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*Research Article*

# Sunlet Decomposition of Certain Equipartite Graphs

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Let $L_{2n}$ stand for the sunlet graph which is a graph that consists of a cycle and an edge terminating in a vertex of degree one attached to each vertex of cycle $C_n$. The necessary condition for the equipartite graph $K_n + I * \overline{K}_m$ to be decomposed into $L_{2n}$ for $n \geq 2$ is that the order of $L_{2n}$ must divide $n^2m^2/2$, the order of $K_n + I * \overline{K}_m$. In this work, we show that this condition is sufficient for the decomposition. The proofs are constructive using graph theory techniques.

## 1. Introduction

Let $C_r$, $K_n$, $\overline{K}_m$ denote cycle of length $r$, complete graph on $n$ vertices, and complement of complete graph on $m$ vertices. For $n$ even, $K_n + I$ denotes the multigraph obtained by adding the edges of a 1-factor to $K_n$, thus duplicating $n/2$ edges. The total number of edges in $K_n + I$ is $n^2/2$. The *lexicographic product*, $G * H$, of graphs $G$ and $H$, is the graph obtained by replacing every vertex of $G$ by a copy of $H$ and every edge of $G$ by the complete bipartite graph $K_{|H|,|H|}$.

For a graph $H$, an $H$-*decomposition* of a graph $G$, $H \mid G$, is a set of subgraphs of $G$, each isomorphic to $H$, whose edge set partitions the edge set of $G$. Note that for any graph $G$ and $H$ and any positive integer $m$, if $H \mid G$ then $(H * \overline{K}_m) \mid (G * \overline{K}_m)$.

Let $G$ be a graph of order $n$ and $H$ any graph. The *corona* (crown) of $G$ with $H$, denoted by $G \odot H$, is the graph obtained by taking one copy of $G$ and $n$ copies of $H$ and joining the $i$th vertex of $G$ with an edge to every vertex in the $i$th copy of $H$. A special corona graph is $C_n \odot K_1$, that is, a cycle with pendant points which has $2n$ vertices. This is called *sunlet* graph and denoted by $L_q$, $q = 2n$.

Obvious necessary condition for the existence of a $k$-*cycle* decomposition of a simple connected graph $G$ is that $G$ has at least $k$ vertices (or trivially, just one vertex), the degree of every vertex in $G$ is even, and the total number of edges in $G$ is a multiple of the cycle length $k$. These conditions have been shown to be sufficient in the case that $G$ is the complete graph $K_n$, the complete graph minus a 1-factor $K_n - I$ [1, 2], and the complete graph plus a 1-factor $K_n + I$ [3].

The study of cycle decomposition of $K_n * \overline{K}_m$ was initiated by Hoffman et al. [4]. The necessary and sufficient conditions for the existence of a $C_p$-decomposition of $K_n * \overline{K}_m$, where $p \geq 5$ ($p$ is prime) that (i) $m(n-1)$ is even and (ii) $p$ divides $n(n-1)m^2$, were obtained by Manikandan and Paulraja [5, 6]. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of a $C_{2p}$-decomposition of $K_n * \overline{K}_m$ were given by Smith [7]. For a prime number $p \geq 3$, Smith [8] showed that $C_{3p}$-decomposition of $K_n * \overline{K}_m$ exists if the obvious necessary conditions are satisfied. In [9], Anitha and Lekshmi proved that the complete graph $K_n$ and the complete bipartite graph $K_{n,n}$ for $n$ even have decompositions into sunlet graph $L_n$. Similarly, in [10], it was shown that the complete equipartite graph $K_n * \overline{K}_m$ has a decomposition into sunlet graph of length $2p$, for a prime $p$.

We extend these results by considering the decomposition of $K_n + I * \overline{K}_m$ into sunlet graphs and prove the following result.

Let $m \geq 2$, $n > 2$, and $q \geq 6$ be even integers. The graph $K_n + I * \overline{K}_m$ can be decomposed into sunlet graph of length $q$ if and only if $q$ divides $n^2m^2/2$, the number of edges in $K_n + I * \overline{K}_m$.

## 2. Proof of the Result

To prove the result, we need the following.

