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Research Article

Generalized Köthe-Toeplitz Duals of Some Vector-Valued Sequence Spaces

Yılmaz Yılmaz

Department of Mathematics, Inonu University, 44280 Malatya, Turkey

Correspondence should be addressed to Yılmaz Yılmaz; yyilmaz44@gmail.com

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We know from the classical sequence spaces theory that there is a useful relationship between continuous and β-duals of a scalar-valued FK-space E originated by the AK-property. Our main interest in this work is to expose relationships between the operator space $\mathscr{L}(E,Y)$ and E^{β} and the generalized β-duals of some X-valued AK-space E where X and Y are Banach spaces and $E^{\beta} = \{(A_k), A_k \in \mathscr{L}(X,Y) : \sum_{k=1}^{\infty} A_k x_k$ converges in Y, for all $x \in E\}$. Further, by these results, we obtain the generalized β-duals of some vector-valued Orlicz sequence spaces.

1. Introduction

The idea of dual sequence space was introduced by Köthe and Toeplitz [1]. Then, Maddox, [2], generalized this notion to X-valued sequence classes where X is a Banach space. This brings an important contribution to the operator matrix transformation of Banach space-valued sequence spaces. Remember that β- and α-duals of a (complex-valued) sequence space E, denoted by E^{β} and E^{α}, respectively, are defined to be

$$E^{\beta} = \left\{(a_k) \in w : \sum_{k=1}^{\infty} a_k x_k \text{ convergent for all } x = (x_k) \in E\right\},$$
$$E^{\alpha} = \left\{(a_k) \in w : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x \in E\right\}, \tag{1}$$

where w is the space of all complex-valued sequences. The (a_k) in the classical definitions of Köthe-Toeplitz duals is replaced by a sequence $(A_k)_{k=1}^{\infty}$ of linear operators, not necessarily continuous, from X into another Banach space Y. Thus, if E is a nonempty set of sequences $x = (x_k)$ with $x_k \in X$, then generalized β- and α-duals of E are defined to be

$$E^{\beta} = \left\{(A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ convergent in the } Y\text{-norm for all } x \in E\right\}, \tag{2}$$

$$E^{\alpha} = \left\{(A_k) : \sum_{k=1}^{\infty} \|A_k x_k\| < \infty \text{ for all } x \in E\right\} \tag{3}$$

respectively. It is clear that this notion depends on the space Y and if $E \subset s(X)$, then

$$E^{\alpha (\text{or } \beta)} \subset s(L(X,Y)), \tag{4}$$

where $L(X,Y)$ is the space of all linear operators from X into Y. Without the loss of generality we can restrict ourselves in this work to continuous operators and $\mathscr{L}(X,Y)$ being the space of all continuous linear operators from X into Y and $s(X)$ being the space of all X-valued sequences which is a natural generalization of $w = s(\mathbb{C})$.

We know from the classical sequence spaces theory that there is a useful relationship between continuous and β-duals of a sequence space whenever it has the AK-property. Related results are also expressed in [3, page 176]. Here, we are going to show that there is an analogue relationship for X-valued sequence spaces in the context of generalized β-duals with respect to another fixed Banach space Y. Further, by applying this result, we obtain generalized β-duals of some vector-valued Orlicz sequence spaces. We think that our results give a fruitful way to find generalized duals of this kind of vector-valued sequence spaces.

2. Prerequisites

We use the notations $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{R}$ for the sets of all positive integers, complex numbers, and real numbers, respectively. For some locally convex (lc, for short) space X, X^* denotes the continuous dual of X and we denote by B_X and S_X the closed unit ball and the sphere of some normed space X, respectively.

An FH-space is an lc Fréchet space E such that E is a vector subspace of a Hausdorff topological vector space H and the topology of E is larger than the restricted topology of H to E; that is, the inclusion map: $E \to H$ is continuous. If $H = w$ then an FH-space is called an FK-space. With a little extension, an X-valued sequence space E is called an FK-space whenever $H = s(X)$ where X is a Banach space. In fact, the theory of FK-spaces can be developed without the local convexity. However, we are interested only in locally convex FK-spaces. Note that, $s(X) = X^{\mathbb{N}}$ and so its topology is the weakest topology such that the projections

$$P_n : s(X) \longrightarrow X, \quad P_n(x) = x_n, \quad n = 1, 2, \ldots \tag{5}$$

are continuous.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(u) > 0$ for all $u > 0$ and $M(u) \to \infty$ as $u \to \infty$. An Orlicz function M can always be represented in the following integral form:

$$M(u) = \int_0^u p(t)\, dt, \tag{6}$$

where p, known as the kernel of M, is right differentiable for $t \geqslant 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, and p is nondecreasing and $p(t) \to \infty$ as $t \to \infty$.

