# Common Fixed Point for Generalized Bose-Mukherjee-Type Fuzzy Mappings 

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We prove a common fixed point theorem for a pair of generalized Bose-Mukherjee-type fuzzy mappings in a complete metric space. An example is also provided to support the main result presented herein.

## 1. Introduction and Preliminaries

In many scientific and engineering applications, the fuzzy set concept plays an important role. The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, the study of fixed point theorems in fuzzy mathematics had been instigated by Weiss [2] and Butnariu [3]. Heilpern [4] introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [5]. Afterwards, several fixed point theorems for fuzzy contractive mappings have appeared in the literature (see [6-13]). Particularly, Vijayaraju and Marudai [6] studied a fixed point result for fuzzy (multivalued) mappings $T: X \rightarrow \mathscr{F}(X)$ in a metric space $X$. This result [6] is significant as it does not require the condition of approximate quantity for $T(x)$ and linearity for $X$. However, Azam and Arshad [7] pointed out that its proof [6] is incorrect and incomplete and presented the right version of this result. In fact, although there exist mistakes in the proof of Theorem 3.1 in [6], its conclusion is correct.

The aim of this work is to establish a common fixed point theorem for a pair of generalized Bose-Mukherjee-type fuzzy mappings in a complete metric space. Also, we give an example to show the validity of our result and by which indicate that our result improves and extends several known results in $[6,7,14]$.

Let $X$ and $Y$ be nonempty sets. A multivalued mapping $T$ from $X$ to $Y$, denoted by $T: X \rightarrow 2^{Y}$, is defined to
be a function that assigns to each element of $X$ a nonempty subset of $Y$. Fixed points of the multivalued mapping $T$ : $X \rightarrow 2^{X}$ will be the points $x \in X$ such that $x \in T(x)$.

Let $(X, d)$ be a metric space and let $\mathscr{C} \mathscr{B}(X)$ denote the set of all nonempty closed and bounded subsets of $X$. For $A, B \in$ $\mathscr{C} \mathscr{B}(X)$, define

$$
\begin{equation*}
H(A, B)=\max \left\{\sup _{x \in A} d(x, B), \sup _{y \in B} d(A, y)\right\} \tag{1}
\end{equation*}
$$

where $d(x, A)=\inf _{y \in A} d(x, y)$.
A fuzzy set in $X$ is a function with domain $X$ and values in $[0,1]$. If $A$ is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of $x$ in $A$. The $\alpha$-level set of $A$ is denoted by $[A]_{\alpha}$ and is defined as follows:

$$
\begin{gather*}
{[A]_{\alpha}=\{x: A(x) \geq \alpha\} \quad \text { if } \alpha \in(0,1],} \\
{[A]_{0}=\overline{\{x: A(x)>0\}} .} \tag{2}
\end{gather*}
$$

Here, $\bar{B}$ denotes the closure of the set $B$. Let $\mathscr{F}(X)$ be the collection of all fuzzy sets in a metric space $X$. For $A, B \in$ $\mathscr{F}(X), A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

A mapping $T$ from $X$ to $\mathscr{F}(Y)$ is called a fuzzy mapping if for each $x \in X, T(x)$ (sometimes denoted by $T x$ ) is a fuzzy set on $Y$ and $T x(y)$ denotes the degree of membership of $y$ in $T x$. Let $\mathscr{W}(X)$ denote the set of all fuzzy sets on $X$ such that each of its $\alpha$-level is a nonempty closed bounded subset of $X$.

The following lemmas are needed in the sequel.
Lemma 1 (Nadler [5]). Let $(X, d)$ be a metric space and $A, B \in$ $\mathscr{C} \mathscr{B}(X)$; then
(1) for each $x \in A, d(x, B) \leq H(A, B)$,
(2) for each $y \in X, d(x, A) \leq d(x, y)+d(y, A)$.

Lemma 2 (Nadler [5]). Let $(X, d)$ be a metric space and $A, B \in$ $\mathscr{C} \mathscr{B}(X)$; then for each $x \in A$ and $\varepsilon>0$ there exists an element $y \in B$ such that $d(x, y) \leq H(A, B)+\varepsilon$.

