Research Article

Characterizations of Strong Strictly Singular Operators

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A new class of operators called strong strictly singular operators on normed spaces is introduced. This class includes the class of precompact operators, and is contained in the class of strictly singular operators. Some properties and characterizations for these operators are derived.

1. Introduction

The spaces X and Y will denote normed spaces, and $T : X \rightarrow Y$ will denote a bounded linear mapping from a normed space X into a normed space Y in this paper. Completeness is assumed only when it is specifically stated. An operator T is called strictly singular if it does not have a bounded inverse on any infinite dimensional subspace contained in X . If $T(B_X)$ is totally bounded in Y , where B_X is the open unit ball in X , then T is called a precompact operator. If $\overline{T(B_X)}$, closure of $T(B_X)$, is compact in Y , then T is called a compact operator. Every precompact operator is strictly singular (cf: [1]). The collection of all strictly singular (precompact) operators from X into Y forms a closed subspace of the normed space $\mathcal{B}(X, Y)$, the collection of all bounded linear operators from X into Y (cf: [1]). The collection of strictly singular (precompact) operators on X (from X into X) forms a closed ideal of the normed algebra $\mathcal{B}(X) (= \mathcal{B}(X, X))$ (cf: [1]). A linear transformation $T : X \rightarrow Y$ has a bounded inverse if and only if $\|x\| \leq c\|Tx\|$ for all $x \in X$, for some $c > 0$. It is easy to see as a consequence of the open mapping theorem that a continuous linear transformation T from a Banach space X into a Banach space Y has closed range, if and only if for given $x \in X$, there is an element $y \in X$ such that $Tx = Ty$ and $\|y\| \leq c\|Ty\|$, for some fixed $c > 0$ (see [2]). This gives a motivation to define a new class of operators called strong strictly singular operators.

Write $\widehat{X} = X/N(T)$, where $N(T)$ is the null space of T , as the quotient space endowed with the quotient norm. Let π_T denote the quotient map from X onto $\widehat{X}$. The 1-1 operator $\widehat{T} : \widehat{X} \rightarrow Y$ induced by T is defined by $\widehat{T}(x + N(T)) = \pi_T(x) = Tx$. Note that $\widehat{T}$ is 1-1 and linear with range same as the range of T . If T is 1-1 then $T = \widehat{T}$. Also, $B_{\widehat{X}} = \pi_T(B_X)$ and $T(B_X) = \widehat{T}(B_{\widehat{X}})$. So, we have the following conclusions.

- (a) T is precompact on X if and only if $\widehat{T}$ is precompact on $\widehat{X}$.
- (b) T is compact on X if and only if $\widehat{T}$ is compact on $\widehat{X}$.

2. Definition

Definition 1. An operator $T : X \rightarrow Y$ is said to be strong strictly singular if for any subspace M of X such that $\dim T(M) = \infty$ and the null space $N(T) \subset M$ there is no positive number c with the property: for a given $x \in M$, there is an element $y \in M$ such that $Tx = Ty$ and $\|y\| \leq c\|Ty\|$.

Lemma 2. Every strong strictly singular operator is strictly singular.

Proof. Let T be strong strictly singular. Suppose M_0 is an infinite dimensional subspace of X such that the restriction of T on M_0 has a bounded inverse. Then, there is a positive constant c such that $\|x\| \leq c\|Tx\|$ for every $x \in M_0$. Since T

is 1-1 on M_0 , we have $N(T) \cap M_0 = \{0\}$. Let us consider that $M = M_0 + N(T)$. Then, $\dim T(M) = \dim T(M_0) = \infty$. Also, for given $x \in M$, there are $y \in M_0$ and $z \in N(T)$ such that $x = y + z$ and $\|y\| \leq c\|Ty\| = c\|Tx\|$. This shows that T is not strong strictly singular. This contradiction shows that T is strictly singular, whenever T is strong strictly singular. $\square$

The converse of Lemma 2 is not true. For example, there is an onto continuous linear operator $T : \ell_1 \rightarrow \ell_2$ which is strictly singular [1, page 89, III.3.7] but not strong strictly singular, because it has a closed range.

A simple characterization for strong strictly singularity is the following lemma.

