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Research Article

On the Classification of Almost Kenmotsu Manifolds of Dimension 3

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This paper deals with the classification of a 3-dimensional almost Kenmotsu manifold satisfying certain geometric conditions. Moreover, by applying our main classification theorem, we obtain some suffcient conditions for an almost Kenmotsu manifold of dimension 3 to be an Einstein-Weyl manifold.

1. Introduction

Contact metric manifolds known as a special class of almost contact metric manifolds are objects of increasing interest both from geometers and physicists [1] recently. We refer the reader to the recent monograph [2] for a wide and detailed overview of the results in this field. From (14) (see Section 3) we know that a normal almost contact metric manifold (which includes Sasakian and Kenmotsu manifolds as its special cases) of dimension 3 satisfies $Q\phi = \phi Q$. But the above property need not be true in an almost contact metric manifold. Blair et al. [3] obtained a classification of 3-dimensional contact metric manifold with $Q\phi = \phi Q$. However, in higher dimensions the classification of contact metric manifold with $Q\phi = \phi Q$ is still open. It is worthy to point out that Ghosh [4] recently proved that a contact metric manifold admitting the Einstein-Weyl structures $W^{\pm} = (g, \pm\omega)$ (see Section 4) and $Q\phi = \phi Q$ is either a K-contact or an Einstein manifold.

On the other hand, in 1972, Kenmotsu [5] introduced a class of almost contact metric manifolds which are known as Kenmotsu manifolds nowadays. Recently, almost Kenmotsu manifolds satisfying η-parallelism and locally symmetries are studied by Dileo and Pastore [6] and [7], respectively. We notice that Dileo and Pastore [8] complete the classification of 3-dimensional almost Kenmotsu manifold with the assumption that ξ belongs to the $(k,\mu)'$-nullity distribution. However, to the best of our knowledge the study of 3-dimensional almost Kenmotsu manifolds is still lacking so far. The object of this paper is to classify the 3-dimensional almost Kenmotsu manifolds satisfying $Q\phi = \phi Q$ and other geometric conditions, providing some results which show the differences between almost Kenmotsu manifolds and the contact metric manifolds of dimension 3 [3, 9]. Moreover, by applying our main classification theorem, we obtain some suffcient conditions for an almost Kenmotsu manifold of dimension 3 to be an Einstein-Weyl manifold.

This paper is organized as the following way. In Section 2, we provide some basic formulas and properties of almost Kenmotsu manifolds. Section 3 is devoted to present our main theorems and their proofs. Finally, in Section 4, we prove that if an almost Kenmotsu manifold of dimension 3 is η-Einstein with certain condition then it admits both Einstein-Weyl structures $W^{+} = (g, +\omega)$ and $W^{-} = (g, -\omega)$.

2. Almost Kenmotsu Manifolds

First of all, we give some basic notions of almost Kenmotsu manifolds which follow from [5, 7]. An almost contact structure on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor, ξ a global vector field, and η a 1-form, such that

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

which implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\operatorname{rank}(\phi) = 2n$. It follows from [2, 10] that a Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold together with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Now let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold. We denote by $l = R(\cdot, \xi)\xi$ and $h = (1/2)\mathscr{L}_\xi \phi$ on M^{2n+1}, where R is the curvature tensor and $\mathscr{L}$ is the Lie differentiation, respectively. Thus, the two $(1, 1)$-type tensors l and h are symmetric and satisfy

$$\begin{gathered} h\xi = 0, \qquad l\xi = 0, \qquad \operatorname{tr} h = 0, \\ \operatorname{tr}(h\phi) = 0, \qquad h\phi + \phi h = 0. \end{gathered} \tag{3}$$

We also have the following formulas following from [5, 7, 8]:

$$\nabla_X \xi = -\phi^2 X - \phi h X \ \left(\Longrightarrow \nabla_\xi \xi = 0\right), \tag{4}$$

$$\phi l \phi - l = 2\left(h^2 - \phi^2\right), \tag{5}$$

$$\operatorname{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \operatorname{tr} h^2, \tag{6}$$

$$\begin{aligned} R(X, Y)\xi = {} & \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) \\ & + (\nabla_Y \phi h) X - (\nabla_X \phi h) Y, \end{aligned} \tag{7}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{8}$$

for any $X, Y \in \Gamma(TM^{2n+1})$, where S, Q, ∇, and $\Gamma(TM)$ denote the Ricci curvature tensor, Ricci operator, the Levi-Civita connection of g, and the Lie algebra of vector fields in M^{2n+1}, respectively. From the above formulas we also have $\nabla_\xi \phi = 0$.

