(x)\right|^p w(x)\, dx \\ &\le C \sum_k \int_{2Q_k} \left|M^\sharp_{2Q_k}\left(g\chi_{2Q_k}\right)(x)\right|^p w(x)\, dx \\ &\quad + C \sum_k w(2Q_k) \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g(x)|\, dx\right)^p. \end{aligned} \tag{19}$$

To complete the proof, we need only to check that $M^\sharp_{2Q_k}(g\chi_{2Q_k})(x) \le C M^\sharp_{\mathrm{loc},4}(g)(x)$ for $x \in 2Q_k$. We have

$$M^\sharp_{\mathrm{loc},4}(g)(x) = \sup_{B \in \mathcal{F}(2Q_k) : B \ni x} \frac{1}{|B \cap 2Q_k|} \int_{B \cap 2Q_k} \left|f - f_{B \cap 2Q_k}\right|. \tag{20}$$

If $r_B \ge 4$, due to $r_{2Q_k} = 2$, $2Q_k \subset B$. Hence, in this situation, we have

$$\begin{aligned} \frac{1}{|B \cap 2Q_k|} \int_{B \cap 2Q_k} \left|f - f_{B \cap 2Q_k}\right| &= \frac{1}{|Q_k|} \int_{2Q_k} \left|f - f_{2Q_k}\right| \\ &\le M^\sharp_{\mathrm{loc},4}(g)(x). \end{aligned} \tag{21}$$

Otherwise, if $r_B < 4$, it is obvious that $|B \cap 2Q_k| \approx |B|$. So we have

$$\begin{aligned} \frac{1}{|B \cap 2Q_k|} \int_{B \cap 2Q_k} \left|f - f_{B \cap 2Q_k}\right| &\le 2 \frac{1}{|B \cap 2Q_k|} \int_{B \cap 2Q_k} \left|f - f_B\right| \\ &\le C \frac{1}{|B|} \int_B \left|f - f_B\right| \le C M^\sharp_{\mathrm{loc},4}(g)(x). \end{aligned} \tag{22}$$

This completes our proof. □

Let $N > 0$. For $\kappa \ge 1$ and $p \ge 1$, we define the following functions for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$G^N_{\kappa,p} f(x) = \sup_{Q \ni x; Q \text{ is critical}} \sum_{k=0}^\infty 2^{-Nk} \left(\frac{1}{\left|2^k \widehat{Q}\right|} \int_{2^k \widehat{Q}} |f(z)|^p dz\right)^{1/p}, \tag{23}$$

where $\widehat{Q} = \kappa Q$.

When $\kappa = 1$, we write G^N_p instead of $G^N_{1,p}$. The following result gives the weighted estimates for $G_{\kappa,p}$.

Proposition 12. *Let $p > s > 1$ and $w \in A^{\theta}_{p/s}$, $\theta \geq 0$. Then we have*

$$\left\|G^N_{\kappa,s} f\right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \tag{24}$$

provided that $N > \theta/s + n/p$.

Without loss of generality, we assume that $\kappa = 1$. Assume that $Q = B(x_0, 1)$. For $x \in Q$, $Q \subset 2B(x,1)$. This implies that

$$\begin{aligned} &G^N_s f(x) \\ &\quad \leq C \sum_{k=0}^{\infty} 2^{-Nk} \left(\frac{1}{\left|2^k B(x,1)\right|} \int_{B_k(x,1)} |f(z)|^s dz \right)^{1/s}, \end{aligned} \tag{25}$$

where $B_k(x,1) = B(x, 2^{k+1})$.

Let $\{Q_j\}$ be the family of critical balls given by Proposition 10. Note that if $x \in Q_j$, $B_k(x,1) \subset Q^k_j$ where $Q^k_j = 2^{k+2} Q_j$. These estimates and Hölder's inequalities give

$$\begin{aligned} &\left\|G^N_p f\right\|_{L^p(w)} \\ &\quad \leq C \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \int_{Q_j} \left(\frac{1}{\left|2^k B(x,1)\right|} \int_{B_k(x,1)} |f(z)|^s dz \right)^{p/s} \right. \\ &\qquad\qquad \left. \times w(x)\, dx \right)^{1/p} \\ &\quad \leq C \sum_{k=0}^{\infty} 2^{-Nk} \\ &\qquad \times \left(\sum_j \int_{Q_j} \left(\frac{1}{\left|2^k Q_j\right|} \int_{Q^k_j} |f(z)|^s dz \right)^{p/s} w(x)\, dx \right)^{1/p} \\ &\quad \leq C \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \frac{w(Q_j)}{\left|2^k Q_j\right|^{p/s}} \left(\int_{Q^k_j} |f(z)|^s dz \right)^{p/s} \right)^{1/p} \\ &\quad \leq C \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_j \frac{w(Q^k_j)}{\left|2^k Q_j\right|^{p/s}} \left(\int_{Q^k_j} w^{-(p/s)'/(p/s)} \right)^{(p/s)/(p/s)'} \right. \\ &\qquad\qquad \left. \times \left(\int_{Q^k_j} |f(z)|^p w(z)\, dz \right) \right)^{1/p}. \end{aligned} \tag{26}$$

Since $w \in A^{\theta}_{p/s}$, by definition of the classes A^{θ}_p, we have

$$w(Q^k_j) \left(\int_{Q^k_j} w^{-(p/s)'/(p/s)} \right)^{(p/s)/(p/s)'} \leq C \left|Q^k_j\right|^{p/s} 2^{k\theta \times (p/s)}. \tag{27}$$

This together with (26) gives

$$\begin{aligned} &\left\|G^N_s f\right\|_{L^p(w)} \\ &\quad \leq C \sum_k 2^{-k(N-\theta/s)} \left(\sum_j \int_{Q^k_j} |f(z)|^p w(z)\, dz \right)^{1/p} \\ &\quad \leq C \sum_k 2^{-k(N-\theta/s-n/p)} \|f\|_{L^p(w)} \\ &\quad \leq C \|f\|_{L^p(w)}. \end{aligned} \tag{28}$$

This completes our proof.