## 2. Main Results

Lemma 3. Let $A_{1}, A_{2}, A_{3}, A_{4}$, and $A_{5}$ be five nonnegative real numbers with $A_{1}+A_{2}+A_{3}+A_{4}+A_{5}=1, A_{5}>0$, and either $A_{1}>A_{2}, A_{3}>A_{4}$ or $A_{1}<A_{2}, A_{3}<A_{4}$. Let $a=\left(A_{1}+A_{3}+A_{5}\right) /\left(A_{1}+A_{4}+A_{5}\right), b=\left(A_{2}+A_{4}+A_{5}\right) /\left(A_{2}+\right.$ $\left.A_{3}+A_{5}\right)$; then $0<a b<1$.

Proof. If $A_{1}>A_{2}, A_{3}>A_{4}$, then $A_{1}>0, A_{3}>0$. Note that $A_{5}>0$; we have $a=\left(A_{1}+A_{3}+A_{5}\right) /\left(A_{1}+A_{4}+A_{5}\right)>0, b=$ $\left(A_{2}+A_{4}+A_{5}\right) /\left(A_{2}+A_{3}+A_{5}\right)>0$; that is, $a b>0$. Moreover, it is evident that $\left(A_{1}-A_{2}\right)\left(A_{3}-A_{4}\right)>0 \Rightarrow A_{1} A_{3}+A_{2} A_{4}>$ $A_{1} A_{4}+A_{2} A_{3}$, which further implies that

$$
\begin{align*}
\left(A_{1}\right. & \left.+A_{4}+A_{5}\right)\left(A_{2}+A_{3}+A_{5}\right) \\
= & A_{1} A_{2}+A_{1} A_{3}+A_{1} A_{5}+A_{4} A_{2}+A_{4} A_{3} \\
& +A_{4} A_{5}+A_{5} A_{2}+A_{5} A_{3}+A_{5} A_{5} \\
> & A_{1} A_{2}+A_{1} A_{4}+A_{1} A_{5}+A_{3} A_{2}+A_{3} A_{4}  \tag{3}\\
& +A_{3} A_{5}+A_{5} A_{2}+A_{5} A_{4}+A_{5} A_{5} \\
= & \left(A_{1}+A_{3}+A_{5}\right)\left(A_{2}+A_{4}+A_{5}\right) .
\end{align*}
$$

That is, $a b=\left(A_{1}+A_{3}+A_{5}\right)\left(A_{2}+A_{4}+A_{5}\right) /\left(A_{1}+A_{4}+\right.$ $\left.A_{5}\right)\left(A_{2}+A_{3}+A_{5}\right)<1$.

Similarly, if $A_{1}<A_{2}, A_{3}<A_{4}$, then $0<a b<1$ holds.

Theorem 4. Let $(X, d)$ be a complete metric space. Let $S, T$ : $X \rightarrow \mathscr{F}(X)$ be two generalized Bose-Mukherjee-type fuzzy mappings. Suppose that, for each $x \in X$, there exists $\alpha(x) \in$ $(0,1]$ such that $[S x]_{\alpha(x)}$ and $[T x]_{\alpha(x)}$ are nonempty closed bounded subsets of $X$ and

$$
\begin{align*}
& H\left([S X]_{\alpha(x),},[T y]_{\alpha(y)}\right) \\
& \quad \leq A_{1} d\left(x,[S x]_{\alpha(x)}\right)+A_{2} d\left(y,[T y]_{\alpha(y)}\right)  \tag{4}\\
& \quad+A_{3} d\left(x,[T y]_{\alpha(y)}\right)+A_{4} d\left(y,[S x]_{\alpha(x)}\right) \\
& \quad+A_{5} d(x, y)
\end{align*}
$$

for all $x, y \in X$, where $A_{1}, A_{2}, A_{3}, A_{4}$, and $A_{5}$ are five nonnegative real numbers with $\sum_{i=1}^{5} A_{i}=1, A_{5}>0$ and either $A_{1}>A_{2}, A_{3}>A_{4}$ or $A_{1}<A_{2}, A_{3}<A_{4}$. Then there exists $z \in X$ such that $z \in[S z]_{\alpha(z)} \cap[T z]_{\alpha(z)}$.