An almost contact manifold is said to be η-Einstein if

$$Q = \alpha \mathrm{id} + \beta \eta \otimes \xi, \tag{9}$$

where α and β are both smooth functions on M^{2n+1}. It is easy to see that an η-Einstein almost Kenmotsu manifold satisfies $Q\phi = \phi Q$ because of $\phi\xi = 0$. We also recall that the k-nullity distribution [11] is defined by

$$\begin{aligned} N_p(k) = \{ & Z \in T_p M : R(X, Y)Z = k(g(Y, Z)X \\ & - g(X, Z)Y) \}, \end{aligned} \tag{10}$$

where k is a real number. When k in (10) is a smooth functions, then the nullity distributions are called the generalized nullity distributions [12]. Also, the sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ-sectional curvature and the sectional curvature $K(X, \phi X)$ of a plane section spanned by vectors X and ϕX with X orthogonal to ξ is called a ϕ-sectional curvature [3].

A Riemanian manifold M of dimension m is conformally flat if and only if the Weyl tensor C defined by

$$\begin{aligned} C(X, Y)Z = {} & R(X, Y)Z - \frac{1}{m-2} \\ & \times \{S(Y, Z)X - S(X, Z)Y \\ & + g(Y, Z)QX - g(X, Z)QY\} \\ & + \frac{r}{(m-1)(m-2)} \{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \tag{11}$$

vanishes for $m > 3$ and

$$\begin{aligned} & (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \\ & \quad = \frac{1}{4}\{X(r)g(Y, Z) - Y(r)g(X, Z)\}, \end{aligned} \tag{12}$$

for $m = 3$, where S, Q, and r denote the Ricci tensor, Ricci operator, and the scalar curvature, respectively.

3. A Classification Theorem

Let $(M^3, \phi, \xi, \eta, g)$ be a normal almost contact metric manifold of dimension 3, then we have $Q\phi = \phi Q$. In fact, it follows from [13, 14] that M^3 satisfies

$$\begin{aligned} S(X, Y) = {} & \left(\frac{r}{2} + \alpha^2 - \beta^2\right) g(\phi X, \phi Y) \\ & - 2\left(\alpha^2 - \beta^2\right)\eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM), \end{aligned} \tag{13}$$

where $\alpha = \operatorname{div}(\xi)$ and $\beta = \operatorname{tr}(\phi \nabla \xi)$ are both constants and r denotes the scalar curvature of M^3. Then from the above equation it is easy to get

$$Q = -\left(\frac{r}{2} + \alpha^2 - \beta^2\right)\phi^2 - 2\left(\alpha^2 - \beta^2\right)\eta \otimes \xi. \tag{14}$$

Noticing that $\phi\xi = 0$, then from (14) we know that $Q\phi = \phi Q$. If $\beta = 1$ (resp., $\alpha = 1$) then M^3 is just the 3-dimensional Sasakian (resp., Kenmotsu) manifold [13]. However, the condition $Q\phi = \phi Q$ need not be true in a contact metric manifold as well as an almost Kenmotsu manifold. In a contact metric manifold M^{2n+1} if the characteristic vector ξ is a Killing vector field, then the manifold is said to be a K-contact manifold. A Sasakian manifold is always a K-contact one, but the converse need not hold only if M is of dimension 3. We present the following result to characterize Kenmotsu manifold of dimension 3 which is analogous to the contact metric manifold of dimension 3.

Lemma 1. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension 3, then the following conditions are equivalent.*

(1) M^3 *is a Kenmotsu manifold.*

(2) $h = 0$.

(3) $\nabla\xi = -\phi^2$.