For a family of balls $\{Q_k\}_k$ given by Proposition 10, we define the operator $\widetilde{M}_s$, $s \geq 1$, as

$$\widetilde{M}_s f = \sum_k \chi_{Q_k} M_s \left(f \chi_{\widetilde{Q}_k} \right), \tag{29}$$

where $\widetilde{Q}_j = 8 Q_j$ and $M_s f = M(|f|^s)^{1/s}$ with M being the Hardy-Littlewood maximal function. We have the following result.

Proposition 13. *If $p > s > 1$ and $w \in A^{\theta}_{p/s}$, $\theta > 0$, then $\widetilde{M}_s$ is bounded on $L^p(w)$.*

Proof. We have

$$\int_{\mathbb{R}^n} \left|\widetilde{M}_s f(x)\right|^p w(x)\, dx = \sum_j \int_{Q_j} \left| M_s \left(f \chi_{\widetilde{Q}_k} \right) \right|^p w(x)\, dx. \tag{30}$$

For each k, if we consider $\widetilde{Q}_k$ as a space of homogeneous type with the Euclidean metric and the Lebesgues measure restricted to $\widetilde{Q}_k$, then $w \in A_{p/s}(\widetilde{Q}_k)$. Moreover, it can be verified that

$$\left\| M_s \left(f \chi_{\widetilde{Q}_k} \right) \right\|_{L^p(w, \widetilde{Q}_k)} \leq C \|f\|_{L^p(w, \widetilde{Q}_k)}, \tag{31}$$

and the constant C is independent of k.

Therefore, by (ii) of Proposition 10,

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\widetilde{M}_s f(x)\right|^p w(x)\, dx &\leq C \sum_j \int_{\widetilde{Q}_k} |f(x)|^p w(x)\, dx \\ &\leq C \|f\|^p_{L^p(w)}. \end{aligned} \tag{32}$$

This completes our proof. □

3. Some Kernel Estimates

Let $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth radial function which is equal to 1 on the unit ball centered at origin and supported on its concentric double. Set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Then, we have

$$\sum_{k=0}^{\infty} \varphi_k(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \tag{33}$$

and supp $\varphi_k \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ for all $k \geq 1$. Moreover, for any multi-index α and $N \geq 0$, we have

$$\left|\partial_\xi^\alpha \varphi_k(\xi)\right| \leq c_\alpha 2^{-k|\alpha|}. \tag{34}$$

Lemma 14. *Let $a \in L^\infty A^m_{\rho,\delta}$ with $m \in \mathbb{R}$, $\rho \in [0,1]$ and $\delta \in [0,1]$. Let $a_k(x,y,\xi) = a(x,y,\xi)\varphi_k(\xi)$ for $k \geq 0$.*

(a) *For each $\ell \geq 0$,*

$$|z|^\ell \left|\int a_k(x,y,\xi) e^{i\langle z,\xi\rangle} d\xi\right| \leq C 2^{k(n+m-\rho\ell)}. \tag{35}$$

(b) *If $a \in L^\infty A^m_{\rho,\delta}$ with $m < n(\rho-1)$ and $\rho, \delta \in [0,1]$, then, for each $N > 0$, there exist $\epsilon, \epsilon' > 0$ so that for any ball $B \subset \mathbb{R}^n$, $y, \overline{y} \in B$, and $x \in S_j(B)$, $j \geq 2$ so that*

$$\begin{aligned}&\left|\int a_k(x,y,\xi) e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &\quad \leq C 2^{-j\epsilon}\left(2^j r_B\right)^{-n} \min\left\{1, \left(2^j r_B\right)^{-N}\right\} 2^{-k\epsilon'}.\end{aligned} \tag{36}$$

(c) *If $a \in L^\infty A^0_{1,\delta}$, $\delta \in [0,1]$, then there exist $\epsilon, \epsilon' > 0$ so that for any ball $B \subset \mathbb{R}^n$, $y, \overline{y} \in B$, and $x \in S_j(B)$, $j \geq 2$ so that*

$$\begin{aligned}&\left|\int a_k(x,y,\xi) e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &\quad \leq C 2^{-j\epsilon}\left(2^j r_B\right)^{-n} \min\left\{1, \left(2^j r_B\right)^{-N}\right\} \left(2^k r_B\right)^{\epsilon'}\end{aligned} \tag{37}$$

as long as $2^k r_B \leq 1$; and

$$\begin{aligned}&\left|\int a_k(x,y,\xi) e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &\quad \leq C 2^{-j\epsilon}\left(2^j r_B\right)^{-n} \min\left\{1, \left(2^j r_B\right)^{-N}\right\} \left(2^k r_B\right)^{-\epsilon'}\end{aligned} \tag{38}$$

as long as $2^k r_B > 1$.

Proof. We refer to Lemma 3.1 in [5] for the proof of (a).

(b) We first note that since $a \in L^\infty A^m_{\rho,\delta}$, we have

$$\left|\partial_\xi^\alpha a_k(x,y,\xi)\right| \leq c_\alpha 2^{k(m-\rho|\alpha|)} \quad \forall k = 1, 2, \ldots. \tag{39}$$

Since $x \in S_j(B)$, $j \geq 2$ and $y, \overline{y} \in B$, we have $x - y \approx x - \overline{y}$. If $|y - \overline{y}| > 2^{-k}$, using (a) with $\ell = n + \epsilon$ so that $m - n(\rho-1) - \rho\epsilon + \epsilon < 0$ gives

$$\begin{aligned}\text{LHS} &:= \left|\int a_k(x,y,\xi) e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &\leq \left|\int a_k(x,y,\xi) e^{i\langle x-y,\xi\rangle} d\xi\right| + \left|\int a_k(x,\overline{y},\xi) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &\leq C|x-y|^{-n-\epsilon} 2^{k(n+m-\rho n-\rho\epsilon)} \\ &\leq C\left(2^j r_B\right)^{-n-\epsilon} 2^{k(m-n(\rho-1)-\rho\epsilon)}.\end{aligned} \tag{40}$$

This together with the fact that $|y - \overline{y}| > 2^{-k}$ gives

$$\begin{aligned}\text{LHS} &\leq C\left(2^j r_B\right)^{-n-\epsilon} 2^{k(m-n(\rho-1)-\rho\epsilon)} \\ &\leq C\left(2^j r_B\right)^{-n+1} 2^{k((m-n(\rho-1))-\rho\epsilon+\epsilon)} |y-\overline{y}|^\epsilon \\ &\leq C 2^{-j\epsilon}\left(2^j r_B\right)^{-n} 2^{-k\epsilon'},\end{aligned} \tag{41}$$

where $\epsilon' = -[(m - n(\rho-1)) - \rho\epsilon + \epsilon] > 0$.