Proof. Let $A_{1}>A_{2}, A_{3}>A_{4}$ and $a=\left(A_{1}+A_{3}+A_{5}\right) /\left(A_{1}+\right.$ $\left.A_{4}+A_{5}\right), b=\left(A_{2}+A_{4}+A_{5}\right) /\left(A_{2}+A_{3}+A_{5}\right)$. By Lemma 3, we know that $a>0, b>0$ and $0<a b<1$. Choosing $x_{0} \in X$, by hypotheses, there exists $\alpha\left(x_{0}\right) \in(0,1]$ such that $\left[S x_{0}\right]_{\alpha\left(x_{0}\right)}$ is nonempty closed bounded subset of $X$. For convenience, we denote $\alpha\left(x_{0}\right)$ by $\alpha_{1}$. Let $x_{1} \in\left[S x_{0}\right]_{\alpha_{1}}$; for this $x_{1}$ there exists $\alpha_{2} \in(0,1]$ such that $\left[T x_{1}\right]_{\alpha_{2}}$ is nonempty closed bounded subset of $X$. Since $A_{1}+A_{4}+A_{5}>0$, by Lemma 2, there exists $x_{2} \in\left[T x_{1}\right]_{\alpha_{2}}$ such that

$$
\begin{equation*}
d\left(x_{1}, x_{2}\right) \leq H\left(\left[S x_{0}\right]_{\alpha_{1}},\left[T x_{1}\right]_{\alpha_{2}}\right)+\left(A_{1}+A_{4}+A_{5}\right) \tag{5}
\end{equation*}
$$

Since $A_{2}+A_{3}+A_{5}>0$, by the same argument, we can find $\alpha_{3} \in(0,1]$ and $x_{3} \in\left[S x_{2}\right]_{\alpha_{3}}$ such that

$$
\begin{equation*}
d\left(x_{2}, x_{3}\right) \leq H\left(\left[S x_{2}\right]_{\alpha_{3}},\left[T x_{1}\right]_{\alpha_{2}}\right)+a b\left(A_{2}+A_{3}+A_{5}\right) \tag{6}
\end{equation*}
$$

By induction, we produce a sequence $\left\{x_{n}\right\}$ of points of $X$,

$$
\begin{equation*}
x_{2 k+1} \in\left[S x_{2 k}\right]_{\alpha_{2 k+1}}, \quad x_{2 k+2} \in\left[T x_{2 k+1}\right]_{\alpha_{2 k+2}}, \quad k=0,1,2, \ldots, \tag{7}
\end{equation*}
$$

such that

$$
\begin{align*}
d\left(x_{2 k+1}, x_{2 k+2}\right) \leq & H\left(\left[S x_{2 k}\right]_{\alpha_{2 k+1}},\left[T x_{2 k+1}\right]_{\alpha_{2 k+2}}\right) \\
& +(a b)^{k}\left(A_{1}+A_{4}+A_{5}\right) \\
d\left(x_{2 k+2}, x_{2 k+3}\right) \leq & \left.H\left(\left[S x_{2 k+2}\right]_{\alpha_{2 k+3}}, T T x_{2 k+1}\right]_{\alpha_{2 k+2}}\right)  \tag{8}\\
& +(a b)^{k+1}\left(A_{2}+A_{3}+A_{5}\right)
\end{align*}
$$

For $k=0,1,2, \ldots$, applying (4), we obtain

$$
\begin{align*}
& d\left(x_{2 k+1}, x_{2 k+2}\right) \\
& \leq H\left(\left[S x_{2 k}\right]_{\alpha_{2 k+1}},\left[T x_{2 k+1}\right]_{\alpha_{2 k+2}}\right)+(a b)^{k}\left(A_{1}+A_{4}+A_{5}\right) \\
& \leq A_{1} d\left(x_{2 k},\left[S x_{2 k}\right]_{\alpha_{2 k+1}}\right)+A_{2} d\left(x_{2 k+1},\left[T x_{2 k+1}\right]_{\alpha_{2 k+2}}\right) \\
&+A_{3} d\left(x_{2 k},\left[T x_{2 k+1}\right]_{\alpha_{2 k+2}}\right)+A_{4} d\left(x_{2 k+1},\left[S x_{2 k}\right]_{\alpha_{2 k+1}}\right) \\
&+A_{5} d\left(x_{2 k}, x_{2 k+1}\right)+(a b)^{k}\left(A_{1}+A_{4}+A_{5}\right) \\
& \leq A_{1} d\left(x_{2 k}, x_{2 k+1}\right)+A_{2} d\left(x_{2 k+1}, x_{2 k+2}\right)+A_{3} d\left(x_{2 k}, x_{2 k+2}\right) \\
&+A_{4} d\left(x_{2 k+1}, x_{2 k+1}\right)+A_{5} d\left(x_{2 k}, x_{2 k+1}\right) \\
&+(a b)^{k}\left(A_{1}+A_{4}+A_{5}\right) \\
& \leq A_{1} d\left(x_{2 k}, x_{2 k+1}\right)+A_{2} d\left(x_{2 k+1}, x_{2 k+2}\right) \\
&+A_{3} d\left(x_{2 k}, x_{2 k+1}\right)+A_{3} d\left(x_{2 k+1}, x_{2 k+2}\right) \\
&+A_{5} d\left(x_{2 k}, x_{2 k+1}\right)+(a b)^{k}\left(A_{1}+A_{4}+A_{5}\right) . \tag{9}
\end{align*}
$$