Proof. It follows from [7, 8] that an almost Kenmotsu is a Kenmotsu manifold if and only if

$$(\nabla_X\phi)Y = -g(X,\phi Y) - \eta(Y)\phi X, \quad \forall X, Y \in \Gamma(TM). \tag{15}$$

Noticing that $\nabla_\xi\phi = 0$ and $(\nabla_X\phi)Y + (\nabla_{\phi X})(\phi Y) = -\eta(Y)\phi X - 2g(X,\phi Y) - \eta(Y)h(X)$ (see [7]) for any $X, Y \in \Gamma(TM)$, thus M^3 is a Kenmotsu manifold if and only if every nonvanishing vector field X in contact distribution $\mathscr{D}$ (defined by $\mathscr{D} = \ker(\eta) = \mathrm{Im}(\phi)$) satisfies the following equations:

$$\begin{gathered}(\nabla_X\phi)X = 0,\\ (\nabla_X\phi)\phi X = g(X,X)\xi,\\ (\nabla_X\phi)\xi = -\phi X,\\ (\nabla_{\phi X}\phi)\xi = X.\end{gathered} \tag{16}$$

On the other hand, on an almost Kenmotsu manifold by using (1)–(4) we obtain

$$\begin{gathered}g((\nabla_X\phi)X, X) = g((\nabla_X\phi)X, \phi X) = 0,\\ g((\nabla_X\phi)X, \xi) = g(X, hX),\\ g((\nabla_X\phi)\phi X, X) = g((\nabla_X\phi)\phi X, \phi X) = 0,\\ g((\nabla_X\phi)\phi X, \xi) = g(X,X) - g(X,\phi hX),\end{gathered} \tag{17}$$

$$\begin{gathered}(\nabla_X\phi)\xi = -\phi X - hX,\\ (\nabla_{\phi X}\phi)\xi = X - h\phi X.\end{gathered} \tag{18}$$

Comparing the above three equations with (16), we complete the proof of equivalence between (1) and (2). The equivalence between (2) and (3) follows from the fact that $\nabla_X\xi = -\phi^2 X - \phi hX$ for any $X \in \Gamma(TM)$. □

Theorem 2. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension 3 for which the characteristic vector field ξ is an eigenvector field of the Ricci operator. If M^3 is conformally flat, then it satisfies*

$$h'Q = Qh', \tag{19}$$

$$X(\mathrm{tr}(l)) = \frac{1}{4}X(r), \tag{20}$$

for any $X \in \Gamma(TM)$, where r denotes the scalar curvature of M^3.

Proof. Let ξ be an eigenvector field of the Ricci operator corresponding to the eigenvalue α, that is, $Q\xi = \alpha\xi$. Substituting the above equation into (6) implies that $\alpha = \mathrm{tr}(l)$. Then we obtain

$$Q\xi = \mathrm{tr}(l)\xi. \tag{21}$$

Differentiating (21) along an arbitrary vector field X and using (4) gives

$$\begin{aligned}(\nabla_X Q)\xi = {} & X(\mathrm{tr}(l))\xi + Q(\phi^2 X + \phi hX)\\ & - \mathrm{tr}(l)(\phi^2 X + \phi hX).\end{aligned} \tag{22}$$

Since M^3 is conformally flat, then substituting $Z = \xi$ into (12) and using $g((\nabla_X Q)Y, Z) = (\nabla_X S)(Y, Z)$ gives

$$g((\nabla_X Q)Y - (\nabla_Y Q)X, \xi) = \frac{1}{4}\{X(r)\eta(Y) - Y(r)\eta(X)\}, \tag{23}$$

for any $X, Y \in \Gamma(TM)$. Using the symmetry $g((\nabla_X Q)Y, Z) = g((\nabla_X Q)Z, Y)$, then from (23) we have

$$\begin{aligned}& g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X)\\ & \quad = \frac{1}{4}\{X(r)\eta(Y) - Y(r)\eta(X)\},\end{aligned} \tag{24}$$

for any $X, Y \in \Gamma(TM)$. Substituting (22) into (24) gives

$$\begin{aligned}& X(\mathrm{tr}(l))\eta(Y) - Y(\mathrm{tr}(l))\eta(X) + g(Y, Q\phi hX)\\ & \quad - g(X, Q\phi hY) = \frac{1}{4}\{X(r)\eta(Y) - Y(r)\eta(X)\},\end{aligned} \tag{25}$$

for any $X, Y \in \Gamma(TM)$. Replacing ϕX and ϕY by X and Y, respectively, in (25) gives

$$g(\phi Y, Q\phi h\phi X) = g(\phi X, Q\phi h\phi Y), \tag{26}$$

for any $X, Y \in \Gamma(TM)$. Thus, it follows from (26) that $h'Q = Qh'$ and hence by substituting $Y = \xi$ in (25) we obtain