If $|y - \overline{y}| \leq 2^{-k}$, we have

$$\begin{aligned}\text{LHS} &\leq \left|\int a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right) e^{i\langle x-y,\xi\rangle} d\xi\right| \\ &\quad + \left|\int \left(a_k(x,y,\xi) - a_k(x,\overline{y},\xi)\right) e^{i\langle x-\overline{y},\xi\rangle} d\xi\right| \\ &:= E_1 + E_2.\end{aligned} \tag{42}$$

We will claim that, for all $\ell \geq 0$, we have

$$E_1 \leq C\left(2^j r_B\right)^\ell 2^{k(m+n-\rho\ell+1)} |y - \overline{y}|. \tag{43}$$

Indeed, we have for all integers $\ell \geq 0$,

$$\begin{aligned}E_1 &\leq |x-y|^{-\ell}|x-y|^\ell \\ &\quad \times \left|\int a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right) e^{i\langle x-y,\xi\rangle} d\xi\right| \\ &\leq \left(2^j r_B\right)^{-\ell} \\ &\quad \times \left|\sum_{|\alpha|=\ell} \int (x-y)^\alpha a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right) e^{i\langle x-y,\xi\rangle} d\xi\right| \\ &\leq \left(2^j r_B\right)^{-\ell} \\ &\quad \times \left|\sum_{|\alpha|=\ell} \int a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right) \partial_\xi^\alpha e^{i\langle x-y,\xi\rangle} d\xi\right|.\end{aligned} \tag{44}$$

Using integration by parts, we get that

$$\begin{aligned}E_1 &\leq \left(2^j r_B\right)^{-\ell} \\ &\quad \times \left|\sum_{|\alpha|=\ell} \int \partial_\xi^\alpha\left[a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right)\right] e^{i\langle x-y,\xi\rangle} d\xi\right|.\end{aligned} \tag{45}$$

We write

$$\begin{aligned}&\sum_{|\alpha|=\ell} \partial_\xi^\alpha\left[a_k(x,y,\xi)\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right)\right] \\ &\quad = \sum_{|\alpha|+|\beta|=\ell} \partial_\xi^\alpha a_k(x,y,\xi)\, \partial_\xi^\beta\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right).\end{aligned} \tag{46}$$

If $|\beta| = 0$, $|1 - e^{i\langle y-\overline{y},\xi\rangle}| \le C|y-\overline{y}||\xi| \le C2^k|y-\overline{y}|$. Therefore, in this situation,

$$\left|\sum_{|\alpha|=\ell} \int \partial_\xi^\alpha \left[a_k(x,y,\xi)\right]\left(1 - e^{i\langle y-\overline{y},\xi\rangle}\right) e^{i\langle x-y,\xi\rangle} d\xi\right| \le C2^{k(n+m+1-\rho|\alpha|)}|y-\overline{y}| = C2^{k(n+m+1-\rho\ell)}|y-\overline{y}|. \tag{47}$$

Otherwise, $|\partial_\xi^\beta(1-e^{i\langle y-\overline{y},\xi\rangle})| \le C|y-\overline{y}|^{|\beta|}$. This together with (39) gives

$$\begin{aligned}\left|\int \partial_\xi^\alpha a_k(x,y,\xi)\,\partial_\xi^\beta\left(1-e^{i\langle y-\overline{y},\xi\rangle}\right)e^{i\langle x-y,\xi\rangle}d\xi\right| &\le C2^{k(n+m-\rho|\alpha|)}|y-\overline{y}|^{|\beta|}\\ &\le C2^{k(n+m+1-\rho|\alpha|-|\beta|)}|y-\overline{y}|\\ &\le C2^{k(n+m+1-\rho\ell)}|y-\overline{y}|.\end{aligned} \tag{48}$$

Therefore,

$$E_1 \le C\left(2^j r_B\right)^{-\ell} 2^{k(m+n-\rho\ell+1)}|y-\overline{y}|. \tag{49}$$

The general statement for noninteger values of ℓ follows by interpolation of the inequality for i and $i+1$, where $i < \ell < i+1$. Therefore, (43) holds for all $\ell > 0$. Now taking $\ell = n+\epsilon$ so that $\epsilon' = -(m+n-\rho n-\rho\epsilon+\epsilon) > 0$, we have

$$\begin{aligned}E_1 &\le C\left(2^j r_B\right)^{-n-\epsilon}2^{k(m+n-\rho n-\rho\epsilon+\epsilon)}|y-\overline{y}|^\epsilon\left(2^k|y-\overline{y}|\right)^{1-\epsilon}\\ &\le C\left(2^j r_B\right)^{-n-\epsilon}2^{-k\epsilon'}|y-\overline{y}|^\epsilon\\ &\le C2^{-j\epsilon}\left(2^j r_B\right)^{-n}2^{-k\epsilon'}.\end{aligned} \tag{50}$$

It remains to take care of the term E_2. Repeating the previous arguments we also obtain

$$E_2 \le \left(2^j r_B\right)^{-\ell} \times\left|\sum_{|\alpha|=\ell}\int \partial_\xi^\alpha\left[a_k(x,y,\xi) - a_k(x,\overline{y},\xi)\right]e^{i\langle y-\overline{y},\xi\rangle}d\xi\right|. \tag{51}$$

At this stage, using the mean value theorem (applied for each component of a_k) and then using the definition of the class $L^\infty A^m_{\rho,\delta}$ give

$$\begin{aligned}E_2 &\le C\left(2^j r_B\right)^{-\ell}|y-\overline{y}|\,2^{k(n+m-\rho\ell+\delta)}\\ &\le C\left(2^j r_B\right)^{-\ell}|y-\overline{y}|\,2^{k(n+m-\rho\ell+1)}\end{aligned} \tag{52}$$

for all integer $\ell \ge 0$. Hence, by interpolation again,

$$E_2 \le C\left(2^j r_B\right)^{-\ell}|y-\overline{y}|\,2^{k(n+m-\rho\ell+1)} \tag{53}$$

for all $\ell \ge 0$. Repeating the arguments used to estimate E_1, we conclude that

$$E_2 \le C2^{-j\epsilon}\left(2^j r_B\right)^{-n}2^{-k\epsilon'}. \tag{54}$$

Therefore, LHS $\le C2^{-j\epsilon}(2^j r_B)^{-n}2^{-k\epsilon'}$. It remains to show that

$$\text{LHS} \le C2^{-j\epsilon}\left(2^j r_B\right)^{-n-N}2^{-k\epsilon'}. \tag{55}$$

To do this, we repeat the arguments above with $\ell = N+n+\epsilon$. Since the proof of this part is analogous to (55), and hence we omit details here. This completes our proof.