Lemma 3. *T is strong strictly singular if and only if $\widehat{T}$ is strictly singular.*

Proof. Suppose that $\widehat{M}_0$ is any subspace of $\widehat{X}$ such that, for all $x + N(T) \in \widehat{M}_0$, we have $\|x + N(T)\| \leq c\|\widehat{T}(x + N(T))\|$, for some fixed $c > 0$. Then, for any fixed $c' > c$, for given $x \in \pi_T^{-1}(\widehat{M}_0)$, there is an element $y \in \pi_T^{-1}(\widehat{M}_0)$ such that $x + N(T) = y + N(T)$ and

$$\begin{aligned} \|y\| &\leq \left(\frac{c'}{c}\right) \|x + N(T)\| \leq c' \|\widehat{T}(x + N(T))\| \\ &= c' \|\widehat{T}(y + N(T))\| = c' \|T(y)\|, \quad Tx = Ty. \end{aligned} \quad (1)$$

This proves that if T is strong strictly singular, then $\widehat{T}$ is strictly singular.

On the other hand, suppose that M is any subspace of X such that $N(T) \subset M$ and such that for a given $x \in M$ there is an element $y \in M$ such that $Tx = Ty$ and $\|y\| \leq c\|Ty\|$, for fixed $c > 0$. Then

$$\begin{aligned} \|x + N(T)\| &\leq \|y\| \leq c\|Ty\| = c\|\widehat{T}(y + N(T))\| \\ &= c\|\widehat{T}(x + N(T))\|, \\ &\text{for any } x + N(T) \in \pi_T(M). \end{aligned} \quad (2)$$

This proves that if $\widehat{T}$ is strictly singular, then T is strong strictly singular. This completes the proof. $\square$

Corollary 4. *If T is 1-1 and strictly singular, then T is strong strictly singular.*

Corollary 5. *Every precompact operator is strong strictly singular.*

Proof. If $T : X \rightarrow Y$ is precompact, then $\widehat{T}$ so is. Hence $\widehat{T}$ is strictly singular, and hence T is strong strictly singular. $\square$

Note that the inclusion map $I : \ell_p \rightarrow \ell_q$, $1 \leq p < q < \infty$, is strong strictly singular but not precompact [3, page 170].

Although the converse of Lemma 2 is not true in general, it is true partially which is seen from the next theorem.

Theorem 6. *Let $T : X \rightarrow Y$ be a strictly singular operator. Suppose that $X = N(T) \oplus Z$, for some subspace Z of X , and*

the corresponding projection $P : X \rightarrow Z$ defined by $p(y+z) = z$ with $y \in N(T)$ and $z \in Z$ is continuous. Then T is strong strictly singular.

Proof. Let us consider a subspace M of X such that $N(T) \subset M$ and such that for each given $x \in X$, there is a $y \in M$ satisfying $Tx = Ty$ and $\|y\| \leq c\|Tx\|$, for some fixed $c > 0$. Let $M_0 = P(M)$. Then T is 1-1 on M_0 . Moreover, for given $x \in M_0$, there are $y_0 \in M$ and $z_0 \in N(T)$ such that $y_0 = x + z_0$, and $Tx = Ty_0$. For this $y_0 \in M$, there is a $y \in M$ such that $\|y\| \leq c\|Ty_0\|$ and $Ty = Ty_0$. For this $y \in M$, let us write $y = x + z_1$, with $z_1 \in N(T)$. Then, we have $\|x\| = \|Py\| \leq \|P\|\|y\| \leq c\|P\|\|Ty_0\| = c\|P\|\|Tx\|$. Thus, we have $\|x\| \leq c\|P\|\|Tx\|$ for every $x \in M_0$. Since T is strictly singular, $\dim M_0 < \infty$. So, $\dim T(M) = \dim M_0 < \infty$. This proves that T is a strong strictly singular operator. $\square$

Corollary 7. *Let $T : X \rightarrow Y$ be a strictly singular operator on a Banach space X such that $X = N(T) \oplus Z$, for some closed subspace Z of X . Then T is strong strictly singular.*

Corollary 8. *Let $T : X \rightarrow Y$ be a strictly singular operator such that $\dim(N(T)) < \infty$. Then T is strong strictly singular.*

3. Characterization

A known classical characterization of strictly singular operators is given in Theorem 9. A new similar characterization for strictly singular operators is found in the paper [4], when the operators are 1-1 and they are from a Banach space X into X itself.

Theorem 9 (cf: [1]). *Suppose $T \in \mathcal{B}(X, Y)$. The following four statements are equivalent.*

- (i) T is strictly singular.
- (ii) For every infinite dimensional subspace $M \subset X$, there exists an infinite dimensional subspace $N \subset M$ such that T is precompact on N .
- (iii) Given $\epsilon > 0$ and given M , an infinite dimensional subspace of X , there exists an infinite dimensional subspace $N \subset M$ such that T restricted to N has norm not exceeding ϵ .
- (iv) Given $\epsilon > 0$ and given M , an infinite dimensional subspace of X , there exists an infinite dimensional subspace $N \subset M$ such that T restricted to N is precompact and it has norm not exceeding ϵ .