$$X(\mathrm{tr}(l)) - \frac{1}{4}X(r) = \left[\xi(\mathrm{tr}(l)) - \frac{1}{4}\xi(r)\right]\eta(X), \tag{27}$$

for any $X \in \Gamma(TM)$. Applying the exterior derivation d on both sides of (27) and using the well-known Poincaré lemma $d^2 = 0$, and then replacing X, Y, respectively, by $\phi X, \phi Y$ in the resulting equation, we obtain

$$\xi(\mathrm{tr}(l)) - \frac{1}{4}\xi(r) = 0. \tag{28}$$

Thus, substituting (28) into (27) gives (20). □

Theorem 3. *Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional almost Kenmotsu manifold satisfying $Q\phi = \phi Q$. Then one of the following cases occurs.*

Case 1. $\xi(\mathrm{tr}(l)) = 0$ *and hence M^3 is a Kenmotsu manifold.*

Case 2. $\xi(\mathrm{tr}(l)) \neq 0$ *and hence the eigenvalues of h are locally given by $\{0, ce^{-2t}, -ce^{-2t}\}$ with coordinate t on R and c a nonzero number.*

Proof. For an almost Kenmotsu manifold the operator l never vanishes. In fact, if $l = 0$, it follows from (6) that $\mathrm{tr}(h^2) = -2$,

there is a contradiction. Now we suppose that M^3 is an almost Kenmotsu manifold of dimension 3 with $Q\phi = \phi Q$. Noticing that $\phi\xi = 0$ then we have $\phi Q\xi = 0$. Denoting by P the projective component of $Q\xi$ on contact distribution $\mathscr{D}$, then from (6) we have $Q\xi = \operatorname{tr}(l)\xi + P$. By using the hypothesis $Q\phi = \phi Q$ on the above equation we get $P = 0$, then we have

$$Q\xi = \operatorname{tr}(l)\,\xi. \tag{29}$$

It is well known that the curvature tensor of a 3-dimensional Riemannian manifold (M^3, g) is given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & - g(QX,Z)Y - \frac{r}{2}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{30}$$

for any $X, Y, Z \in \Gamma(TM)$, where r denotes the scalar curvature of M^3. Noticing (29) and replacing Y, Z by ξ in (30) yields

$$\begin{aligned} QX = {} & lX - \left(\operatorname{tr}(l) - \frac{r}{2}\right)X \\ & - \left(\frac{r}{2} - 2\operatorname{tr}(l)\right)\eta(X)\,\xi, \quad \forall X \in \Gamma(TM). \end{aligned} \tag{31}$$

Using $\phi\xi = 0$ and $\phi Q = Q\phi$, it follows from (31) that $\phi lX = \phi Q + (\operatorname{tr}(l) - r/2)\phi X = l\phi X$ for any $X \in \Gamma(TM)$, that is,

$$\phi l = l\phi. \tag{32}$$

On the other hand, substituting (32) into (5) gives that

$$l = \phi^2 - h^2. \tag{33}$$

From (8) and (33) we obtain that

$$\nabla_\xi h = -2h, \qquad \nabla_\xi l = 4h^2. \tag{34}$$

On the other hand, since ϕ is antisymmetric on $\mathscr{D}$ then from (32) we have $g(\phi X, lX) = 0$ for any $X \in \mathscr{D}$. Thus, noticing that $g(\xi, lX) = 0$, we obtain $lX = (\operatorname{tr}(l)/2)X$ for any $X \in \mathscr{D}$, that is, $lX = -(\operatorname{tr}(l)/2)\phi^2 X$ for any $X \in \Gamma(TM)$. Substituting the above equation into (31) gives

$$QX = \frac{r - \operatorname{tr}(l)}{2}X + \frac{3\operatorname{tr}(l) - r}{2}\eta(X)\,\xi, \quad \forall X \in \Gamma(TM). \tag{35}$$

Differentiating (35) along $Y \in \Gamma(TM)$ we obtain

$$\begin{aligned} (\nabla_Y Q)X = {} & Y\left(\frac{r - \operatorname{tr}(l)}{2}\right)X + \frac{3\operatorname{tr}(l) - r}{2}\eta(X)\nabla_Y\xi \\ & + \frac{1}{2}\{Y(3\operatorname{tr}(l) - r)\eta(X) \\ & + (3\operatorname{tr}(l) - r)\,g(X, \nabla_Y\xi)\}\,\xi, \\ & \forall X, Y \in \Gamma(TM). \end{aligned} \tag{36}$$