(c) If $2^{-k} \le r_B$, using the argument as in (b), we have

$$\begin{aligned}\text{LHS} &:= \left|\int a_k(x,y,\xi)e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi)e^{i\langle x-\overline{y},\xi\rangle}d\xi\right|\\ &\le \left|\int a_k(x,y,\xi)e^{i\langle x-y,\xi\rangle}d\xi\right| + \left|\int a_k(x,\overline{y},\xi)e^{i\langle x-\overline{y},\xi\rangle}d\xi\right|\\ &\le C|x-y|^{-n-\epsilon}2^{-k\epsilon}\\ &\le C\frac{\left(2^j r_B\right)^{-n-\epsilon}}{r_B^\epsilon}\frac{2^{-k\epsilon}}{r_B^\epsilon} = C2^{-j\epsilon}\left(2^j r_B\right)^n\left(\frac{1}{r_B 2^k}\right)^\epsilon.\end{aligned} \tag{56}$$

If $r_B < 2^{-k}$, we have

$$\begin{aligned}\text{LHS} &\le \left|\int a_k(x,y,\xi)\left(1-e^{i\langle y-\overline{y},\xi\rangle}\right)e^{i\langle x-y,\xi\rangle}d\xi\right|\\ &\quad + \left|\int \left(a_k(x,y,\xi) - a_k(x,\overline{y},\xi)\right)e^{i\langle x-\overline{y},\xi\rangle}d\xi\right|\\ &:= E_1 + E_2.\end{aligned} \tag{57}$$

The previous arguments in (b) show that

$$\begin{aligned}E_1 + E_2 &\le C\left(2^j r_B\right)^{-n-\epsilon}2^{k(-\epsilon+1)}|y-\overline{y}|\\ &\le C\left(2^j r_B\right)^{-n-\epsilon}2^{k(-\epsilon+1)}r_B\\ &= C\left(2^j r_B\right)^{-n-\epsilon}r_B^\epsilon\left(r_B 2^k\right)^{(-\epsilon+1)}\\ &\le C2^{-j\epsilon}\left(2^j r_B\right)^{-n}\left(r_B 2^k\right)^{(1-\epsilon)}.\end{aligned} \tag{58}$$

Hence,

$$\begin{aligned}&\left|\int a_k(x,y,\xi)e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi)e^{i\langle x-\overline{y},\xi\rangle}d\xi\right|\\ &\quad\le C2^{-j\epsilon}\left(2^j r_B\right)^{-n}(2^k r_B)^{\epsilon'} \quad \text{if } 2^k r_B \le 1,\\ &\left|\int a_k(x,y,\xi)e^{i\langle x-y,\xi\rangle} - a_k(x,\overline{y},\xi)e^{i\langle x-\overline{y},\xi\rangle}d\xi\right|\\ &\quad\le C2^{-j\epsilon}\left(2^j r_B\right)^{-n}\left(2^k r_B\right)^{-\epsilon'} \quad \text{if } 2^k r_B > 1.\end{aligned} \tag{59}$$

By taking $\ell = n+N+\epsilon$ and repeating the previous arguments, we obtain that

$$\begin{aligned} &\left|\iint a_k(x,y,\xi)\, e^{i\langle x-y,\xi\rangle} - a_k(x,\bar{y},\xi)\, e^{i\langle x-\bar{y},\xi\rangle} d\xi\right| \\ &\quad \leq C2^{-j\epsilon}\left(2^j r_B\right)^{-n-N}\left(2^k r_B\right)^{\epsilon'} \quad \text{if } 2^k r_B \leq 1, \\ &\left|\iint a_k(x,y,\xi)\, e^{i\langle x-y,\xi\rangle} - a_k(x,\bar{y},\xi)\, e^{i\langle x-\bar{y},\xi\rangle} d\xi\right| \\ &\quad \leq C2^{-j\epsilon}\left(2^j r_B\right)^{-n-N}\left(2^k r_B\right)^{-\epsilon'} \quad \text{if } 2^k r_B > 1. \end{aligned} \tag{60}$$

This completes the proof of (c). □

Since the associated kernel $K(x,y)$ of the operator T_a is given by

$$\begin{aligned} K(x,y) &= \frac{1}{(2\pi)^n}\int a(x,y,\xi)\, e^{i\langle x-y,\xi\rangle} d\xi \\ &= \sum_{k\geq 0}\frac{1}{(2\pi)^n}\int a_k(x,y,\xi)\, e^{i\langle x-y,\xi\rangle} d\xi \end{aligned} \tag{61}$$

with $a_k(x,\xi)$ as in Lemma 14, from Lemma 14 we deduce directly the following result.

Lemma 15. *Let $a \in L^\infty A^m_{\rho,\delta}$ with $m < n(\rho-1)$ or $a \in L^\infty A^0_{1,\delta}$, $\delta \in [0,1]$, and let $K^*(x,y)$ be the associated kernel of the operator T^*_a, the conjugate of T_a.*

(a) *For any $N > 0$, we have*

$$\left|K^*(x,y)\right| \leq \frac{C}{|x-y|^{-N}}, \quad x \neq y. \tag{62}$$

(b) *For any $N > 0$, there exists $\epsilon > 0$ so that any ball $B \subset \mathbb{R}^n$, $y, \bar{y} \in B$, $x \in S_j(B)$, $j \geq 2$, we have*

$$\begin{aligned} &\left|K^*(y,x) - K^*(\bar{y},x)\right| \\ &\quad \leq C2^{-j\epsilon}\left(2^j r_B\right)^{-n}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}. \end{aligned} \tag{63}$$

4. Proof of Theorem 3

Note that, by duality argument, the linear operator T is bounded on $L^p(w)$, $1 < p < \infty$ if and only if its conjugate T^* is bounded on $L^{p'}(w^{1-p'})$. Moreover, by Hölder's inequality, it can be verified that $w \in A^\infty_p$ if and only if $w^{1-p'} \in A^\infty_{p'}$. Therefore, it suffices to prove (a) and (b) for T^*_a and $T^{*,b}_a = [b, T^*_a]$ with $b \in \mathrm{BMO}_\infty$. Before coming to the proof of Theorem 3, we need the following results.

