

# Research Article

## Subdividing the Trefoil by Origami

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Received 7 November 2012; Accepted 21 November 2012

Academic Editor: Michel Planat

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In 2005, David Cox and Jerry Shurman proved that the curves they call  $m$ -clovers can be subdivided into  $n$  equal lengths (for certain values of  $n$ ) by origami, in the cases where  $m = 1, 2, 3$ , and  $4$ . In this paper, we expand their work to include the 6-clover.

### 1. Historical Background

From antiquity, it was known that regular polygons with  $n$  sides could be constructed with compass and (unmarked) straightedge for  $n$  of one of the forms  $2^k$ ,  $2^k \cdot 3$ ,  $2^k \cdot 5$ , and  $2^k \cdot 15$ . In 1801, Gauss showed that the list could be expanded to include powers of two times any product of distinct Fermat primes, primes of the form  $2^{2^u} + 1$ . He claimed to have a proof of the converse statement, but as Pierpont noted ([1], p.79), he never actually provided it. Pierpont gives an elementary proof (i.e., without Galois theory) in his paper.

In 1837, the French mathematician Pierre Wantzel resolved three celebrated ancient mathematical problems definitively, when he proved the impossibility of trisecting an arbitrary angle, duplicating the cube, or constructing a regular polygon with  $n$  sides for values of  $n$  other than those of Gauss using only a compass and (unmarked) straightedge.

Remarkably, these same constructions can be achieved by the technique of *origami* (paper folding). In fact, using origami, it is also possible to trisect angles, duplicate cubes, and generally construct roots of cubic equations. This was observed by Beloch in a publication in 1936 [2]. An explanation of Beloch's work, including a survey of the history, can be found in [3].

Alternatively, with a *marked* straightedge, one can achieve the same result. Generalizing the notion of construction to include this or an equivalent tool and using Galois theory [4], the values of  $n$  for which a regular polygon can be constructed consist of all numbers of the form  $n = 2^a 3^b p_1 \cdots p_n$  where  $a, b \geq 0$  and  $p_1, \dots, p_n$  are distinct primes of the form  $2^u 3^v + 1$  with  $u, v \geq 0$ . Such primes are known as *Pierpont* primes.

Meanwhile, Abel showed in 1828 that the *lemniscate* can also be divided into  $n$  pieces of equal length with straightedge and compass for the same values of  $n$  as for the circle. See [5] for a modern proof of this result, including the converse; see also [6].

The 2005 paper of Cox and Shurman [7] expands the family of divisible curves to include the *clover*. The  $m$ -clover is the plane curve defined by the polar equation:

$$r^{m/2} = \cos\left(\frac{m}{2}\theta\right), \quad (1)$$

where  $m$  is a positive integer. This is a subfamily of the *sinusoidal* or *sinus spirals* ([8], p.194). For  $m = 1$ , the curve is the cardioid;  $m = 2$  is the circle;  $m = 3$  is the clover;  $m = 4$  is the Bernoulli lemniscate. In their paper, they prove that these first four curves can be divided into  $n$  arcs of equal length by origami (paper-folding) construction for certain values of  $n$ , as follows.

**Theorem 1** (see [7]). *For any  $n$ , the cardioid can be divided into  $n$  arcs of equal length by straightedge and compass. The circle can be divided into  $n$  equal lengths by origami if and only if  $n = 2^a 3^b p_1 \cdots p_n$  where  $a, b \geq 0$  and  $p_1, \dots, p_n$  are distinct Pierpont primes. In the case of the lemniscate, the Pierpont primes must satisfy  $p_i = 7$  or  $p_i \equiv 1 \pmod{4}$ . The clover can be divided into  $n$  equal lengths by origami if and only if  $n = 2^a 3^b p_1 \cdots p_n$  where  $a, b \geq 0$  and  $p_1, \dots, p_n$  are distinct Pierpont primes such that  $p_i = 5, p_i = 17$ , or  $p_i \equiv 1 \pmod{3}$ .*

There are only five Fermat primes known, but more than 4000 Pierpont primes have been found; as of June 2012, the

largest known Pierpont prime is  $3 \cdot 2^{7033641} + 1$ , which has 2117338 digits. ([9]; 16th on the list of largest primes.) See [10] for up-to-date information.