Also, it is well known that

$$\frac{1}{2}\operatorname{grad}(r) = (\nabla_X Q)X + (\nabla_\phi XQ)\phi X + (\nabla_\xi Q)\xi, \tag{37}$$

for any unit vector filed X in contact distribution $\mathscr{D}$, where $\operatorname{grad}(r)$ denotes the gradient of scalar curvature of M^3. Now letting $Y = X$ in (36) be unit vector fields in $\mathscr{D}$ gives

$$(\nabla_X Q)X = X\left(\frac{r - \operatorname{tr}(l)}{2}\right)X + \frac{3\operatorname{tr}(l) - r}{2}g(X, X + h\phi X)\,\xi. \tag{38}$$

Similarly, we have

$$\begin{aligned} (\nabla_\phi XQ)\phi X = {} & \phi X\left(\frac{r - \operatorname{tr}(l)}{2}\right)\phi X \\ & + \frac{3\operatorname{tr}(l) - r}{2}g(X, X - h\phi X)\,\xi. \end{aligned} \tag{39}$$

Substituting (38) and (39) and $(\nabla_\xi Q)\xi = \xi(\operatorname{tr}(l))\xi$ into (37) implies

$$\begin{aligned} \frac{1}{2}\operatorname{grad}(r) = {} & X\left(\frac{r - \operatorname{tr}(l)}{2}\right)X \\ & + \phi X\left(\frac{r - \operatorname{tr}(l)}{2}\right)\phi X \\ & + \{(3\operatorname{tr}(l) - r) + \xi(\operatorname{tr}(l))\}\,\xi. \end{aligned} \tag{40}$$

By using (40) and taking an inner product with X and ϕX, respectively, we obtain $X(\operatorname{tr}(l)) = 0$ and $\phi X(\operatorname{tr}(l)) = 0$ for any $X \in \mathscr{D}$, that is, $\operatorname{tr}(l)$ is a constant on $\mathscr{D}$.

Case 1. If $\xi(\operatorname{tr}(l)) = 0$, then we find that $\operatorname{tr}(l)$ is a constant on M and hence from (6) we see that $\operatorname{tr}(h^2)$ is also a constant on M. Let $X \in \mathscr{D}$ be a unit eigenvector filed of h with eigenvalue λ, that is, $hX = \lambda X$ (and hence $h\phi X = -\lambda\phi X$), then λ is a constant since that $\operatorname{tr}(h^2)$ is a constant. Using the first term of (34) gives $g((\nabla_\xi h)X, X) = g(\xi(\lambda)X, X) = -2\,g(hX, X) = -2\lambda$; thus, noting that λ is a constant then we obtain $\lambda = 0$, that is, $h = 0$ and hence from Lemma 1 we see that M^3 is a Kenmotsu manifold.

Case 2. If $\xi(\operatorname{tr}(l)) \neq 0$, it follows from Case 1 that $\xi(\lambda) = -2\lambda \neq 0$, where $\pm\lambda$ denotes the eigenvalues of h on $\mathscr{D}$. Locally, we can write $\xi = \partial/\partial t$ and hence $\lambda = ce^{-2t}$ with coordinate t on R and c a nonzero number following the fact that λ is a constant on contact distribution $\mathscr{D}$, which completes the proof. □

Corollary 4. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension 3. Then the following assertions are equivalent:*

(a) *M^3 is an η-Einstein Kenmotsu manifold.*

(b) *$Q\phi = \phi Q$.*

(c) *ξ belongs to the generalized k-nullity distribution.*

Moreover, if one of the above conditions holds, then the ξ-sectional curvature of M^3 is -1 and the ϕ-sectional curvature is $2 + r/2$, where r denotes the scalar curvature of M^3.

Proof. Suppose that M^3 is an η-Einstein almost Kenmotsu manifold of dimension 3; from (9) it is easy to see (a) $\Rightarrow$ (b).

If $Q\phi = \phi Q$, then replacing Z by ξ in (30) and using (35) we see that ξ belongs to the generalized k-nullity distribution, which means that $(b) \Rightarrow (c)$. Now letting ξ belongs to the generalized k-nullity distribution, then by a straightforward calculation we know that ξ is an eigenvector field of Ricci operator. Replacing $Y = Z$ by ξ in (30) implies that $(b) \Rightarrow (a)$.