Consider the kernel $p(t)$ associated with Orlicz function $M(u)$, and let

$$q(s) = \sup\{t : p(t) \leqslant s\}. \tag{7}$$

Then q possesses the same properties as the function p. Suppose now

$$N(v) = \int_0^v q(s)\, ds. \tag{8}$$

Then N is an Orlicz function. The functions M and N are called mutually complementary Orlicz functions, and they satisfy the Young inequality,

$$uv \leqslant M(u) + N(v) \quad \text{for } u, v \geqslant 0. \tag{9}$$

An Orlicz function M is said to satisfy the Δ_2-condition for small u at 0 if for each $k > 0$ there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leqslant R_k M(u)$, for all $u \in (0, u_k]$ [4].

The space ℓ_M consists of all sequences (x_k) of scalars such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0, \tag{10}$$

and it becomes a Banach space which is called an Orlicz sequence space with the Luxemburg norm

$$\|x\|_{(M)} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leqslant 1\right\}. \tag{11}$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(u) = u^p$, $(1 \leqslant p < \infty)$.

Another definition of ℓ_M, [4], is given by the complementary function to M as follows:

$$\ell_M = \left\{x = (x_k) \in w : \sum_{k=1}^{\infty} x_k y_k \text{ converges, for all } y \in \widetilde{\ell_N}\right\}, \tag{12}$$

where N is the complementary function to M and $\widetilde{\ell_N}$ is the collection of all x in w with $\sum_{k=1}^{\infty} N(|x_k|) < \infty$. Clearly, $\widetilde{\ell_N} \subseteq \ell_N$ and ℓ_M are normed by the Orlicz norm

$$\|x\|_M = \sup\left\{\left|\sum_{k=1}^{\infty} x_k y_k\right| : \sum_{k=1}^{\infty} N(|y_k|) \leqslant 1\right\}. \tag{13}$$

It was shown that these two norms on ℓ_M are equivalent.

An important closed subspace of ℓ_M, introduced by Y. Garibanov, is h_M which is defined by

$$h_M = \left\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for all } \rho > 0\right\}. \tag{14}$$

Immediately, we can introduce the vector-valued extension of the spaces ℓ_M and h_M for any Banach space X. Therefore,

$$\ell_M(X) = \left\{x \in s(X) : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}, \tag{15}$$

where $s(X)$ is the space of all X-valued sequences and $\|\cdot\|$ is the norm of X. $\ell_M(X)$ is a Banach space with the Luxemburg norm

$$\|x\|_{(M)} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \leqslant 1\right\}, \tag{16}$$

and it coincides with ℓ_M whenever $X = \mathbb{C}$. Further, define the closed subspace $h_M(X)$ of $\ell_M(X)$ by $x = (x_k) \in h_M(X)$ if and only if

$$\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty \quad \forall \rho > 0. \tag{17}$$

If M satisfies the Δ_2-condition then $h_M(X) = \ell_M(X)$.

3. Relative Weak Topologies

An operator $T \in \mathscr{L}(V, Y)$, for Banach spaces V and Y, is called a Hahn-Banach operator if for every Banach space X containing V as a subspace there exists an operator $\widehat{T} \in \mathscr{L}(V, Y)$ such that $\|\widehat{T}\| = \|T\|$ and $\widehat{T}x = Tx$, for every $x \in V$. Thus, the classical Hahn-Banach theorem can be restated in the following way for operators (see also [5]).

Theorem 1. *Let X, Y be Banach spaces and let $T : X \to Y$ be a continuous linear operator of rank 1. Then T is a Hahn-Banach operator.*

By some modification on the assertion: norm preserving, the result remains true if X is taken as a locally convex space. The evidence of this assertion can be found in [6, Section 7.2.].

Hence, by using Theorem 1 we derived some tools for later sections as in the way that is similar to classical treatments. The proof of the following result is also given in [7]. Nevertheless, it will be convenient to restate it here.

Corollary 2. *Let X, Y be Banach spaces and $x \in X$. Then, for some $a \in S_Y$, there exists a corresponding operator $T_a (\in \mathscr{L}(X, Y))$ such that*

$$\|T_a\| = 1, \qquad T_a(x) = \|x\| a. \tag{18}$$

Proof. If x is not null, take $V = \{\lambda x\}$. Then V is a closed subspace of X hence is a Banach space with the same norm. Define $T_a^{'}(\lambda x) = \lambda \|x\| a$, for some $a \in S_Y$, from V into Y. Then $T_a^{'}$ satisfies the required condition of Theorem 1 on V. Hence it is a Hahn-Banach operator such that $\|T_a^{'}\| = 1$. The norm preserving extension T_a of $T_a^{'}$ to X has the desired properties. The result is obvious for $x = 0$. ∎