In this paper, we observe that these results on origami construction can be extended to the case  $m = 6$ , the *trefoil*. See Figure 1.

**Theorem 2.** *The trefoil can be divided into  $n$  equal lengths by origami if and only if  $n = 2^a 3^b p_1 \cdots p_n$  where  $a, b \geq 0$  and  $p_1, \dots, p_n$  are distinct Pierpont primes such that  $p_i = 5$ ,  $p_i = 17$ , or  $p_i \equiv 1 \pmod{3}$ .*

Cox and Shurman define the  $m$ -clover function  $\varphi_m$  by  $r = \varphi_m(s)$ ,  $0 \leq s \leq \omega_m$ ; here  $s$  is the arclength,  $r$  the radial distance from the origin, and  $\omega_m$  is the length of one leaf of the  $m$ -clover. The function  $\varphi_m$  is found by inverting the arclength integral:

$$r = \varphi_m(s) \iff s = \int_0^r \frac{1}{\sqrt{1-t^m}} dt. \quad (2)$$

For  $m = 1$  and  $m = 2$ , these integrals are elementary. For  $m = 3$  and  $m = 4$ , these are *elliptic integrals*, and the corresponding  $m$ -clover functions are elliptic functions. For  $m \geq 5$ , the integral is no longer an elliptic integral, and the corresponding clover function is not an elliptic function. However,  $(\varphi_6)^2$  is an elliptic function, and this turns out to be enough to prove Theorem 2.

*Note.* Practically all of the hard work can be found in [7], to which we refer the reader for the details of our arguments. For detailed information about Galois theory, especially its application to subdividing the lemniscate, see [4]. A discussion of origami numbers can be found in [11].

## 2. Origami Constructibility

Viewing the plane as the complex numbers  $\mathbb{C}$ , the set  $\mathbb{O}$  of points which can be constructed by origami is the smallest subfield containing the rational numbers and closed under rational operations and under square roots and cube roots. If  $z$  is a root of a polynomial of degree less than five with coefficients in  $\mathbb{O}$ , then  $z$  itself is in  $\mathbb{O}$ . To subdivide a leaf of the trefoil into equal lengths, it suffices to show that the  $x$  and  $y$  coordinates of the division points are numbers in  $\mathbb{O}$ .

In their proof of the clover theorem, Cox and Shurman show that the values of the clover function  $\varphi(u) = \varphi_3(u)$  lie in  $\mathbb{O}$  when  $u = \ell(\omega_3/n)$  with  $\ell = 0, 1, \dots, n-1$  and  $n = 2^a 3^b p_1 \cdots p_n$  where  $a, b \geq 0$  and  $p_1, \dots, p_n$  are distinct Pierpont primes such that  $p_i = 5$ ,  $p_i = 17$ , or  $p_i \equiv 1 \pmod{3}$ . This is the main fact we need to extend the result to the trefoil curve. If  $\varphi(u)$  is in  $\mathbb{O}$ , then  $\varphi'(u)$  is also in  $\mathbb{O}$ , since  $\varphi$  satisfies the differential equation:

$$\left(\frac{d\varphi}{dt}\right)^2 = 1 - \varphi(t)^3, \quad \varphi(0) = 0. \quad (3)$$

Moreover, any rational expression in  $\varphi(u)$  and  $\varphi'(u)$  is in  $\mathbb{O}$ .

## 3. The Trefoil Curve

The trefoil curve is given in polar coordinates by the equation

$$r^3 = \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta. \quad (4)$$

The rectangular equation is

$$(x^2 + y^2)^3 = x^3 - 3xy^2. \quad (5)$$

Thus,  $r^6 + (3x)r^2 - 4x^3 = 0$ , and of course  $y^2 = r^2 - x^2$ , from which we conclude the following.