Finally, if one of the above conditions holds, then from the above statements we have $Q\phi = \phi Q$. We choose a unit nonvanishing vector field X in contact distribution $\mathscr{D}$. Replacing $Y = Z = \phi X$ in (30) gives an equation, taking an inner product with X on the resulting equation and taking into account $Q\phi = \phi Q$ we obtain

$$R(X, \phi X, X, \phi X) = g(R(X, \phi X)\phi X, X) = 2S(X, X) - \frac{r}{2}. \tag{41}$$

On the other hand, by the definition of Ricci curvature tensor we have

$$\begin{aligned} S(X, X) &= R(X, \xi, X, \xi) + R(X, \phi X, X, \phi X) \\ &= g(lX, X) + R(X, \phi X, X, \phi X). \end{aligned} \tag{42}$$

Using $lX = (\mathrm{tr}(l)/2)X$ for any $X \in \mathscr{D}$ (see Theorem 3), then it follows from (41) into (42) that $S(X, X) = 1 + r/2$. Then from (41) we know that the ϕ-sectional curvature of M^3 is $2 + r/2$, which completes the proof. □

Kenmotsu [5] proved that if a Kenmotsu manifold M^{2n+1} is a space of constant ϕ-holomorphic sectional curvature H then M^{2n+1} is a space of constant curvature and $H = -1$. Thus, the following result follows from Corollary 4 and Theorem 3.

Corollary 5. *Let $(M^3, \phi, \xi, \eta, g)$ be an η-Einstein almost Kenmotsu manifold of dimension 3 with $\xi(\mathrm{tr}(l)) = 0$. If the scalar curvature of M^3 is a constant, then M^3 is locally isometric to a hyperbolic space $\mathbb{H}^3(-1)$ with constant scalar curvature -6.*

4. Einstein-Weyl Structures

Recall that a Weyl structure [4, 15] on a Riemannian manifold (M, g) of dimension ≥ 3 is defined by the pair $W^+ = (g, \omega)$ satisfying

$$Dg = \omega \otimes g, \tag{43}$$

where D is a unique torsion-free connection which preserves the conformal class $[g] = \{\lambda g : \lambda \in C^\infty(M)\}$ on M and ω is an 1-form on M. It follows from (43) that

$$D_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)E, \tag{44}$$

where ∇ and E denote the Levi-Civita connection of g and the dual vector field of ω with respect to g, respectively. The Weyl structure is said to be Einstein-Weyl if the symmetrized Ricci tensor associated with the Weyl connection is proportional to the Riemannian metric g, that is,

$$S^D(X, Y) + S^D(Y, X) = \Lambda g(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{45}$$

where S^D denotes the Ricci tensor associated with D and Λ is a smooth function on M. Notice that Narita [15] proved that an η-Einstein almost contact metric manifold satisfying $\nabla_X \xi = -\phi X$ admits an Einstein-Weyl structure $W^+ = (g, \omega)$. However, (4) implies that an almost Kenmotsu manifold never satisfies Narita's condition even if $h = 0$. Since then, we present the following sufficient conditions to characterize Einstein-Weyl structure on an almost Kenmotsu manifold of dimension 3.

Theorem 6. *Let $(M^3, \phi, \xi, \eta, g)$ be an η-Einstein almost Kenmotsu manifold of dimension 3 with $\xi(\mathrm{tr}(l)) = 0$, that is, $S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y)$. If β is a constant $\leq 1/4$, then M^3 admits an Einstein-Weyl structure $W^+ = (g, \omega)$.*

Proof. We define 1-form ω by $\omega = \lambda\eta$, where λ is a nonvanishing function on M^3. Then the dual vector field of ω respective to g is $\lambda\xi$. We also define a connection D on M^3 by

$$D_X Y = \nabla_X Y - \frac{1}{2}\gamma\eta(X)Y - \frac{1}{2}\gamma\eta(Y)X + \frac{1}{2}\gamma g(X, Y)\xi, \tag{46}$$

then from (46) it is easy to verify that D is torsion free and $Dg = \lambda\eta \otimes g$, that is, (g, ω) is a Weyl structure on M^3. It follows from [15, 16] that