It implies that

$$
\begin{align*}
d\left(x_{2 k+1}, x_{2 k+2}\right) \leq & \frac{A_{1}+A_{3}+A_{5}}{1-A_{2}-A_{3}} d\left(x_{2 k}, x_{2 k+1}\right) \\
& +(a b)^{k} \frac{A_{1}+A_{4}+A_{5}}{1-A_{2}-A_{3}} \tag{10}
\end{align*}
$$

Note that $\sum_{i=1}^{5} A_{i}=1$; we have

$$
\begin{align*}
d\left(x_{2 k+1}, x_{2 k+2}\right) \leq & \frac{A_{1}+A_{3}+A_{5}}{A_{1}+A_{4}+A_{5}} d\left(x_{2 k}, x_{2 k+1}\right) \\
& +(a b)^{k} \frac{A_{1}+A_{4}+A_{5}}{A_{1}+A_{4}+A_{5}}  \tag{11}\\
= & a d\left(x_{2 k}, x_{2 k+1}\right)+(a b)^{k}
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
d\left(x_{2 k+2}, x_{2 k+3}\right) \leq b d\left(x_{2 k+1}, x_{2 k+2}\right)+(a b)^{k+1} \tag{12}
\end{equation*}
$$

Using inductive method, for $k=0,1,2, \ldots$, by (11) and (12), we can obtain

$$
\begin{align*}
d & \left(x_{2 k+1}, x_{2 k+2}\right) \\
& \leq a d\left(x_{2 k}, x_{2 k+1}\right)+(a b)^{k} \\
& \leq a b d\left(x_{2 k-1}, x_{2 k}\right)+a(a b)^{k}+(a b)^{k} \\
& \leq a b a d\left(x_{2 k-2}, x_{2 k-1}\right)+(a b)^{k}+a(a b)^{k}+(a b)^{k} \\
& \leq(a b)^{2} d\left(x_{2 k-3}, x_{2 k-2}\right)+2(a+1)(a b)^{k} \\
& \leq \cdots \leq(a b)^{k} d\left(x_{1}, x_{2}\right)+k(a+1)(a b)^{k} \\
& \leq(a b)^{k} a d\left(x_{0}, x_{1}\right)+(a b)^{k}+k(a+1)(a b)^{k} \\
& \leq(a b)^{k} d\left(x_{0}, x_{1}\right)+(k+1)(a+1)(a b)^{k}, \\
d & \left(x_{2 k+2}, x_{2 k+3}\right) \\
& \leq b\left((a b)^{k} a d\left(x_{0}, x_{1}\right)+(k+1)(a+1)(a b)^{k}\right)+(a b)^{k+1} \\
& =(a b)^{k+1} d\left(x_{0}, x_{1}\right)+b(k+1)(a+1)(a b)^{k}+(a b)^{k+1} \\
& =(a b)^{k+1} d\left(x_{0}, x_{1}\right)+(k+1)\left(\frac{a+1}{a}\right)(a b)^{k+1}+(a b)^{k+1} \\
& \leq(a b)^{k+1} d\left(x_{0}, x_{1}\right)+(k+2)\left(\frac{a+1}{a}\right)(a b)^{k+1} . \tag{13}
\end{align*}
$$