Using Lemma 3 and Theorem 9, one can easily verify the following.

Theorem 10. *Suppose $T \in \mathcal{B}(X, Y)$. The following four statements are equivalent.*

- (i) T is strong strictly singular.
- (ii) For every infinite dimensional subspace $M \subset X$ satisfying $N(T) \subset M$ and $\dim T(M) = \infty$, there exists an infinite dimensional subspace N of X satisfying

$N(T) \subset N \subset M$ and $\dim T(N) = \infty$ such that T is precompact on N .

- (iii) Given $\epsilon > 0$ and given M , an infinite dimensional subspace of X , satisfying $N(T) \subset M$ and $\dim T(M) = \infty$, there exists an infinite dimensional subspace N of X satisfying $N(T) \subset N \subset M$ and $\dim T(N) = \infty$ such that T restricted to N has norm not exceeding ϵ .
- (iv) Given $\epsilon > 0$ and given M , an infinite dimensional subspace of X satisfying $N(T) \subset M$ and $\dim T(M) = \infty$, there exists an infinite dimensional subspace N of X satisfying $N(T) \subset N \subset M$ and $\dim T(N) = \infty$ such that T restricted to N is precompact and its norm not exceeding ϵ .

The following interesting theorem on automatic continuity is due to van Dulst [5].

Theorem 11 (see [5]). Let $T : X \rightarrow Y$ be a linear transformation (need not be continuous). Suppose that there exists a constant $c > 0$ with the property that every infinity dimensional subspace of X contains a vector x such that $\|x\| = 1$ and $\|Tx\| < c$. Then there exists a subspace $L \subset X$ with $\dim X/L < \infty$ such that T_L , the restriction of T to L , is continuous.

Corollary 12. Let $T : X \rightarrow Y$ be a bounded linear transformation. Then the following statements are equivalent.

- (a) T is strong strictly singular.
- (b) For a given subspace $M \subset X$ with $\dim T(M) = \infty$, there is no positive constant c such that for any subspace M_1 of M with $\dim T(M_1) = \infty$, there is an element $x \in M_1 + N(T)$ with $\|x\| \leq c\|Tx\|$.

Proof. (a) $\Rightarrow$ (b): Suppose that (b) fails to be true. Then there is a subspace M of X satisfying $\dim T(M) = \infty$ and there is a positive constant c such that for every subspace M_1 of M satisfying $\dim T(M_1) = \infty$ there is an element $x \in M_1 + N(T)$ such that $\|x\| \leq c\|Tx\|$. Then, there is a subspace $\widehat{M}_0$ of $\pi_T(M)$ such that $\dim \pi_T(M)/\widehat{M}_0 < \infty$ and $\widehat{T}$ restricted to $\widehat{M}_0$ is a homeomorphism; which follows from Theorem 11. So, $\widehat{T}$ is not strictly singular. Hence, T is not strong strictly singular. This proves (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Trivial. $\square$

4. Properties

Proposition 13. Let $T_n : X \rightarrow Y$ be a sequence of strong strictly singular operators such that $\|T_n - T\| \rightarrow 0$ for some operator $T : X \rightarrow Y$. Suppose that $N(T) = N(T_i) = N(T_j)$, for all i, j . Then T is also strong strictly singular.

Proof. Note that $\|\widehat{T}_n - \widehat{T}\| \rightarrow 0$ and each $\widehat{T}_n$ is strictly singular, by Lemma 3. So $\widehat{T}$ is strictly singular. Hence T is strong strictly singular, by Lemma 3. $\square$

The following technical lemma is on vector spaces.

Lemma 14. Let M be an infinite dimensional subspace of $X \times Y$, where X and Y are vector spaces over the same field. Then at least one of the following is true.

- (i) There is an infinite dimensional subspace X_1 of X such that $X_1 \times \{0\} \subset M$.
- (ii) There is an infinite dimensional subspace Y_1 of Y such that $\{0\} \times Y_1 \subset M$.
- (iii) There are infinite dimensional subspaces X_1 and Y_1 of X and Y , respectively, and there is an onto isomorphism $L : X_1 \rightarrow Y_1$ such that $\{(x, Lx) : x \in X_1\} \subset M$.