$$\begin{aligned} S^D(X, Y) = {} & S(X, Y) + (\nabla_X \omega)Y \\ & - \frac{1}{2}(\nabla_Y \omega)X + \frac{1}{4}\omega(X)\omega(Y) \\ & + \left(\frac{1}{2}\operatorname{div}(\gamma\xi) - \frac{1}{4}\gamma^2\right)g(X, Y), \end{aligned} \tag{47}$$

for any $X, Y \in \Gamma(TM^3)$. By using (4) then a simple computation gives that

$$\begin{aligned} & (\nabla_X \omega)Y + (\nabla_Y \omega)X \\ & \quad = X(\lambda)\eta(Y) + Y(\lambda)\eta(X) \\ & \qquad + 2f[g(X, Y) - \eta(X)\eta(Y) + g(h\phi X, Y)], \end{aligned} \tag{48}$$

for any $X, Y \in \Gamma(TM^3)$ and

$$\begin{aligned} \operatorname{div}(\lambda\xi) &= g(\nabla_\xi(\lambda\xi), \xi) + g(\nabla_e(\lambda\xi), e) + g(\nabla_{\phi e}(\lambda\xi), \phi e) \\ &= \xi(\lambda) + 2\lambda, \end{aligned} \tag{49}$$

where $e \in \mathscr{D}$ is a unit vector field on M^3. Thus, substituting (48) and (49) into (47) yields that

$$\begin{aligned} S^D(X,Y) + S^D(Y,X) &= 2S(X,Y) + \frac{1}{2}\left((\nabla_X\omega)Y + (\nabla_Y\omega)X\right) \\ &\quad + \frac{1}{2}\omega(X)\omega(Y) + \left(\operatorname{div}(\lambda\xi) - \frac{1}{2}\lambda^2\right)g(X,Y) \\ &= \left(2\alpha + 3\lambda + \xi(\lambda) - \frac{1}{2}\lambda^2\right)g(X,Y) \\ &\quad + \left(2\beta - \lambda + \frac{1}{2}\lambda^2\right)\eta(X)\eta(Y) \\ &\quad + \frac{1}{2}\left(X(\lambda)\eta(Y) + Y(\lambda)\eta(X)\right) \\ &\quad + \lambda g(h\phi X, Y), \quad \forall X, Y \in \Gamma(TM^3). \end{aligned} \tag{50}$$

On the other hand, noticing Theorem 3 and Corollary 5, we know that $h = 0$ and M^3 is a Kenmotsu manifold. We set $\lambda = 1 + \sqrt{1-4\beta}$ or $\lambda = 1 - \sqrt{1-4\beta}$, then it follows from (50) that

$$\begin{aligned} S^D(X,Y) + S^D(Y,X) &= 2(\alpha + \lambda + \beta)g(X,Y), \quad \forall X, Y \in \Gamma(M^3), \end{aligned} \tag{51}$$

which completes the proof. □

Remark 7. From Corollary 5 we know that an η-Einstein almost Kenmotsu manifold of dimension 3 with $\xi(\operatorname{tr}(l)) = 0$ is a Kenmotsu manifold and $h = 0$; however, this property need not be true in higher dimensions more than 3. Thus, our result cannot be generalized to the case of higher dimensions.

A Weyl structure $W^- = (g, -\omega)$ on a Riemannian manifold M of dimension $(2n+1)$ is defined by $Dg = -\omega \otimes g$ for a unique torsion-free connection D and a 1-form ω. The Weyl structure $W^- = (g, -\omega)$ is said to be an Einstein-Weyl structure if (45) holds for a smooth function Λ on M. For an Einstein-Weyl structure $W^- = (g, -\omega)$ it follows from [4] that

$$\begin{aligned} D_X Y &= \nabla_X Y + \frac{1}{2}\gamma\eta(X)Y + \frac{1}{2}\gamma\eta(Y)X - \frac{1}{2}\gamma g(X,Y)\xi, \\ S^D(X,Y) &= S(X,Y) - n(\nabla_X\omega)Y \\ &\quad + \frac{1}{2}(\nabla_Y\omega)X + \frac{1}{4}\omega(X)\omega(Y) \\ &\quad + \left(\frac{1}{2}\operatorname{div}(E) - \frac{1}{4}\omega^2\right)g(X,Y), \end{aligned} \tag{52}$$

for any $X, Y \in \Gamma(TM^{2n+1})$, where E denotes the dual vector field of ω with respect to g. Thus, a straightforward calculation which is similar to the proof of Theorem 6 gives the following.