Corollary 3. *Let X be an lc space, Y be a Banach space, S be a vector subspace of X, and $x \in X \setminus \overline{S}$. Then there exists an $a \in S_Y$ and a corresponding operator $T_a (\in L(X, Y))$ such that*

$$T_a(x) = a, \quad T_a = 0 \text{ on } S. \tag{19}$$

Proof. Let $V = \{\lambda x\}$ and consider $Z = S \oplus V$. Fix some $a \in S_Y$ and define the operator

$$T_a^{'} : Z \longrightarrow Y, \qquad T_a^{'}(z) = T_a^{'}(s + \lambda x) = \lambda a, \tag{20}$$

for $z \in Z$ (equivalently, for some $s \in S$ and $\lambda \in \mathbb{K}$ (= $\mathbb{C}$ or $\mathbb{R}$) such that $z = s + \lambda x$). Clearly, the hyphothesis $x \in X \setminus \overline{S}$ says that S is not dense in Z. Thus, S must be closed in Z since it is a maximal subspace of Z (see [6, Prob. 4.2.5]). But $S = \operatorname{Ker} T_a^{'} = \{z \in Z : T(z) = 0\}$, hence $T_a^{'}$ is continuous. Further, $T_a^{'}$ is an operator of rank 1 such that $T_a^{'}(x) = a$ and $T_a^{'} = 0$ on S. Thus, the extension T_a of $T_a^{'}$ is the desired operator. ∎

Let us establish an lc topology on a Banach space X with respect to another Banach space Y. Let w_Y be a topology on X such that, for each net $x = (x_\delta)$ in X, $x_\delta \to 0(w_Y)$ if and only if $\|T(x_\delta)\|_Y \to 0$ for each $T \in \mathscr{L}(X, Y)$. It is an lc topology generated by the family $P_Y = \{\|\cdot\|_Y \circ T : T \in \mathscr{L}(X, Y)\}$ of the seminorms $\|\cdot\|_Y \circ T$ on X. Obviously, the norm topology of X is stronger than w_Y, in general. If Y is a scalar field of X then w_Y coincide with the usual weak topology. It is clear that, a net which is w_Y-convergent to 0 is also weak convergent to 0. The converse of this assertion is not true.

Example 4. Let $X = Y = \ell_2$. Then the sequence $(e_n)_{n=1}^{\infty}$ of unit vectors is weak convergent to 0 in ℓ_2 [8, page 99]. But, it is not w_{ℓ_2}-convergent to 0. Therefore, for the identity operator on ℓ_2, we have $\|Ie_n\| = \|e_n\| = 1 \nrightarrow 0$.

However, we cannot work this example in ℓ_1 (in fact, in a Banach space which has the Schur property) since weak convergence implies the norm convergence in this case. Hence the following result is obvious from the definition of the Schur property.

Theorem 5. *Let X be a Banach space having the Schur property. Then weak convergence implies w_Y-convergence in X for every Banach space Y.*

Now consider the canonical embedding $X \to \mathscr{L}^2(X, Y)$, where $\mathscr{L}^2(X, Y)$ is the space of all continuous operators from $\mathscr{L}(X, Y)$ into Y and X and Y are Banach spaces, which assigns each $x \in X$ to the operator F_x on $\mathscr{L}(X, Y)$ defined by

$$F_x(T) = Tx, \quad \text{for each } T \in \mathscr{L}(X, Y). \tag{21}$$

Clearly,

$$\|F_x(T)\|_Y = \|Tx\|_Y \leq \|T\| \|x\|_X \tag{22}$$

so that $F_x \in \mathscr{L}^2(X, Y)$ and

$$\|F_x\| \leq \|x\|_X. \tag{23}$$

Theorem 1 and the succeeding corollary assert that the canonical embedding is a linear isometry from X into $\mathscr{L}^2(X, Y)$ as is in the classical case.

Now, let us investigate how do the bounded subsets of the X in the w_Y-topology behave. Note that a subset A of X is called w_Y-bounded if $T(A)$ is bounded in Y for each $T \in \mathscr{L}(X, Y)$. It is clear that, for every pair of the Banach spaces X and Y, $A \subseteq X$ is w_Y-bounded if it is norm bounded. The converse of this assertion is the following theorem.

Theorem 6. *Let X and Y be Banach spaces. Then w_Y-bounded sets are norm bounded.*

Proof. Let $V \subset X$ be w_Y-bounded and $\widehat{V}$ be canonical embedding of V into $\mathscr{L}^2(X, Y)$. A hypothesis says that $\widehat{V}$ is pointwise bounded so it is uniformly (norm) bounded by the uniform boundedness principle. Hence there exists a $K > 0$ such that $\|F_x\| \leq K$ for each $x \in V$. So

$$\|x\| = \|F_x\| \leq K \tag{24}$$

for each $x \in V$. ∎

Theorem 7. *Let X be an lc space and Y be a Banach space. Then w_Y-bounded sets are also bounded in the lc topology of X.*

Proof. Let $V \subset X$ be w_Y-bounded. We are going to show that $p[V]$ is bounded for each seminorm p in P where P is the family of all seminorms generating the lc topology of X. For an arbitrary p, $Z = (X, p)$ is a seminormed space. Thus we can show as in Theorem 6 that $\widehat{V}$ is bounded in $\mathscr{L}^2(Z, Y)$, whence, V is bounded in Z, that is, $p[V]$ is a bounded subset of $\mathbb{R}$. ∎

We conclude this section with a brief discussion of equicontinuity. A set Ω of linear maps from one topological vector space X into another one Y is called equicontinuous if, for each neighborhood N of 0 in Y,

$$\bigcap \left\{T^{-1}[N] : T \in \Omega\right\} \tag{25}$$

is a neighborhood of 0 in X. Equicontinuity is a generalization of the uniform boundedness of the family of linear maps between seminormed spaces.