Lemma 16. *Let $a \in L^\infty A^m_{\rho,\delta}$ with $m < n(\rho-1)$ or $a \in L^\infty A^0_{1,\delta}$, $\delta \in [0,1]$, and $b \in BMO_\theta$, $\theta \geq 0$. If T_a is bounded on L^p for all $1 < p < \infty$, then for any $p > 1$ and $N > 0$ there exists $C > 0$ such that for all balls $Q = Q(x_0, 1)$,*

(a)

$$\frac{1}{|Q|}\int_Q\left|T^*_a f(x)\right| dx \leq C\inf_{y\in Q} G^N_p(y); \tag{64}$$

(b)

$$\frac{1}{|Q|}\int_Q\left|T^{*,b}_a f(x)\, dx\right| \leq C\inf_{y\in Q} G^{N-n-\theta}_p f(y)\, \|b\|_\theta; \tag{65}$$

Proof. (a) We split $f = f_1 + f_2$ where $f_1 = f\chi_{4Q}$. For each $j \geq 0$, we have

$$\begin{aligned} \frac{1}{|Q|}\int_Q\left|T^*_a f(x)\right| dx &\leq \frac{1}{|Q|}\int_Q\left|T^*_a f_1\right| + \frac{1}{|Q|}\int_Q\left|T^*_a f_2\right| \\ &:= I_1 + I_2. \end{aligned} \tag{66}$$

Using Hölder's inequality and the fact that T^*_a is bounded on L^p, $1 < p < \infty$, we write

$$\begin{aligned} I_1 &\leq C\left(\frac{1}{|Q|}\int_Q\left|T^*_a f_1\right|^p\right)^{1/p} \leq \left(\frac{1}{|4Q|}\int_{4Q}|f|^p\right)^{1/p} \\ &\leq C\inf_{y\in Q} G^N_p f(y). \end{aligned} \tag{67}$$

For the term I_2 we have, for $x \in Q$,

$$\begin{aligned} T^*_a f_2(x) &= \int_{R^n\setminus 4Q} K^*(x,y) f(y)\, dy \\ &= \int_{R^n\setminus 4Q} K^*(y,x) f(y)\, dy \\ &= \sum_{k\geq 3}\int_{S_k(Q)} K^*(y,x) f(y)\, dy. \end{aligned} \tag{68}$$

Applying (a) of Lemma 15, we have

$$\begin{aligned} T^*_a f_2(x) &= \sum_{k\geq 3}\int_{S_k(Q)} K^*(x,y) f(y)\, dy \\ &\leq \sum_{k\geq 3}\int_{S_k(Q)}\frac{f(y)}{|x-y|^{n+N}} dy \\ &\leq C\inf_{y\in Q} G^N f(y) \leq C\inf_{y\in Q} G^N_p f(y). \end{aligned} \tag{69}$$

This completes the proof of (a).

(b) Taking $1 < r < p$, we write

$$T^{*,b}_a f = \left(b - b_Q\right) T^*_a f - T^*_a\left(\left(b - b_Q\right) f\right). \tag{70}$$

So, we have

$$\begin{aligned} \frac{1}{|Q|}\int_Q\left|T^{*,b}_a f(x)\right| dx &\leq \frac{1}{|Q|}\int_Q\left|\left(b - b_Q\right) T^*_a f\right| dx \\ &\quad + \frac{1}{|Q|}\int_Q\left|T^*_a\left(\left(b - b_Q\right) f\right)(x)\right| dx \\ &:= I_1 + I_2. \end{aligned} \tag{71}$$

We now take care of I_1. By Hölder's inequality, we can write

$$\begin{aligned} I_1 &\le C\|b\|_\theta \left(\frac{1}{|Q|} \int_Q |T_a^* f|^p \right)^{1/p} \\ &\le C\|b\|_\theta \left(\left(\frac{1}{|Q|} \int_Q |T_a^* f_1|^p \right)^{1/p} + \left(\frac{1}{|Q|} \int_Q |T_a^* f_2|^p \right)^{1/p} \right) \\ &:= I_{11} + I_{12}, \end{aligned} \tag{72}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{4Q}$.

Due to L^p boundedness of T_a^*, one has

$$I_{11} \le C \left(\frac{1}{|4Q|} \int_{4Q} |f|^p \right)^{1/p} \le C \inf_{y\in Q} G_p^N f(y). \tag{73}$$

To estimate I_{12}, using (69) gives $I_{12} \le C \inf_{y\in Q} G_p^N f(y)$.

The estimate for I_2 can be proceeded in the same method. Indeed, we write

$$\begin{aligned} &\frac{1}{|Q|} \int_Q \left| T_a^* \left(\left(b - b_Q \right) f \right)(x) \right| dx \\ &\quad \le \frac{1}{|Q|} \int_Q \left| T_a^* \left(\left(b - b_Q \right) f_1 \right)(x) \right| dx \\ &\qquad + \frac{1}{|Q|} \int_Q \left| T_a^* \left(\left(b - b_Q \right) f_2 \right)(x) \right| dx \\ &\quad := I_{21} + I_{22}, \end{aligned} \tag{74}$$

where $f = f_1 + f_2$ and $f_1 = f\chi_{4Q}$.