Theorem 8. *Let $(M^3, \phi, \xi, \eta, g)$ be an η-Einstein almost Kenmotsu manifold of dimension 3 with $\xi(\operatorname{tr}(l)) = 0$, that is, $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$. If β is a constant $\leq 1/4$, then M^3 admits both Einstein-Weyl structures $W^\pm = (g, \pm\omega)$.*

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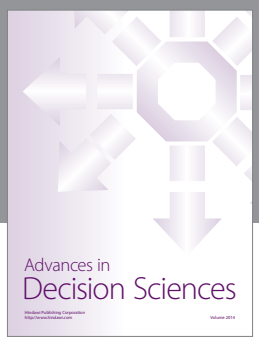

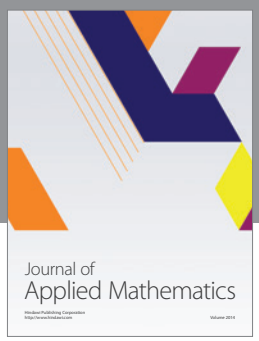

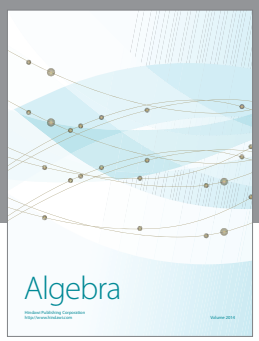

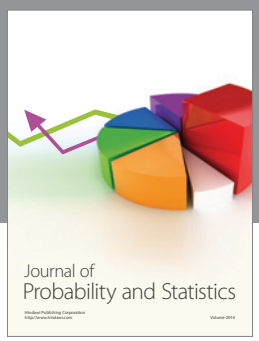

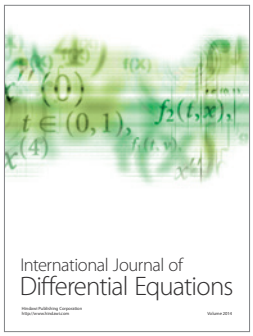

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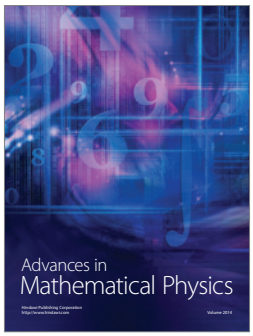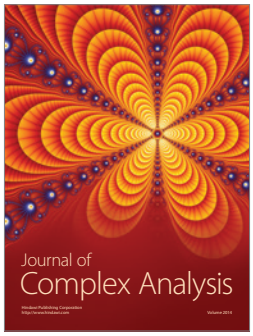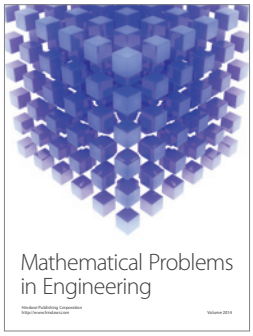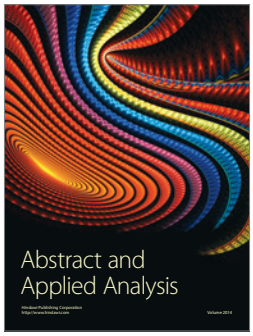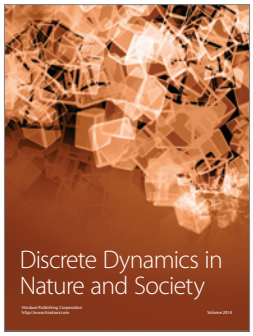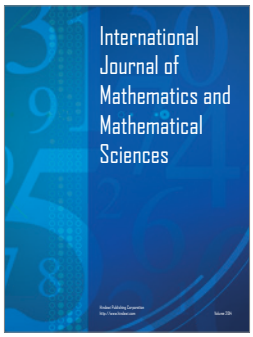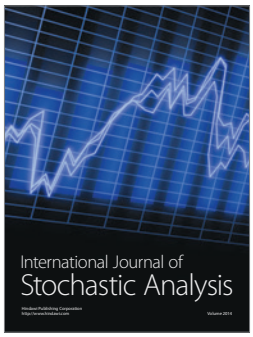
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