**Proposition 3.** (1)  *$r$  is origami constructible if and only if  $x$  is origami constructible.*

(2)  *$x$  is origami constructible implies that  $y$  is origami constructible.*

From the polar coordinate formula, it is easy to derive the arclength formula in terms of  $r$  ([7], p.686):

$$s = \int_0^r \frac{1}{\sqrt{1-t^6}} dt. \quad (6)$$

Now, if we make the miraculous substitution

$$t^2 = \frac{3z^2}{4-z^3}, \quad (7)$$

then the following equations hold:

$$\begin{aligned} 1-t^6 &= \frac{(4-z^3)^3 - 27z^6}{(4-z^3)^3} = \frac{(1-z^3)(8+z^3)^2}{(4-z^3)^3}, \\ 2tdt &= \frac{24z + 3z^4}{(4-z^3)^2} dz = \frac{3z(8+z^3)}{(4-z^3)^2} dz, \\ 4dt^2 &= \frac{3(8+z^3)^2}{(4-z^3)^3} dz^2, \quad \frac{4dt^2}{1-t^6} = \frac{3dz^2}{1-z^3}. \end{aligned} \quad (8)$$

Therefore,

$$s = \int_0^r \frac{1}{\sqrt{1-t^6}} dt = \frac{\sqrt{3}}{2} \int_0^v \frac{1}{\sqrt{1-z^3}} dz, \quad (9)$$

where

$$r^2 = \frac{3v^2}{4-v^3}. \quad (10)$$

**Proposition 4.**  *$r$  is origami constructible if and only if  $v$  is origami constructible.*

*Proof.* If  $v$  is in  $\mathbb{O}$ , then  $r$  is the square root of an element of  $\mathbb{O}$  and, therefore, an element of  $\mathbb{O}$ . Conversely, if  $r^2$  is in  $\mathbb{O}$ , then  $v$  satisfies a cubic equation with coefficients in  $\mathbb{O}$ , so it is in  $\mathbb{O}$ .  $\square$

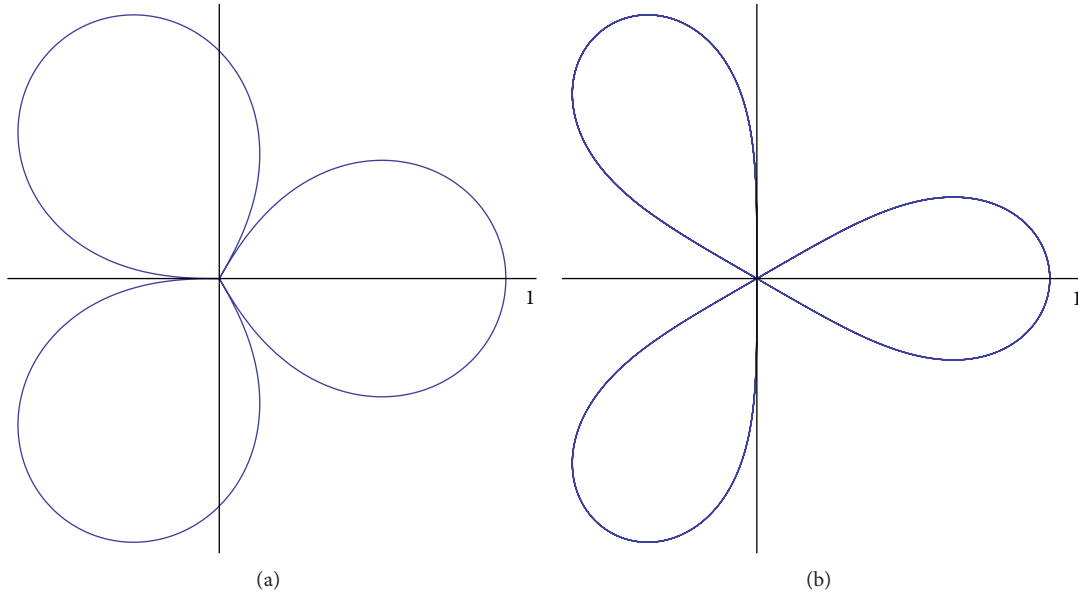


FIGURE 1: (a) The clover. (b) The trefoil.