Proof. Let M_X, M_Y be the projections of M on X and Y , respectively. Then M_X is of infinite dimension, or M_Y is of infinite dimension. Suppose that M_X is of infinite dimension. Let S_X be a basis of M_X . To each $x \in S_X$, find an element $Lx \in M_Y$ such that $(x, Lx) \in M$. Extend the map $x \mapsto Lx$ to $L : M_X \rightarrow M_Y$ as a linear mapping. If $\dim L(M_X) < \infty$, then we select X_1 as $\ker L$ so that (i) is true. If $\dim L(M_X) = \infty$, then find a subspace X_1 of M_X such that $L(X_1) = L(M_X)$ and L is 1-1 on X_1 . In this case (iii) is true with $Y_1 = L(M_X)$. Similarly, we can show that either (ii) or (iii) is true if $\dim M_Y = \infty$. $\square$

Remark 15. It is possible to select X_1 and Y_1 such that cardinalities of bases for X_1 and Y_1 are equal to the cardinality of a basis for M .

Theorem 16. Let $T_1 : X_1 \rightarrow Y_1, T_2 : X_2 \rightarrow Y_2$ be bounded linear operators between normed spaces. Define a bounded linear operator $T = T_1 \times T_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ by $T(x_1, x_2) = (T_1 x_1, T_2 x_2)$. Then one has the following.

- (i) If T_1 and T_2 are precompact (compact), then T is precompact (compact).
- (ii) If T_1 and T_2 are strictly singular operators, then T is a strictly singular operator.
- (iii) If T_1 and T_2 are strong strictly singular operators, then T is a strong strictly singular operator.

Proof. (i) Is obvious. To prove (ii), suppose that T_1 and T_2 are strictly singular operators. Let M be infinite dimensional subspace of $X_1 \times X_2$. If there is an infinite dimensional subspace X'_1 of X_1 such that $X'_1 \times \{0\} \subset M$, then we can find an infinite dimensional subspace X''_1 of X'_1 such that T_1 is precompact on X''_1 . In this case, there is an infinite dimensional subspace $X''_1 \times \{0\}$ of M such that T is precompact on $X''_1 \times \{0\}$. Similarly, if there is an infinite dimensional subspace X'_2 of X_2 such that $\{0\} \times X'_2 \subset M$, then T is precompact on an infinite dimensional subspace of M . Suppose that there are infinite dimensional subspaces X'_1 and X'_2 , of X_1 and X_2 , respectively, and an onto linear isomorphism $L : X'_1 \rightarrow X'_2$ such that $\{(x_1, Lx_1) : x_1 \in X'_1\} \subset M$. Find an infinite dimensional subspace X''_1 of X'_1 such that T_1 is precompact on X''_1 . Find an infinite dimensional subspace X''_2 of $L(X''_1)$ such that T_2 is precompact on X''_2 . Then, T is precompact on an infinite dimensional subspace $L^{-1}(X''_2) \times X''_2$ of M . This proves (ii). To prove (iii), suppose that T_1 and T_2 are strong strictly singular operators. Note that

$N(T) = N(T_1) \times N(T_2)$ and $(X_1 \times X_2)/N(T) = X_1/N(T_1) \times X_2/N(T_2)$. So the natural mapping from $(X_1 \times X_2)/N(T)$ to $Y_1 \times Y_2$ coincides with the natural mapping from $X_1/N(T_1) \times X_2/N(T_2)$ to $Y_1 \times Y_2$. Since, by (ii), the later is strictly singular and the former is also strictly singular. So, T is strong strictly singular. This proves (iii). $\square$

The following corollary is well known.

Corollary 17. *If $T_1 : X \rightarrow Y$, $T_2 : X \rightarrow Y$ are two strictly singular (precompact) operators. Then $T_1 + T_2$ is strictly singular (precompact).*

Proof. It follows from the continuous mappings $x \mapsto (x, x)$ from X into $X \times X$, $(y_1, y_2) \mapsto y_1 + y_2$ from $Y \times Y$ into Y and strictly singular (precompact) mapping $(x_1, x_2) \mapsto (T_1 x_1, T_2 x_2)$ from $X \times X$ into $Y \times Y$. $\square$

Lemma 18. *Let $T : X \rightarrow Y$ be a strong strictly singular operator. Let $S : Y \rightarrow Z$ be a 1-1 continuous operator. Then $S \circ T$ is strong strictly singular.*

Proof. Note that $N(T) = N(S \circ T)$. Since the natural mapping $S_1 : X/N(T) \rightarrow Z$ is strictly singular, the mapping $S_1 : X/N(S \circ T) \rightarrow Z$ is strictly singular. Then, by Lemma 3, $S \circ T$ is strong strictly singular. $\square$

Corollary 17, Lemma 18, and Proposition 13 lead to natural questions about limits of sequences of strong strictly singular operators and sum of two strong strictly singular operators.

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