Next, we prove that the sequence $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$. For $k<p$, we have

$$
\begin{aligned}
& d\left(x_{2 k+1}, x_{2 p+1}\right) \\
& \quad \leq d\left(x_{2 k+1}, x_{2 k+2}\right)+\cdots+d\left(x_{2 p}, x_{2 p+1}\right) \\
& \quad \leq\left(\sum_{i=k}^{p-1}(a b)^{i}+\sum_{i=k+1}^{p}(a b)^{i}\right) d\left(x_{0}, x_{1}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\left((a+1) \sum_{i=k}^{p-1}(i+1)(a b)^{i}+\left(\frac{a+1}{a}\right) \sum_{i=k+1}^{p}(i+1)(a b)^{i}\right) \\
\leq & \left(\frac{(a b)^{k}}{1-a b}+\frac{(a b)^{k+1}}{1-a b}\right) d\left(x_{0}, x_{1}\right) \\
& +\left((a+1) \sum_{i=k}^{\infty}(i+1)(a b)^{i}+\left(\frac{a+1}{a}\right) \sum_{i=k+1}^{\infty}(i+1)(a b)^{i}\right) \\
\leq & 2 \frac{(a b)^{k}}{1-a b} d\left(x_{0}, x_{1}\right)+2 M \sum_{i=k}^{\infty}(i+1)(a b)^{i}, \tag{14}
\end{align*}
$$

where $M=\max \{a+1,(a+1) / a\}$. By the similar reasoning process, we can obtain

$$
\begin{align*}
& d\left(x_{2 k}, x_{2 p+1}\right) \leq 2 \frac{(a b)^{k}}{1-a b} d\left(x_{0}, x_{1}\right)+2 M \sum_{i=k}^{\infty}(i+1)(a b)^{i}, \\
& d\left(x_{2 k}, x_{2 p}\right) \leq 2 \frac{(a b)^{k}}{1-a b} d\left(x_{0}, x_{1}\right)+2 M \sum_{i=k}^{\infty}(i+1)(a b)^{i}, \\
& d\left(x_{2 k+1}, x_{2 p}\right) \leq 2 \frac{(a b)^{k}}{1-a b} d\left(x_{0}, x_{1}\right)+2 M \sum_{i=k}^{\infty}(i+1)(a b)^{i} . \tag{15}
\end{align*}
$$

Then, there exists $k$ with $(n-1) / 2 \leq k \leq n / 2$, for any $0<n<$ $m$, such that

$$
\begin{equation*}
d\left(x_{m}, x_{n}\right) \leq 2 \frac{(a b)^{k}}{1-a b} d\left(x_{0}, x_{1}\right)+2 M \sum_{i=k}^{\infty}(i+1)(a b)^{i} . \tag{16}
\end{equation*}
$$

Since $0<a b<1$, it follows from Cauchy's root test that $\Sigma(n+$ 1) $(a b)^{n}$ is convergent and hence $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$. Since $X$ is a complete metric space, then there exists $z \in X$ such that $x_{n} \rightarrow z$ as $n \rightarrow \infty$. Then, by (4) and Lemma 1, we have

$$
\begin{align*}
d(z, & {\left.[T(z)]_{\alpha(z)}\right) } \\
\leq & d\left(z, x_{2 n+1}\right)+d\left(x_{2 n+1},[T(z)]_{\alpha(z)}\right) \\
\leq & d\left(z, x_{2 n+1}\right)+H\left(\left[S x_{2 n}\right]_{\alpha_{2 n+1}},[T(z)]_{\alpha(z)}\right) \\
\leq & d\left(z, x_{2 n+1}\right)+A_{1} d\left(x_{2 n},\left[S x_{2 n}\right]_{\alpha_{2 n+1}}\right) \\
& +A_{2} d\left(z,[T(z)]_{\alpha(z)}\right)+A_{3} d\left(x_{2 n},[T(z)]_{\alpha(z)}\right) \\
& +A_{4} d\left(z,\left[S x_{2 n}\right]_{\alpha_{2 n+1}}\right)+A_{5} d\left(x_{2 n}, z\right) \\
\leq & d\left(z, x_{2 n+1}\right)+A_{1} d\left(x_{2 n}, x_{2 n+1}\right)+A_{2} d\left(z,[T(z)]_{\alpha(z)}\right) \\
& +A_{3} d\left(x_{2 n}, z\right)+A_{3} d\left(z,[T(z)]_{\alpha(z)}\right)+A_{4} d\left(z, x_{2 n+1}\right) \\
& +A_{5} d\left(x_{2 n}, z\right) . \tag{17}
\end{align*}
$$

Therefore,

$$
\begin{align*}
(1- & \left.A_{2}-A_{3}\right) d\left(z,[T(z)]_{\alpha(z)}\right) \\
\leq & d\left(z, x_{2 n+1}\right)+A_{1} d\left(x_{2 n}, x_{2 n+1}\right) \\
& +A_{3} d\left(x_{2 n}, z\right)+A_{4} d\left(z, x_{2 n+1}\right)  \tag{18}\\
& +A_{5} d\left(x_{2 n}, z\right)
\end{align*}
$$

and hence $d\left(z,[T(z)]_{\alpha(z)}\right) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $z \in$ $[T z]_{\alpha(z)}$.