To estimate I_{21}, using Hölder's inequality, we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q \left| T_a^* \left(\left(b - b_Q \right) f_1 \right)(x) \right| dx \\ &\quad \le \left(\frac{1}{|Q|} \int_Q \left| T_a^* \left(\left(b - b_Q \right) f_1 \right)(x) \right|^r dx \right)^{1/r} \\ &\quad \le \left(\frac{1}{|Q|} \int_Q \left| \left(\left(b - b_Q \right) f_1 \right)(x) \right|^r dx \right)^{1/r} \\ &\quad \le \left(\frac{1}{|4Q|} \int_{4Q} |f(x)|^p dx \right)^{1/p} \\ &\qquad \times \left(\frac{1}{|4Q|} \int_{4Q} \left| b(x) - b_Q \right|^\nu dx \right)^{1/\nu} \quad \left(\nu = \frac{pr}{p-r} \right) \\ &\quad \le C\|b\|_\theta \inf_{y\in Q} G_p^N f(y). \end{aligned} \tag{75}$$

For the term I_{22}, due to (a) of Lemma 15, we can write

$$\begin{aligned} &T_a^* \left(\left(b - b_Q \right) f_2 \right)(x) \\ &\quad = \sum_{k\ge 3} \int_{S_k(Q)} K^*(x,y) \left(\left(b - b_Q \right) f \right)(y)\, dy \\ &\quad \le C \sum_{k\ge 3} 2^{-kN} \int_{S_k(Q)} \left| \left(b(y) - b_Q \right) f(y) \right| dy. \end{aligned} \tag{76}$$

By Hölder's inequality and Proposition 9, we give

$$\begin{aligned} &\int_{S_k(Q)} \left| \left(b(y) - b_Q \right) f(y) \right| dy \\ &\quad \le \left| 2^k Q \right| \left(\frac{1}{\left| 2^k Q \right|} \int_{2^k Q} |f|^p \right)^{1/p} \\ &\qquad \times \left(\frac{1}{\left| 2^k Q \right|} \int_{2^k Q} \left| b - b_Q \right|^{p'} \right)^{1/p'} \\ &\quad \le k 2^{k\theta} \left| 2^k Q \right| \|b\|_\theta \left(\frac{1}{\left| 2^k Q \right|} \int_{2^k Q} |f|^p \right)^{1/p} \\ &\quad \le k 2^{k(\theta+n)} \|b\|_\theta \left(\frac{1}{\left| 2^k Q \right|} \int_{2^k Q} |f|^p \right)^{1/p}. \end{aligned} \tag{77}$$

From (77) and (76) we obtain that

$$T_a^* \left(\left(b - b_Q \right) f_2 \right)(x) \le C\|b\|_\theta \inf_{y\in Q} G_p^{N-n-\theta} f(y). \tag{78}$$

This completes our proof. □

Remark 17. The result in Lemma 16 still holds if we replace the critical ball Q by 2Q.

Lemma 18. *Let $a \in L^\infty A_{\rho,\delta}^m$ with $m < n(\rho - 1)$ or $a \in L^\infty A_{1,\delta}^0$, $\delta \in [0,1]$ and $b \in \mathrm{BMO}_\theta$, $\theta \ge 0$. If T_a is bounded on L^p for all $1 < p < \infty$, then for any $p > 1$ and $N > 0$ there exists $C > 0$ so that, for all f and $x, y \in B = B(x_B, r_B)$ with $r_B < 4$, we have*

(a)

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus 2B} \left| \left(K^*(x,z) - K^*(y,z) \right) f(z) \right| dz \\ &\quad \le C \left(\inf_{u\in B} G_{4,p}^N f(u) + \inf_{u\in B} \widetilde{M}_p f(u) \right); \end{aligned} \tag{79}$$

(b)

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus 2B} \left| \left(K^*(x,z) - K^*(y,z) \right) \left(\left(b - b_B \right) f \right)(z) \right| dz \\ &\quad \le C\|b\|_\theta \inf_{u\in B} \left(\inf_{u\in B} G_{4,p}^{N-n-\theta} f(u) + \inf_{u\in B} \widetilde{M}_p f(u) \right). \end{aligned} \tag{80}$$

Proof. (a) Using (b) of Lemma 15, we write

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B} \left|\left(K^*(x,z)-K^*(y,z)\right)f(z)\right| dz \\
&\quad\leq C\sum_{k\geq 2}\int_{S_k(B)} \left|\left(K^*(x,z)-K^*(y,z)\right)f(z)\right| dz \\
&\quad\leq C\sum_{k\geq 2}2^{-k\epsilon}\left(2^k r_B\right)^{-n}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}\int_{S_k(B)}|f(z)|\,dz \\
&\quad\leq C\sum_{k\geq 2}2^{-k\epsilon}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}\frac{1}{|2^k B|}\int_{S_k(B)}|f(z)|\,dz \\
&\quad=\sum_{k=2}^{k_0}\cdots+\sum_{k>k_0}\cdots := I_1+I_2,
\end{aligned} \tag{81}$$

where k_0 is the smallest integer so that $2^{k_0+1}r_B > 4$.

To estimate I_1, let $\{Q_l\}$ and $\{\widetilde{Q}_l\}$ be families of balls as in (29). If $x\in Q_l\cap B$, then $2^k B\subset\widetilde{Q}_l$ for all $k=1,2,\ldots,k_0$. This implies that

$$\frac{1}{|2^k B|}\int_{2^k B}|f(z)|\,dz \leq \inf_{u\in B}\widetilde{M}_p f(u) \tag{82}$$

for all $k=1,2,\ldots,k_0$.

Hence

$$I_1\leq\sum_{k=2}^{k_0}2^{-k\epsilon}\inf_{u\in B}\widetilde{M}_p f(u)\leq C\inf_{u\in B}\widetilde{M}_p f(u). \tag{83}$$

For the term I_2, since $2^{k_0}r_B\geq 4$ we have

$$\begin{aligned}
I_2&\leq\sum_{k\geq k_0}2^{-k\epsilon}\left(2^k r_B\right)^{-N}\frac{1}{|2^k B|}\int_{S_k(B)}|f(z)|\,dz \\
&\leq\sum_{k\geq k_0}2^{-k\epsilon}\left(2^{k-k_0}2^{k_0}r_B\right)^{-N}\frac{1}{|2^{k-k_0}2^{k_0}B|}\int_{2^{k-k_0}2^{k_0}B}|f(z)|\,dz \\
&\leq\sum_{k\geq k_0}2^{-k\epsilon}\left(2^{k-k_0}\right)^{-N}\frac{1}{|2^{k-k_0}2^{k_0}B|}\int_{2^{k-k_0}2^{k_0}B}|f(z)|\,dz \\
&\leq\sum_{k\geq 0}2^{-k\epsilon}2^{-kN}\frac{1}{|2^k 2^{k_0}B|}\int_{2^k 2^{k_0}B}|f(z)|\,dz.
\end{aligned} \tag{84}$$