**Lemma 1** (see [10]). *For $r \geq 3$, $L_{2r}$ decomposes $C_r * \overline{K}_2$.*

**Lemma 2.** *For any integer $r > 2$ and a positive even integer $m$, the graph $C_r * \overline{K}_m$ has a decomposition into sunlet graph $L_q$, for $q = rm$.*

*Proof*

*Case 1* ($r$ is even). First observe that $C_r * \overline{K}_2$ can be decomposed into 2 sunlet graphs with $2r$ vertices. Now, set $m = 2t$ and decompose $C_r * \overline{K}_t$ into cycles $C_{rt}$. To decompose $C_r * \overline{K}_t$ into $t$-cycles $C_{rt}$, denote vertices in $i$th part of $C_r * \overline{K}_t$ by $x_{i,j}$ for $j = 1, \ldots, t$, $i = 1, 2, \ldots, r$ and create $t$ base cycles $x_{1,j}x_{2,j}x_{3,j} \cdots x_{r-1,j}x_{r,j}$. Next, combine these base cycles into one cycle $C_{rt}$ by replacing each edge $x_{1,j}x_{2,j}$ with $x_{1,j}x_{2,j+1}$. To create the remaining cycles $C_{rt}$, we apply mappings $\phi_s$ for $s = 0, 1, \ldots, t-1$ defined on the vertices as follows.

*Subcase 1.1* ($i$ odd). Consider

$$\phi_s\left(x_{i,j}\right) = x_{i,j}. \tag{1}$$

This is the desired decomposition into cycles $C_{rt}$.

*Subcase 1.2* ($i$ even). Consider

$$\phi_s\left(x_{i,j}\right) = x_{i,j+s}. \tag{2}$$

This is the desired decomposition into cycles $C_{rt}$.

Now take each cycle $C_{rt}$, and make it back into $C_{rt} * \overline{K}_2$. Each $C_{rt} * \overline{K}_2$ decomposes into 2 sunlet graphs $L_{2rt}$ (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs with length $rm$ for $r$ even. Note that

$$C_r * \overline{K}_{2t} = \left(C_r * \overline{K}_t\right) * \overline{K}_2. \tag{3}$$

*Case 2* ($r$ is odd)

*Subcase 2.1* ($m \equiv 2 \pmod 4$). Set $m = 2t$. First create $t$ cycles $C_{(r-1)t}$ in $C_{r-1} * \overline{K}_t$ as in Case 1. Then, take complete tripartite graph $K_{t,t,t}$ with partite sets $X_i = \{x_{i,j}\}$ for $i = 1, r-1, r$ and $j = 1, \ldots, t$ and decompose it into triangles using well-known construction via Latin square, that is, construct $t \times t$ Latin square and consider each element in the form $(a, b, c)$ where $a$ denotes the row, $b$ denotes the column, and $c$ denotes the entry with $1 \leq a, b, c \leq t$. Each cycle is of the form $x_{(1,a)}$, $x_{(r-1,b)}$, $x_{(r,c)}$. Then, for every triangle $x_{1,a}x_{r-1,b}x_{r,c}$, replace the edge $x_{1,a}x_{r-1,b}$ in each $C_{(r-1)t}$, by the edges $x_{r-1,b}x_{r,c}$ and $x_{r,c}x_{1,a}$ to obtain cycles $C_{rt}$. Therefore, $C_{rt} \mid C_r * \overline{K}_t$. Now take each cycle $C_{rt}$, make it into $C_{rt} * \overline{K}_2$, and by Lemma 1, $C_{rt} * \overline{K}_2$ has a decomposition into sunlet graphs $L_{2rt} = L_q$.

*Subcase 2.2* ($m \equiv 0 \pmod 4$). Set $m = 2t$. The graph $C_r * \overline{K}_t$ decomposes into Hamilton cycle $C_{rt}$ by [11]. Next, make each cycle $C_{rt}$ into $C_{rt} * \overline{K}_2$. Each graph $C_{rt} * \overline{K}_2$ decomposes into sunlet graph $L_{2rt}$ by Lemma 1. □

**Theorem 3.** *Let $r$, $m$ be positive integers satisfying $r, m \equiv 0 \pmod 4$, then $L_r$ decomposes $C_r * \overline{K}_m$.*

*Proof.* Let the partite sets (layers) of the $r$-partite graph $C_r * \overline{K}_m$ be $U_1, U_2, \ldots, U_r$. Set $m = 2t$. Obtain a new graph from $C_r * \overline{K}_m$ as follows.

Identify the subsets of vertices $\{x_{i,j}\}$, for $1 \leq i \leq r$ and $1 \leq j \leq m/2$ into new vertices $x_i^1$, and identify the subset of vertices $\{x_{i,j}\}$ for $1 \leq i \leq r$ and $m/2 + 1 \leq j \leq m$ into new vertices $x_i^2$ and two of these vertices $x_i^k$, where $k = 1, 2$, are adjacent if and only if the corresponding subsets of vertices in $C_r * \overline{K}_m$ induce $K_{t,t}$. The resulting graph is isomorphic to $C_r * \overline{K}_2$. Next, decompose $C_r * \overline{K}_2$ into cycles $C_{r/2}$ as follows: $x_{k,1}x_{k+1,1}, \ldots, x_{d,1}x_{d-1,2}, \ldots, x_{k+1,2}, x_{k,1}$

$$k = 1, \frac{r}{4} + 1, \frac{r}{2} + 1, \frac{3r}{4} + 1, \ldots, r - \frac{r}{4} + 1, \qquad d = \frac{r}{4} + k, \tag{4}$$

where $k$, $d$ are calculated modulo $r$.