Theorem 8 (see [9]). *Let Ω be a collection of continuous linear mappings T from the Fréchet space X into the topological vector space Y. Then Ω is equicontinuous if and only if the set*

$$\Omega(x) = \{Tx : T \in \Omega\} \tag{26}$$

is bounded in Y, for each $x \in X$.

Lemma 9 (see [6]). *Let (T_δ) be a net of continuous operators T_δ from a Fréchet space X into the topological vector space Y. Then the set $\{x : T_\delta x \to 0\}$ is a closed subspace of X.*

4. Sectional Properties and Operator Spaces

For an $x \in s(X)$,

$$x^{(n)} = (x_1, x_2, \ldots, x_n, 0, \ldots) \tag{27}$$

is called nth section of x. Further, $\phi(X)$ denotes the space of all finite sequences in $s(X)$.

Definition 10. Let $E \supset \phi(X)$ be an FK-space. If, for each $x \in E$,

$$x^{(n)} \longrightarrow x \quad \text{in } E, \tag{28}$$

then E is called an AK-space. Further, E is called an AD-space whenever $\phi(X)$ is dense in E. If, for each $x \in E$, the sequence $(x^{(n)})$ is bounded in E then E is called an AB-space.

An AK-(AD-, AB-) space is also called to have AK-(AD-, AB-) property.

Let $E \supset \phi(X)$ be an FK-space and define the set

$$B = \left\{x \in E : \left(x^{(n)}\right) \text{ is bounded in } E\right\}. \tag{29}$$

Clearly, $B \subset E$ and $B = E$ whenever E is an AB-space.

To define another important classes we consider the mappings

$$I_k : X \longrightarrow E \qquad I_k(a) = \Big(0, \ldots, 0, \overset{k\text{th position}}{a}, 0, \ldots\Big) \tag{30}$$

and define the set W_Y, for some Banach space Y, by

$$W_Y = \left\{x \in E : Tx = \sum (T \circ I_k)(x_k) \ \forall T \in \mathscr{L}(E, Y)\right\}. \tag{31}$$

Proposition 11. *For each Banach space Y, $W_Y \subset B$.*

Proof. Let $x \in W_Y$ and $T \in \mathscr{L}(E, Y)$ then we can write $Tx = \sum (T \circ I_k)(x_k)$, that is,

$$\sum_{k=1}^{n} (T \circ I_k)(x_k) \longrightarrow Tx \ (n \longrightarrow \infty) \quad \text{in } E. \tag{32}$$

Since

$$\sum_{k=1}^{n} (T \circ I_k)(x_k) = Tx^{(n)}, \tag{33}$$

$Tx^{(n)} \to Tx$; that is, the sequence $\{x^{(n)}\}$ is w_Y-convergent hence it is w_Y-bounded. Thus, it is also bounded in the lc Fréchet topology of E by Theorem 7. ∎

Proposition 12. *For each Banach space Y,*

$$W_Y \subset \overline{\phi(X)}, \tag{34}$$

where $\overline{\phi(X)}$ is the closure of $\phi(X)$ in E.

Proof. Let $x \in W_Y$. Then, for every $T \in \mathscr{L}(E, Y)$ such that $T = 0$ on $\phi(X)$,

$$\begin{aligned} Tx &= \sum (T \circ I_k)(x_k) = \sum T\Big(0, \ldots, 0, \overset{k\text{th}}{x_k}, 0, \ldots\Big) \\ &= \sum 0 = 0. \end{aligned} \tag{35}$$

This implies $x \in \overline{\phi(X)}$. If this is not so, then there exists an $a \in S_Y$ and a corresponding operator $T_a (\in \mathscr{L}(E, Y))$ such that

$$T_a(x) = a \neq 0, \quad T_a = 0 \ \text{ on } \phi(X) \tag{36}$$

by Corollary 3. This is a contradiction. ∎

Proposition 13. *Let $E \supset \phi(X)$ be an FK-space with AD- and AB-property. Then E also has the AK-property.*

Proof. Define

$$A_n : E \longrightarrow E, \qquad A_n(x) = x^{(n)} - x, \quad \text{for } n = 1, 2, \ldots. \tag{37}$$