Similarly, we can prove that $z \in[S z]_{\alpha(z)}$. Hence, $z \in$ $[S z]_{\alpha(z)} \cap[T z]_{\alpha(z)}$.

If $A_{1}<A_{2}, A_{3}<A_{4}$, by the same argument, we can prove that the conclusion holds.

Corollary 5 (Vijayaraju and Marudai [6]). Let (X,d) be a complete metric space. Let $S, T: X \rightarrow \mathscr{F}(X)$ be two fuzzy mappings. Suppose that, for each $x \in X$, there exists $\alpha(x) \in(0,1]$ such that $[S x]_{\alpha(x)}$ and $[T x]_{\alpha(x)}$ are nonempty closed bounded subsets of $X$ and

$$
\begin{align*}
& H\left([S x]_{\alpha(x)},[T y]_{\alpha(y)}\right) \\
& \quad \leq a_{1} d\left(x,[S x]_{\alpha(x)}\right)+a_{2} d\left(y,[T y]_{\alpha(y)}\right) \\
& \quad+a_{3} d\left(x,[T y]_{\alpha(y)}\right)+a_{4} d\left(y,[S x]_{\alpha(x)}\right)+a_{5} d(x, y) \tag{19}
\end{align*}
$$

for all $x, y \in X$, where $a_{1}, a_{2}, a_{3}, a_{4}$, and $a_{5}$ are nonnegative real numbers with $\sum_{i=1}^{5} a_{i}<1$ and either $a_{1}=a_{2}$ or $a_{3}=a_{4}$. Then, there exists $z \in X$ such that $z \in[S z]_{\alpha(z)} \cap[T z]_{\alpha(z)}$.

Proof. If $a_{1}=a_{2}, a_{3}>a_{4}$, by $\sum_{i=1}^{5} a_{i}<1$, we can take $2 \delta=1-$ $\left(\sum_{i=1}^{5} a_{i}\right)>0$. And let $A_{1}=a_{1}+\delta, A_{2}=a_{2}, A_{3}=a_{3}, A_{4}=a_{4}$, and $A_{5}=a_{5}+\delta$; then we have $\Sigma_{i=1}^{5} A_{i}=1, A_{1}>A_{2}, A_{3}>$ $A_{4}, A_{5}>0$, and for all $x, y \in X$,

$$
\begin{align*}
& H\left([S x]_{\alpha(x)},[T y]_{\alpha(y)}\right) \\
& \quad \leq A_{1} d\left(x,[S x]_{\alpha(x)}\right)+A_{2} d\left(y,[T y]_{\alpha(y)}\right) \\
& \quad+A_{3} d\left(x,[T y]_{\alpha(y)}\right)+A_{4} d\left(y,[S x]_{\alpha(x)}\right)+A_{5} d(x, y) \tag{20}
\end{align*}
$$

which implies the conditions of Theorem 4 are satisfied. Similarly, we can prove that some cases of $a_{1}=a_{2}, a_{3}<a_{4}$ or $a_{1}>a_{2}, a_{3}=a_{4}$ or $a_{1}<a_{2}, a_{3}=a_{4}$, respectively. Therefore, by Theorem 4, the corollary is proved.

Remark 6. Corollary 5 shows that, although there exist mistakes in the proof of Theorem 3.1 in [6], its conclusion is correct. Moreover, Corollary 5 also shows that Theorem 4 in [7] is not the right version of Theorem 3.1 in [6] but the special case of Theorem 3.1 in [6]. In addition, we give a correct proof of Theorem 3.1 in [6].

Corollary 7 (Azam and Arshad [7]). Let $(X, d)$ be a complete metric space. Let $S, T: X \rightarrow \mathscr{F}(X)$ be two fuzzy mappings.