Recall that for the clover function,  $\varphi_3(s) = r \Leftrightarrow s = \int_0^r (1/\sqrt{1-z^3})dz$ . So, formula (9) shows that

$$v(s) = \varphi\left(\frac{2}{\sqrt{3}}s\right). \quad (11)$$

Now, combining Propositions 3 and 4 and formula (11) with Cox and Shurman's result about  $\varphi$  yields the proof of Theorem 2.

The arclength parametrization of the curve can be derived using lines through the origin.

Letting  $y = mx/\sqrt{3}$ , where  $m$  is a parameter, we have

$$\left(1 + \frac{m^2}{3}\right)^3 x^6 = (1 - m^2)^3 x^3, \quad (12)$$

and we may parametrize the curve by

$$x = 3 \frac{\sqrt[3]{1-m^2}}{3+m^2}, \quad y = \frac{mx}{\sqrt{3}}. \quad (13)$$

If we then replace  $m$  by  $\varphi'(t)$ , we have

$$x = \frac{3\varphi(t)}{3 + \varphi'(t)^2}, \quad y = \frac{\sqrt{3}\varphi(t)\varphi'(t)}{3 + \varphi'(t)^2}. \quad (14)$$

Now, observe that

$$x^2 + y^2 = r^2 = \frac{3\varphi^2}{3 + \varphi'^2} = \frac{3\varphi^2}{4 - \varphi^3}. \quad (15)$$

Therefore, replacing  $t$  by  $(2/\sqrt{3})s$  in (14) and comparing with (10) and (11), we see that this is the arclength parametrization. We have shown the following.

**Theorem 5.** *The coordinates of the arclength parametrization of the trefoil are rational expressions in the elliptic functions  $\varphi$  and  $\varphi'$ .*

## 4. Concluding Remarks

As is the case for the clover, also a leaf of the trefoil cannot be subdivided into, for example, three equal arcs using straightedge and compass. In fact, it can be shown that the circle of radius  $1/\sqrt[3]{2}$  centered at the origin trisects each leaf of the trefoil. Construction of this circle amounts to the construction of the Delian number  $\sqrt[3]{2}$ . In other words, the problem of trisecting one leaf of the trefoil is equivalent to the classical problem of duplication of the cube!

It is also worthwhile to note that the algebraic problem of division into  $n$  equal pieces by radicals (i.e., *solubility*) was achieved for the circle by Gauss, for the lemniscate by Abel (with the help of Liouville after Abel's untimely death), and likewise it holds for the clover and the trefoil.

## References

- [1] J. Pierpont, "On an undemonstrated theorem of the Disquisitiones Arithmeticae," *Bulletin of the American Mathematical Society*, vol. 2, no. 3, pp. 77–83, 1895.
- [2] M. P. Beloch, "Sul metodo del ripiegamento della carta per la risouzione dei problemi geometrici," *Periodico di Matematiche Serie 4*, vol. 16, pp. 104–108, 1936.
- [3] T. C. Hull, "Solving cubics with creases: the work of Beloch and Lill," *American Mathematical Monthly*, vol. 118, no. 4, pp. 307–315, 2011.
- [4] D. A. Cox, *Galois Theory*, Pure and Applied Mathematics, John Wiley & Sons, Hoboken, NJ, USA, 2004.
- [5] M. Rosen, "Abel's theorem on the lemniscate," *The American Mathematical Monthly*, vol. 88, no. 6, pp. 387–395, 1981.
- [6] V. Prasolov and Y. Solov'yev, *Elliptic Functions and Elliptic Integrals*, vol. 170 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1997.
- [7] D. A. Cox and J. Shurman, "Geometry and number theory on clovers," *American Mathematical Monthly*, vol. 112, no. 8, pp. 682–704, 2005.

- [8] C. Zwikker, *The Advanced Geometry of Plane Curves and Their Applications*, Dover, New York, NY, USA, 2005.
- [9] C. Caldwell, Prime pages, <http://primes.utm.edu/primes/lists/short.txt>.
- [10] N. J. A. Sloane, Ed., "The online encyclopedia of integer sequences," <http://oeis.org/A005109>.
- [11] R. C. Alperin, "A mathematical theory of origami constructions and numbers," *New York Journal of Mathematics*, vol. 6, pp. 119–133, 2000.