Then the sequence $\{A_n(x)\}$ is bounded by the AB-property. Therefore $\{A_n\}$ is equicontinuous by Theorem 8. On the other hand $A_n(x) \to 0$ for each $x \in \phi(X)$, That is,

$$\phi(X) \subset \{x \in E;\ A_n(x) \longrightarrow 0\} = \Lambda. \tag{38}$$

Since Λ is a closed subspace of E from Lemma 9, we obtain that $\overline{\phi(X)} \subseteq \Lambda$. Thus

$$A_n(x) = x^{(n)} - x \longrightarrow 0 \quad (39)$$

for each $x \in \overline{\phi(X)}$ ($= E$ by the AD-property), whence, E has the AK-property. ∎

Theorem 14. *Let $E \supset \phi(X)$ be an FK-space. Then E is an AK-space if and only if E^β and $\mathscr{L}(E, Y)$ are isomorphic for every Banach space Y.*

Proof. Let $(T_k) \in E^\beta$ where each $T_k \in \mathscr{L}(X, Y)$ and define $A[(T_k)] = T$ by

$$Tx = \sum_{k=1}^{\infty} T_k x_k \quad (40)$$

for each $x \in E$. Write

$$\eta_n = \sum_{k=1}^{n} T_k \circ P_k, \quad (41)$$

where $P_k : E \to X$, $k = 1, 2, \ldots$ is the kth (continuous) projection defined by $P_k(x) = x_k$. Then each η_n is continuous and the sequence (η_n) is pointwise convergent since the series $\sum T_k x_k$ is convergent. So, the operator T, which is also defined by

$$Tx = \lim \eta_n(x), \quad (42)$$

is continuous by the Banach-Steinhauss closure theorem, whence, $T \in \mathscr{L}(E, Y)$. That A is injective comes from the following discussion. Let $A[(T_k)] = T = 0$. Then, for each $a \in X$,

$$(T \circ I_k)(a) = T_k a = 0. \quad (43)$$

This implies each $T_k = 0$, that is, $(T_k) = 0$. Further, for each $T \in \mathscr{L}(E, Y)$, let us consider

$$T_k = T \circ I_k, \quad \text{for } k = 1, 2, \ldots, \quad (44)$$

from X to Y. For each $x \in E$,

$$\sum_{k=1}^{n} T_k x_k = \sum_{k=1}^{n} (T \circ I_k)(x_k) \longrightarrow Tx \quad (n \longrightarrow \infty) \quad (45)$$

since E is an AK-space, whence, $(T_k) \in E^\beta$. This means that A is surjective.

Conversely, let E^β and $\mathscr{L}(E, Y)$ be isomorphic. Then each $T \in \mathscr{L}(E, Y)$ has the representation (T_k) such that each

$$T_k = T \circ I_k \in \mathscr{L}(E, Y) \quad (46)$$

and also, for each $x \in E$,

$$Tx = \sum T_k x_k. \quad (47)$$

This shows that $x \in W_Y$, that is,

$$E \subseteq W_Y. \quad (48)$$

Thus, we obtain $E = \overline{\phi(X)}$ by the Proposition 12, whence, E has the AD-property. Also, E has the AB-property by Proposition 11. Hence, E is an AK-space by Proposition 13. ∎

5. Applications on Vector-Valued Orlicz Sequence Spaces

It is not hard to see as in the classical case, [4], that another definition of $\ell_M(X)$ by the complementary function N to M is

$$\ell_M(X) = \Big\{ x \in s(X) : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges, for all } f = (f_k) \in \widetilde{\ell_N}(X^*) \Big\}, \quad (49)$$

where $\widetilde{\ell_N}(X^*)$ is the class of all sequences $f = (f_k)$ such that $\sum_{k=1}^{\infty} N(\|f_k\|) < \infty$ and each $f_k \in X^*$. Further, for each $x \in \ell_M(X)$,

$$\|x\|_M = \sup \left\{ \left| \sum_{k=1}^{\infty} f_k(x_k) \right| : \sum_{k=1}^{\infty} N(\|f_k\|) \leqslant 1 \right\} < \infty \quad (50)$$

defines a norm on $\ell_M(X)$. This norm is said to be Orlicz norm on $\ell_M(X)$.

Lemma 15. *On $\ell_M(X)$, the norms $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ are equivalent, and $\|x\|_{(M)} \leqslant \|x\|_M \leqslant 2\|x\|_{(M)}$.*

Proofs of this lemma and the above assertion can be given in a similar way followed in [4, Theorem 8.9], by using the inequality

$$\left| \sum_{k=1}^{\infty} f_k(x_k) \right| \leqslant \sum_{k=1}^{\infty} \|f_k\| \, \|x_k\|, \quad (51)$$

and by using the fact that $x = (x_k) \in \ell_M(X)$ if and only if $(\|x_k\|)_{k=1}^{\infty} \in \ell_M$.