Suppose that, for each $x \in X$, there exists $\alpha(x) \in(0,1]$ such that $[S x]_{\alpha(x)}$ and $[T x]_{\alpha(x)}$ are nonempty closed bounded subsets of $X$ and

$$
\begin{align*}
& H\left([S x]_{\alpha(x)},[T y]_{\alpha(y)}\right) \\
& \quad \leq a_{1} d\left(x,[S x]_{\alpha(x)}\right)+a_{2} d\left(y,[T y]_{\alpha(y)}\right) \\
& \quad+a_{3}\left[d\left(x,[T y]_{\alpha(y)}\right)+d\left(y,[S x]_{\alpha(x)}\right)\right]+a_{4} d(x, y) \tag{21}
\end{align*}
$$

for all $x, y \in X$, where $a_{1}, a_{2}, a_{3}$, and $a_{4}$ are nonnegative real numbers with $a_{1}+a_{2}+2 a_{3}+a_{4}<1$. Then, there exists $z \in X$ such that $z \in[S z]_{\alpha(z)} \cap[T z]_{\alpha(z)}$.

Theorem 8. Let $(X, d)$ be a complete metric space. Let $S, T$ : $X \rightarrow \mathscr{C} \mathscr{B}(X)$ be two generalized Bose-Mukherjee-type multivalued mappings. Suppose that, for all $x, y \in X$,

$$
\begin{align*}
H(S x, T y) \leq & A_{1} d(x, S x)+A_{2} d(y, T y) \\
& +A_{3} d(x, T y)+A_{4} d(y, S x)  \tag{22}\\
& +A_{5} d(x, y),
\end{align*}
$$

where $A_{1}, A_{2}, A_{3}, A_{4}$, and $A_{5}$ are five nonnegative real numbers with $\sum_{i=1}^{5} A_{i}=1, A_{5}>0$ and either $A_{1}>A_{2}, A_{3}>$ $A_{4}$ or $A_{1}<A_{2}, A_{3}<A_{4}$. Then, there exists $z \in X$ such that $z \in S z \cap T z$.

Proof. Let the fuzzy mappings $S, T: X \rightarrow \mathscr{F}(X)$ be defined as $S(x)=\chi_{S(x)}$ and $T(x)=\chi_{T(x)}$, where $\chi_{A}$ is the characteristic function on any subset $A$ of $X$. Using the facts $[S x]_{\alpha(x)}=S(x)$ and $[T x]_{\alpha(x)}=T(x)$ for any $\alpha(x) \in$ $(0,1]$, it is evident that $S$ and $T$ satisfy the condition of Theorem 4.

Similarly, as the proof of Corollary 5, from Theorem 8, we can obtain that common fixed point theorem for Bose-Mukherjee-type multivalued mappings in [14].

Corollary 9 (Bose and Mukherjee [14]). Let (X,d) be a complete metric space. Let $S, T: X \rightarrow \mathscr{C} \mathscr{B}(X)$ be two multivalued mappings. Suppose that, for all $x, y \in X$,

$$
\begin{align*}
H(S x, T y) \leq & a_{1} d(x, S x)+a_{2} d(y, T y) \\
& +a_{3} d(x, T y)+a_{4} d(y, S x)  \tag{23}\\
& +a_{5} d(x, y)
\end{align*}
$$

where $a_{1}, a_{2}, a_{3}, a_{4}$, and $a_{5}$ are nonnegative real numbers with $\sum_{i=1}^{5} a_{i}<1$ and either $a_{1}=a_{2}$ or $a_{3}=a_{4}$. Then, there exists $z \in X$ such that $z \in S z \cap T z$.

Example 10. Let $X=\{0,1,2\} ; d$ is an ordinary metric; then $(X, d)$ is a complete metric space. Define two fuzzy mappings $S, T: X \rightarrow \mathscr{F}(X)$ as follows:
$(S x)(z)=\left\{\begin{array}{ll}1, & \text { if } z=0, \\ 0, & \text { if } z=1 \text { or } 2,\end{array} \quad \forall x \in X ;\right.$
$(T 2)(z)=\left\{\begin{array}{ll}1, & \text { if } z=1, \\ 0, & \text { if } z=0 \text { or } 2,\end{array}\right.$ and for $y \in X \backslash\{2\}$,

$$
(T y)(z)= \begin{cases}1, & \text { if } z=0  \tag{24}\\ 0, & \text { if } z=1 \text { or } 2\end{cases}
$$