Note that $2^{k_0}B\subset\widehat{Q}=4Q$ here $Q=B(x_0,1)$ and $|Q|\approx|2^{k_0}B|$. So, we have

$$\begin{aligned}
I_2&\leq\sum_{k\geq 0}2^{-k\epsilon}2^{-kN}\left(\frac{1}{|2^k\widehat{Q}|}\int_{2^k\widehat{Q}}|f(z)|\,dz\right) \\
&\leq C\inf_{u\in B}G^N_{4,p}f(u).
\end{aligned} \tag{85}$$

Hence, we get (a).

(b) Using Hölder's inequality and (b) of Lemma 15, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B}\left|\left(K^*(x,z)-K^*(y,z)\right)\left(\left(b-b_B\right)f\right)(z)\right| dz \\
&\quad=\sum_{k\geq 2}\int_{S_k(B)}\left|\left(K^*(x,z)-K^*(y,z)\right)\left(\left(b-b_B\right)f\right)(z)\right| dz \\
&\quad\leq C\sum_{k\geq 2}2^{-k\epsilon}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}\frac{1}{|2^k B|} \\
&\qquad\times\int_{S_k(B)}\left|\left(\left(b-b_B\right)f\right)(z)\right| dz \\
&\quad\leq C\sum_{k\geq 2}2^{-k\epsilon}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}\left(\frac{1}{|2^k B|}\int_{2^k B}|f(z)|^p dz\right)^{1/p} \\
&\qquad\times\left(\frac{1}{|2^k B|}\int_{2^k B}\left|b(z)-b_B\right|^{p'}dz\right)^{1/p'}.
\end{aligned} \tag{86}$$

Now using Proposition 9, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B}\left|\left(K^*(x,z)-K^*(y,z)\right)\left(\left(b-b_B\right)f\right)(z)\right| dz \\
&\quad\leq C\sum_{k\geq 2}k2^{-k\epsilon}\left(2^j r_B\right)^{\theta}\min\left\{1,\left(2^j r_B\right)^{-N}\right\}\|b\|_{\theta} \\
&\qquad\times\left(\frac{1}{|2^k B|}\int_{2^k B}|f(z)|^p dz\right)^{1/p}.
\end{aligned} \tag{87}$$

At this stage, repeating the same argument as in (a), we complete the proof of (b). □

We are now in position to prove Theorem 3.

Proof of Theorem 3. (a) Using the standard argument, see, for example, [13], fix $1<p<\infty$ and $w\in A_p^{\infty}$. Let $N>0$ which will be fixed later. So, by Proposition 8, we can pick $r>1$ and $\nu\geq 0$ so that $w\in A^{\nu}_{p/r}$. By Lemma 11 we have

$$\begin{aligned}
\|T_a^* f\|^p_{L^p(w)}&\leq\left\|M_{\mathrm{loc},\beta}T_a^* f\right\|^p_{L^p(w)} \\
&\leq C\left\|M^{\sharp}_{\mathrm{loc},4}T_a^* f\right\|^p_{L^p(w)} \\
&\quad+C\sum_k w\left(Q_k\right)\left(\frac{1}{2Q_k}\int_{2Q_k}\left|T_a^* f\right|\right)^p \\
&:= I_1+I_2.
\end{aligned} \tag{88}$$

Let us estimate I_1 first. By Lemma 16 and Remark 17, we have

$$\frac{1}{2Q_k}\int_{2Q_k}\left|T_a^* f\right|\leq C\inf_{y\in Q_k}G_r^N f(y). \tag{89}$$

Invoking Proposition 12, we conclude that

$$\begin{aligned}\sum_k w(Q_k)\left(\frac{1}{2Q_k}\int_{2Q_k}|T_a^* f|\right)^p &\le \sum_k \int_{Q_k}\left|G_r^N f(x)\right|^p w(x)\,dx \\ &\le C\int_{\mathbb{R}^n}\left|G_r^N f(x)\right|^p w(x)\,dx \\ &\le C\|f\|_{L^p(w)}^p\end{aligned} \tag{90}$$

as long as $N > n/p + \nu/r$. We now take care of I_2. For any ball $B(x_0, r_B)$ with $r_B \le 4$ and $x \in B$, we write

$$\begin{aligned}\frac{1}{|B|}\int_B |T_a^* f(x) - (T_a^* f)_B|\,dx &\le \frac{2}{|B|}\int_B |T_a^* f_1(x)|\,dx \\ &\quad + \frac{1}{|B|}\int_B |T_a^* f_2(x) - (T_a^* f_2)_B|\,dx \\ &:= E_1 + E_2,\end{aligned} \tag{91}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

For E_1, since T_a^* is bounded on L^r, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |T_a^* f_1(x)|\,dx &\le \left(\frac{1}{|B|}\int_B |T_a^* f_1(x)|^r dx\right)^r \\ &\le C\left(\frac{1}{|2B|}\int_{2B}|f|^r dx\right)^r \\ &\le C\inf_{u\in B}\widetilde{M}_r f(u).\end{aligned} \tag{92}$$

Due to Lemma 18, we can write

$$\begin{aligned}E_2 &\le \frac{1}{|B|^2} \\ &\quad\times \iint_B \left(\int_{\mathbb{R}^n\setminus 2B} |(K^*(u,z) - K^*(y,z))\, f(z)|\,dz\right) dy\,du \\ &\le C\left(\inf_{u\in B} G_{4,r}^N f(u) + \inf_{u\in B}\widetilde{M}_r f(u)\right).\end{aligned} \tag{93}$$

These two estimates of E_1 and E_2 tell us that

$$M_{\mathrm{loc},4}^{\sharp} T_a^* f(x) \le C\left(G_{4,r}^N f(x) + \widetilde{M}_r f(x)\right). \tag{94}$$

Applying Proposition 12 and the weighted estimates of $\widetilde{M}_r$, we get that

$$\left\|M_{\mathrm{loc},4}^{\sharp} T_a^* f\right\|_{L^p(w)} \le C\|f\|_{L^p(w)} \tag{95}$$

provided that $M > n/p + \nu/r$.