Lemma 16. *Let M be an Orlicz function. The sets*

$$\Lambda_1 = \left\{ x \in s(X) : \sum_{k=1}^{\infty} M(\|x_k\|) \leqslant 1 \right\}, \qquad \Lambda_2 = \left\{ x \in s(X) : \|x\|_{(M)} \leqslant 1 \right\} \quad (52)$$

are identical.

Proof. Let $x \in \Lambda_1$, this means $\sum_{k=1}^{\infty} M(\|x_k\|/\rho) \leqslant 1$ for $\rho = 1$. Hence, $\|x\|_{(M)} \leqslant 1$, that is, $x \in \Lambda_2$. Conversely, let $x \in \Lambda_2$, that is

$$\inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho} \right) \leqslant 1 \right\} \leqslant 1. \quad (53)$$

This means $\sum_{k=1}^{\infty} M(\|x_k\|/\rho) \leqslant 1$ for some $\rho \leqslant 1$. Therefore $\sum_{k=1}^{\infty} M(\|x_k\|) \leqslant 1$ since M is nondecreasing. ∎

In general $h_M(X)$ has no Schauder basis in classical manner. In [10] we introduce a new kind basis notion. Let us give this definition and prove that $h_M(X)$ has a basis in this manner.

Definition 17 (see [10]). Let X and Y be Banach spaces and $\mathbb{A}$ be a set. A family $\{\eta_a : a \in \mathbb{A}\}$ of continuous linear functions $\eta_a : Y \to X$ is called Y-basis for X if the following condition is satisfied. There exists a directed subset $\mathscr{D}$ (by some relation $\ll$) of $\mathscr{F}$ satisfying the property; for each $a \in \mathbb{A}$ there is some $F \in \mathscr{D}$ such that $a \in F$, and there exists a unique family $\{R_a : a \in \mathbb{A}\}$ of linear functions R_a from X onto Y such that, for each $x \in X$, the net $(\pi_F(x) : \mathscr{D}, \ll)$ converges to x in X where

$$\pi_F(x) = \sum_{a \in F} (\eta_a \circ R_a)(x), \tag{54}$$

for each $F \in \mathscr{D}$ and $\mathscr{F}$ is the family of all finite subsets of the index set $\mathbb{A}$ which is directed by the inclusion relation $\subseteq$. Furthermore, $\{\eta_a\}$ is called a Y-Schauder basis for X whenever each R_a is continuous.

Thus we say that each $x \in X$ has the representation

$$x = \sum_{a \in \mathbb{A}} (\eta_a \circ R_a)(x), \tag{55}$$

in this case.

Definition 18. The Y-basis $\{\eta_a : a \in \mathbb{A}\}$ in the above definition is called unconditional whenever $\mathscr{D} = \mathscr{F}$ with the inclusion relation $\subseteq$.

By taking $\mathbb{A} = \mathbb{N}$ in the Definition [10] we now prove that $h_M(X)$ has an unconditional X-Schauder basis.

Theorem 19. *For $k \in \mathbb{N}$ consider again the operators $I_k : X \to h_M(X)$ such that*

$$I_k(u) = \Big(0, 0, \dots, 0, \overset{k\text{th position}}{u}, 0, \dots\Big). \tag{56}$$

Then, the sequence $\{I_k\}$ is an unconditional X-Schauder basis for $h_M(X)$.

Proof. Let us take $R_k = P_k : h_M(X) \to X$, $P_k(x) = x_k$ as a coordinate projection in the Definition [10]. We should prove that the net $(\pi_F(x) : \mathscr{F}, \subseteq)$ converges to x in $h_M(X)$. This means, for each $\epsilon > 0$, we should find an $F_0 \in \mathscr{F}$ such that $\|x - \pi_F(x)\|_{(M)} < \epsilon$ for $F_0 \subseteq F$. Now, let $\epsilon > 0$ be given. Since $\sum_{k=1}^{\infty} M(\|x_k\|/\rho) < \infty$ for every $\rho > 0$, especially for $\epsilon > 0$, the series $\sum_{k=1}^{\infty} M(\|x_k\|/\epsilon)$ is absolutely convergent and hence it is unconditional convergent in real numbers. Hence we can find an $n_0(\epsilon)$ such that $\sum_{k=n_0+1}^{\infty} M(\|x_k\|/\epsilon) \le 1$. Now let $F_0 = \{1, 2, \dots, n_0\}$. Obviously, F_0 is dependent on ϵ and the set

$$\left\{\rho > 0 : \sum_{k \in \mathbb{N} \setminus F_0} M\left(\frac{\|x_k\|}{\rho}\right) \leqslant 1\right\} \tag{57}$$

includes the ϵ. This means

$$\inf\left\{\rho > 0 \sum_{k \in \mathbb{N} \setminus F} M\left(\frac{\|x_k\|}{\rho}\right) \leqslant 1\right\} \leqslant \epsilon. \tag{58}$$