Then, we have

$$
\begin{gather*}
{[S x]_{1}=[S x]_{\alpha}=\{0\} \quad \forall x \in X, \alpha \in(0,1]} \\
{[T y]_{1}=[T y]_{\alpha}=\left\{\begin{array}{ll}
\{1\}, & \text { if } y=2, \\
\{0\}, & \text { if } y=0 \text { or } 1,
\end{array} \quad \forall \alpha \in(0,1] .\right.} \tag{25}
\end{gather*}
$$

Now we take $A_{1}=2 / 45, A_{2}=22 / 45, A_{3}=1 / 9, A_{4}=23 /$ 90, $A_{5}=1 / 10$; then we have $\sum_{i=1}^{5} A_{i}=1, A_{5}>0$ and $A_{1}<$ $A_{2}, A_{3}<A_{4}$.

Moreover, if $x \in X$ and $y=0$ or 1 , then, for all $\alpha \in(0,1]$,

$$
\begin{align*}
H\left([S x]_{\alpha},[T y]_{\alpha}\right)= & 0 \\
\leq & \frac{2}{45} d\left(x,[S x]_{\alpha}\right)+\frac{22}{45} d\left(y,[T y]_{\alpha}\right) \\
& +\frac{1}{9} d\left(x,[T y]_{\alpha}\right)+\frac{23}{90} d\left(y,[S x]_{\alpha}\right)  \tag{26}\\
& +\frac{1}{10} d(x, y)
\end{align*}
$$

If $x=0$ and $y=2$, then, for all $\alpha \in(0,1]$,

$$
\begin{align*}
H\left([S 0]_{\alpha},[T 2]_{\alpha}\right)= & 1 \\
< & \frac{2}{45} d(0,\{0\})+\frac{22}{45} d(2,\{1\}) \\
& +\frac{1}{9} d(0,\{1\})+\frac{23}{90} d(2,\{0\})+\frac{1}{10} d(0,2) . \tag{27}
\end{align*}
$$

If $x=1$ and $y=2$, then, for all $\alpha \in(0,1]$,

$$
\begin{align*}
H\left([S 1]_{\alpha},[T 2]_{\alpha}\right)= & 1 \\
< & \frac{2}{45} d(1,\{0\})+\frac{22}{45} d(2,\{1\}) \\
& +\frac{1}{9} d(1,\{1\})+\frac{23}{90} d(2,\{0\})+\frac{1}{10} d(1,2) . \tag{28}
\end{align*}
$$

If $x=2$ and $y=2$, then, for all $\alpha \in(0,1]$,

$$
\begin{align*}
H\left([S 2]_{\alpha},[T 2]_{\alpha}\right)= & 1 \\
< & \frac{2}{45} d(2,\{0\})+\frac{22}{45} d(2,\{1\}) \\
& +\frac{1}{9} d(2,\{1\})+\frac{23}{90} d(2,\{0\})+\frac{1}{10} d(2,2) . \tag{29}
\end{align*}
$$

Hence, the conditions of Theorem 4 are satisfied, and there exists $0 \in X$ such that $0 \in\{0\}=[S 0]_{\alpha} \cap[T 0]_{\alpha}$. But, for any nonnegative real numbers $a_{1}, a_{2}, a_{3}, a_{4}$, and $a_{5}$ with $a_{1}+a_{2}+$ $2 a_{3}+a_{5}<1$, we have

$$
\begin{align*}
H\left([S 1]_{\alpha},[T 2]_{\alpha}\right)= & 1 \\
> & a_{1} d(1,\{0\})+a_{2} d(2,\{1\}) \\
& +a_{3}[d(1,\{1\})+d(2,\{0\})]+a_{5} d(1,2) \tag{30}
\end{align*}
$$

for all $\alpha \in(0,1]$. Thus, $S, T$ cannot satisfy the general contractive condition $a_{1}+a_{2}+2 a_{3}+a_{5}<1$.

## 3. Conclusion

The aim of this work is to establish a common fixed point theorem for a pair of generalized Bose-Mukherjee-type fuzzy mappings in a complete metric space. Also, we give an example to show the validity of our result and by which indicate that our result improves and extends several known results in $[6,7,14]$. Moreover, we give a correct proof of Theorem 3.1 in [6] and point out the conclusion of Theorem 4.1 in [7] is the special case of Theorem 3.1 in [6]. Finally, we hope that this theory would provide a mathematical background to the ongoing work in the problems of scientific and engineering applications.

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