From (90) and (95), we obtain that

$$\|T_a^* f\|_{L^p(w)} \le C\|f\|_{L^p(w)}. \tag{96}$$

This completes our proof.

(b) Fixed $1 < p < \infty$, $b \in \mathrm{BMO}_\theta$, $\theta \ge 0$ and $w \in A_p^\infty$. So, we can pick $r > 1$ and $\nu \ge 0$ so that $w \in A_{p/r}^\nu$. Then we have by Lemma 11

$$\begin{aligned}\left\|T_a^{*,b} f\right\|_{L^p(w)}^p &\le \int_{\mathbb{R}^n}\left|M_{\mathrm{loc},\beta}\left(T_a^{*,b} f\right)(x)\right|^p w(x)\,dx \\ &\le C\int_{\mathbb{R}^n}\left|M_{\mathrm{loc},4}^{\sharp}\left(T_a^{*,b} f\right)(x)\right|^p w(x)\,dx \\ &\quad + \sum_k w(Q_k)\left(\frac{1}{|2Q_k|}\int_{2Q_k}\left|T_a^{*,b} f\right|\right)^p,\end{aligned} \tag{97}$$

where $\{Q_k\}$ is a family of critical balls given in Lemma 11.

The analogous argument to that in (a) gives

$$\begin{aligned}\sum_k w(Q_k)\left(\frac{1}{|2Q_k|}\int_{2Q_k}\left|T_a^{*,b} f\right|\right)^p \\ \le C\|b\|_\theta^p \|f\|_{L^p(w)}^p.\end{aligned} \tag{98}$$

It remains to estimate $\int_{\mathbb{R}^n}|M_{\mathrm{loc},4}^{\sharp}(T_a^{*,b} f)(x)|^p w(x)dx$. For any ball $B(x_0, r_B)$ with $r_B \le 4$ and $x \in B$, we write

$$\begin{aligned}\frac{1}{|B|}\int_B \left|T_a^{*,b} f(x) - \left(T_a^{*,b} f\right)_B\right| dx &\le \frac{2}{|B|}\int_B |(b - b_B)\, T_a^* f(x)|\,dx \\ &\quad + \frac{2}{|B|}\int_B |T_a^*((b - b_B) f_1)(x)|\,dx \\ &\quad + \frac{1}{|B|}\int_B |T_a^*((b - b_B) f_2)(x) \\ &\qquad -(T_a^*((b - b_B) f_2))_B|\,dx \\ &:= E_1 + E_2 + E_3,\end{aligned} \tag{99}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

Hölder's inequality and Proposition 9 show that

$$\begin{aligned}E_1 &\le C\left(\frac{1}{|B|}\int_B |b - b_B|^{r'}\right)^{1/r'}\left(\frac{1}{|B|}\int_B |T_a^*|^r\right)^{1/r} \\ &\le C\|b\|_\theta\left(\frac{1}{|B|}\int_B |T_a^* f|^r\right)^{1/r}.\end{aligned} \tag{100}$$

For any critical ball Q_j such that $x \in Q_j \cap B$. It can be verified that $B \subset \widetilde{Q}_j := 8Q_j$. This yields that

$$E_1 \le C\|b\|_\theta \times \inf_{y\in B}\widetilde{M}_r\left(T_a^* f\right)(y). \tag{101}$$

Using Hölder's inequality and Proposition 9 again, we have, for $1 < s < r$,

$$\begin{aligned} E_2 &\leq C\left(\frac{1}{|B|}\int_B \left|T_a^*\left((b-b_B)f_1\right)\right|^s\right)^{1/s} \\ &\leq C\left(\frac{1}{|B|}\int_{2B}\left|(b-b_B)f_1\right|^s\right)^{1/s} \\ &\lesssim \left(\frac{1}{|B|}\int_{2B}\left|(b-b_B)\right|^\gamma\right)^{1/\gamma}\left(\frac{1}{|B|}\int_{2B}|f|^r\right)^{1/r} \\ &\qquad\qquad \text{for some } \gamma > \mathrm{s} \\ &\lesssim \|b\|_\theta \times \inf_{y\in B}\widetilde{M}_r(f)(y). \end{aligned} \tag{102}$$

To estimate E_3, using Lemma 18, we conclude that

$$\begin{aligned} E_3 &\leq C\frac{1}{|B|^2}\iint_B\left(\int_{R^n\setminus 2B}\left|K^*(u,z)-K^*(y,z)\right|\right. \\ &\qquad \left.\times\left|b(z)-b_B\right|\left|f(z)\right|dz\right)dy\,du \\ &\leq C\|b\|_\theta\left(G_{4,r}^{N-\theta-n}f(x)+\widetilde{M}_r(f)(x)\right). \end{aligned} \tag{103}$$

These three estimates of E_1, E_2, and E_3 give

$$\begin{aligned} M_{\mathrm{loc},4}^{\sharp}\left(T_a^{*,b}f\right)(x) &\leq C\|b\|_\theta\left(\widetilde{M}_r\left(T_a^*f\right)(x)\right. \\ &\qquad \left.+G_{4,r}^{N-\theta-n}(x)+\widetilde{M}_r(f)(x)\right). \end{aligned} \tag{104}$$

This implies that

$$\begin{aligned} &\left\|M_{\mathrm{loc},4}^{\sharp}\left(T_a^{*,b}f\right)\right\|_{L^p(w)} \\ &\quad\leq C\|b\|_\theta\left(\left\|\widetilde{M}_{p_0}\left(T_a^*f\right)\right\|_{L^p(w)}+\left\|G_{4,r}^{N-\theta-n}f\right\|_{L^p(w)}\right. \\ &\qquad \left.+\left\|\widetilde{M}_r(f)\right\|_{L^p(w)}\right). \end{aligned} \tag{105}$$

Since $\widetilde{M}_r$, $G_{4,r}^{N-\theta-n}$, and T_a^* are bounded on $L^p(w)$ as long as $N > n+\theta+n/p+\nu/s$, we obtain the desired results.

This completes our proof. □

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