Now, remember that

$$\pi_F(x) = \sum_{k \in F} (I_k \circ P_k)(x). \tag{59}$$

Hence, for some $F \in \mathscr{F}$ such that $F_0 \subseteq F$, we have

$$\begin{aligned}
&\|x - \pi_F(x)\|_{(M)} \\
&\quad = \big\|\big(x_1, \dots, x_{n_1-1}, 0, x_{n_1+1} \dots x_{n_2-1}, 0, \\
&\qquad x_{n_2+1}, \dots x_{n_m-1}, 0, x_{n_m+1} \dots\big)\big\|_{(M)} \\
&\quad = \inf\left\{\rho > 0 : \sum_{k \in \mathbb{N} \setminus F} M\left(\frac{\|x_k\|}{\rho}\right) \leqslant 1\right\} \\
&\quad \leqslant \epsilon.
\end{aligned} \tag{60}$$

The continuity of each P_k and uniqueness of the sequence $\{P_k\}$ in the representation can be done similarly in the classical case. This completes the proof. ∎

One of our main results is the following theorem which states the generalized β-dual of $h_M(X)$ with respect to the Banach space Y. The above theorem brings that $h_M(X)$ is an AK-space and we can use Theorem 14 to find the generalized β-dual of $h_M(X)$.

Theorem 20. *Let X, Y be Banach spaces and M, N be mutually complementary Orlicz functions. Then, $h_M(X)^\beta$ is isomorphic by the mapping $T \to (T \circ I_k)$ to the Banach space*

$$\begin{aligned}
V_N = \Big\{ & A = (A_k) \in s(\mathscr{B}(X, Y)), \\
& \|A\| = \sup_{f \in B_{Y^*}} \left\|(A_k^* f)_{k=1}^{\infty}\right\|_N < \infty \Big\},
\end{aligned} \tag{61}$$

where each I_k is defined as in Theorem 19.

Proof. We prove that $\mathscr{L}(h_M(X), Y)$ is isometrically isomorphic to V_N.

A routine calculation shows that $\|A\| = \sup_{f \in B_{Y^*}} \|(A_k^* f)_{k=1}^{\infty}\|_N$ really defines a norm on V_N and it is a Banach space with this norm. Let $T \in \mathscr{L}(h_M(X), Y)$ and say $A_k = T \circ I_k$ for each k. This implies $\|(A_k \circ P_k)(x)\| = 0$ so that $A_k(x_k) = 0$ for each k. Since each $x \in h_M(X)$ has the unconditional representation $x = \sum_{k=1}^{\infty} (I_k \circ P_k)(x)$, we can write

$$Tx = \sum_{k=1}^{\infty} (T \circ I_k)(x_k) = \sum_{k=1}^{\infty} A_k x_k. \tag{62}$$

Immediately each $A_k \in \mathscr{L}(X, Y)$ since $\|A_k\| \leqslant \|T\| \|I_k\| = \|T\|$. Now, let us define the mapping

$$\begin{gathered}
\Psi : \mathscr{L}(h_M(X), Y) \longrightarrow V_N, \\
\text{by } \Psi(T) = A = (A_k)_{k=1}^{\infty}; \quad A_k = T \circ I_k.
\end{gathered} \tag{63}$$

$\Psi(T) = 0$ if and only if each $T \circ I_k = 0$ so $T = 0$ by the definition of each I_k, that is, Ψ is one to one. Also, for an arbitrary $A \in V_N$, if we define the operator T by

$$Tx = \sum_{k=1}^{\infty} A_k x_k \tag{64}$$

on $h_M(X)$ then, by using the Young inequality, we have

$$\begin{aligned}\left\|\sum_{k=m}^{n} A_k x_k\right\| &= \sup_{f\in B_{Y^*}} \left|f\left(\sum_{k=m}^{n} A_k x_k\right)\right| \\ &\leqslant \sup_{f\in B_{Y^*}} \sum_{k=m}^{n} \|A_k^* f\| \|x_k\| \\ &\leqslant \sup_{f\in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N}\right) \\ &\quad + \sup_{f\in B_{Y^*}} \sum_{k=m}^{n} M\left(\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \|x_k\|\right) \\ &\leqslant \sup_{f\in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N}\right) \\ &\quad + \sum_{k=m}^{n} M\left(\frac{\|x_k\|}{1/\|A\|}\right).\end{aligned} \tag{65}$$

Since $(A_k^* f)_{k=1}^{\infty} \in \ell_N(\mathscr{L}(X,Y))$ for each $f \in Y^*$ and

$$\sum_k N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N}\right) \leqslant 1 \tag{66}$$

from [4, Prop. 8.12], we have

$$\sup_{f\in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N}\right) \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty. \tag{67}$$