To construct the remaining cycles, apply mapping $\phi$ defined on the vertices.

*Subcase 1.1* ($i$ odd in each cycle). Consider

$$\phi\left(x_{i,j}\right) = x_{i,j+1}. \tag{5}$$

This is the desired decomposition of $C_r * \overline{K}_2$ into cycles $C_{r/2}$.

*Subcase 1.2* ($i$ even in each cycle). Consider

$$\phi\left(x_{i,j}\right) = x_{i,j}. \tag{6}$$

This is the desired decomposition of $C_r * \overline{K}_2$ into cycles $C_{r/2}$.

By lifting back these cycles $C_{r/2}$ of $C_r * \overline{K}_2$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_{r/2} * \overline{K}_t$. Obtain a new graph again from $C_{r/2} * \overline{K}_t$ as follows.

For each $j$, $1 \leq j \leq t/2$, identify the subsets of vertices $\{x_{i,2j-1}, x_{i,2j}\}$, where $1 \leq i \leq r/2$ into new vertices $x_i^j$, and two of these vertices $x_i^j$ are adjacent if and only if the corresponding subsets of vertices in $C_{r/2} * \overline{K}_t$ induce $K_{2,2}$. The resulting graph is isomorphic to $C_{r/2} * \overline{K}_{t/2}$. Then, decompose $C_{r/2} * \overline{K}_{t/2}$ into cycles $C_{r/2}$. Each $C_{r/2} * \overline{K}_{t/2}$ decomposes into cycles $C_{r/2}$ by [12]. By lifting back these cycles $C_{r/2}$ of $C_{r/2} * \overline{K}_{t/2}$ to $C_{r/2} * \overline{K}_t$, we get edge-disjoint subgraph isomorphic to $C_{r/2} * \overline{K}_2$. Finally, each $C_{r/2} * \overline{K}_2$ decomposes into two sunlet graphs $L_r$ (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs $L_r$ as required. □

**Theorem 4** (see [12]). *The cycle $C_m$ decomposes $C_k * \overline{K}_m$ for every even $m > 3$.*

**Theorem 5** (see [12]). *If $m$ and $k \geq 3$ are odd integers, then $C_m$ decomposes $C_k * \overline{K}_m$.*

**Theorem 6.** *The sunlet graph $L_m$ decomposes $C_r * \overline{K}_m$ if and only if either one of the following conditions is satisfied.*

(1) *$r$ is a positive odd integer, and $m$ is a positive even integer.*

(2) *$r$, $m$ are positive even integers with $m \equiv 0 \pmod 4$.*

*Proof.* (1) Set $m = 2t$, where $t$ is a positive integer. Let the partite sets (layers) of the $r$-partite graph $C_r * \overline{K}_m$ be $U_1, U_2, \ldots, U_r$. For each $j$, where $1 \le j \le t$, identify the subsets of vertices $\{x_{i,2j-1}, x_{i,2j}\}$, for $1 \le i \le r$ into new vertices $x_i^j$, and two of these vertices $x_i^j$ are adjacent if and only if the corresponding subsets of vertices in $C_r * \overline{K}_m$ induce $K_{2,2}$. The resulting graph is isomorphic to $C_r * \overline{K}_t$. Then, decompose $C_r * \overline{K}_t$ into cycles $C_t$, where $t$ is a positive integer.

Now, $C_t \mid C_r * \overline{K}_t$ by Theorems 4 and 5.

By lifting back these $t$-cycles of $C_r * \overline{K}_t$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_t * \overline{K}_2$. Each copy of $C_t * \overline{K}_2$ decomposes into sunlet graphs of length $2t$ (by Lemma 1), and we have $C_r * \overline{K}_m$ decomposing into sunlet graphs of length $m$ as required.

(2) Set $m = 2t$, where $t$ is an even integer since $m \equiv 0 \pmod 4$.

Obtain a new graph $C_r * \overline{K}_t$ from the graph $C_r * \overline{K}_m$ as in Case 1. By Theorem 4, $C_t \mid C_r * \overline{K}_t$. By lifting back these $t$-cycles of $C_r * \overline{K}_t$ to $C_r * \overline{K}_{2t}$, we get edge-disjoint subgraphs isomorphic to $C_t * \overline{K}_2$. Each copy of $C_t * \overline{K}_2$ decomposes into sunlet graph of length $2t$ (by Lemma 1). Therefore, $L_m \mid C_r * \overline{K}_m$ as required. □

*Remark 7.* In [10], it was shown that

$$L_{2r} * \overline{K}_l \text{ can be decomposed into } l^2 \text{ copies of } L_{2r}. \quad (7)$$

This, coupled with Lemma 1, gives the following.

**Theorem 8** (see [10]). *The graph $C_r * \overline{K}_{2l}$ decomposes into sunlet graphs $L_{2r}$ for any positive integer $l$.*

**Lemma 9** (see [3]). *Let $n \ge 4$ be an even integer. Then, $K_n + I$ is $C_n$-decomposable.*

**Lemma 10** (see [3]). *Let $m$ and $n$ be integers with $m$ odd, $n \equiv 2 \pmod 4$, $3 \le m \le n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is $C_m$-decomposable.*

**Lemma 11** (see [3]). *Let $m$ and $n$ be integers with $m$ odd, $n \equiv 0 \pmod 4$, $3 \le m \le n < 2m$, and $n^2 \equiv 0 \pmod{2m}$. Then, $K_n + I$ is $C_m$-decomposable.*

We can now prove the major result.

**Theorem 12.** *For any even integers $m \ge 2$, $n > 2$, and $q \ge 6$, the sunlet graph $L_q$ decomposes $K_n + I * \overline{K}_m$ if and only if $n^2m^2/2 \equiv 0 \pmod q$.*

*Proof.* The necessity of the condition is obvious, and so we need only to prove its sufficiency. We split the problem into the following two cases.

*Case 1* ($q \mid n$)

*Subcase 1.1* ($n > q$). Cycle $C_n$ decomposes $K_n + I$ by Lemma 9, and we have

$$C_n * \overline{K}_m \mid K_n + I * \overline{K}_m. \quad (8)$$

Each graph $C_n * \overline{K}_m$ decomposes into sunlet graph $L_q$, where $q = nm$ by Lemma 2, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph $L_q$, where $q > n$.

*Subcase 1.2* ($q = n$). First, consider $n \equiv 0 \pmod 4$.

Cycle $C_q$ decomposes $K_q + I$ by Lemma 9, and we have

$$C_q * \overline{K}_m \mid K_q + I * \overline{K}_m. \quad (9)$$

Now, sunlet graph $L_q \mid (C_q * \overline{K}_m)$ by Theorem 3, and hence sunlet graph $L_q$ decomposes $K_n + I * \overline{K}_m$.

Also, consider $n \equiv 2 \pmod 4$.

Suppose $m = 2t$. Cycle $C_{q/2}$ decomposes $K_q + I$ by Lemma 10, and we have

$$C_{q/2} * \overline{K}_{2t} \mid K_q + I * \overline{K}_{2t}. \quad (10)$$

Now, sunlet graph $L_q$ decomposes $C_{q/2} * \overline{K}_{2t}$ by Theorem 8, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph of length $q$.

*Case 2* ($q \mid m$)

*Subcase 2.1* ($m \equiv 0 \pmod 4$). Suppose $m = q$, and by Lemma 9, cycle $C_n$ decomposes $K_n + I$, and we have

$$C_n * \overline{K}_q \mid K_n + I * \overline{K}_q. \quad (11)$$

Also, sunlet graph $L_q$ decomposes each $C_n * \overline{K}_q$ by Theorem 6, and we have sunlet graph $L_q$ decomposing $K_n + I * \overline{K}_m$.

*Subcase 2.2* ($m \equiv 2 \pmod 4$). Let $m = q$ and $r \le n$ an odd integer. Cycle $C_r$ decomposes $K_n + I$, by Lemmas 9, 10, and 11, and we have

$$C_r * \overline{K}_q \mid K_n + I * \overline{K}_q. \quad (12)$$

Now, each $C_r * \overline{K}_q$ decomposes into sunlet graph $L_q$ by Theorem 6, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph $L_q$ as required.

*Subcase 2.3* ($m > q$). Set $m = wq$, where $w$ is any positive integer, then by Subcases 2.1 and 2.2, we have

$$L_q * \overline{K}_w \mid \left(K_n + I * \overline{K}_q\right) * \overline{K}_w. \quad (13)$$

Each graph $L_q * \overline{K}_w$ decomposes into sunlet graph $L_q$ by Remark 7, and we have $K_n + I * \overline{K}_m$ decomposing into sunlet graph $L_q$. □

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