Also

$$\sum_{k=m}^{n} M\left(\frac{\|x_k\|}{1/\|A\|}\right) \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty \tag{68}$$

since $x \in h_M(X)$. This means the series $\sum A_k(x_k)$ is convergent, that is, T is well defined. Further, that the mapping Ψ is onto, that is, $T \in \mathscr{B}(h_M(X), Y)$ comes from the following equalities:

$$\begin{aligned}\|T\| &= \sup_{x\in B_M} \|Tx\| = \sup_{x\in B_M} \left\|\sum_{k=1}^{\infty} A_k(x_k)\right\| \\ &= \sup_{x\in B_M} \sup_{f\in B_{Y^*}} \left|f\left(\sum_{k=1}^{\infty} A_k(x_k)\right)\right| \\ &= \sup_{f\in B_{Y^*}} \sup_{x\in B_M} \left|\sum_{k=1}^{\infty} (A_k^* f)(x_k)\right| \\ &= \sup_{f\in B_{Y^*}} \sup \left\{\left|\sum_{k=1}^{\infty} (A_k^* f)(x_k)\right| : \sum_{k=1}^{\infty} M(\|x_k\|) \leqslant 1\right\}, \\ &\quad \text{(by Lemma 16)} \\ &= \sup_{f\in B_{Y^*}} \left\|(A_k^* f)_{k=1}^{\infty}\right\|_N = \|A\|.\end{aligned} \tag{69}$$

This shows at the same time that Ψ is an isometry. □

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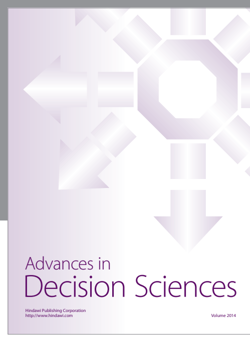
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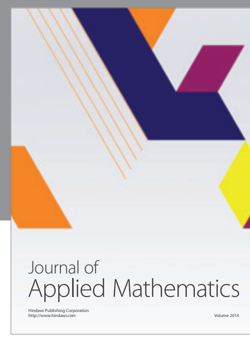
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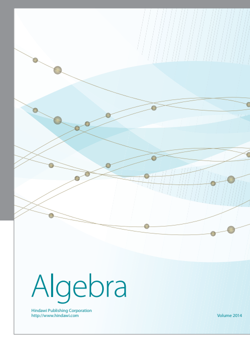
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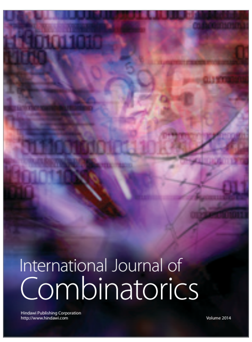
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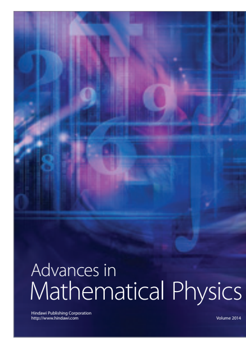
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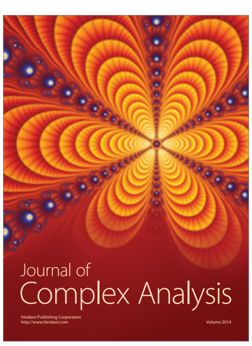
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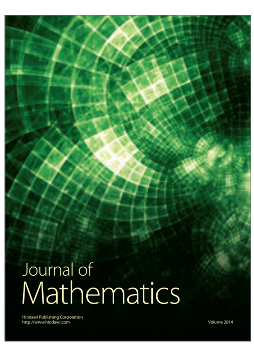
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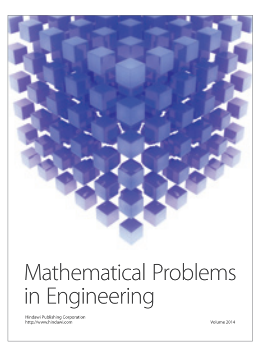
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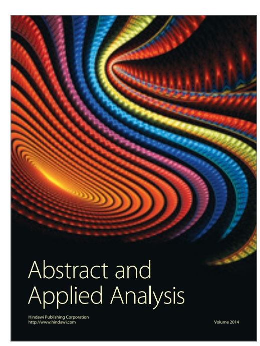
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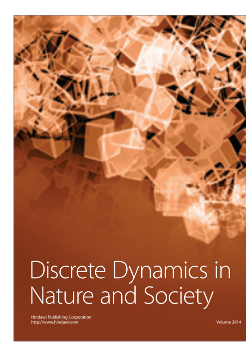
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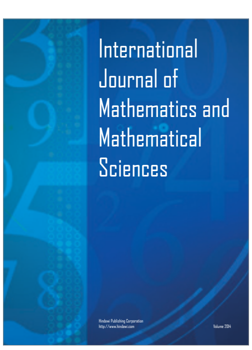
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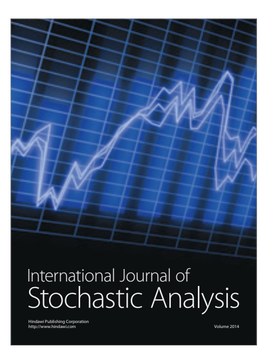
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