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Research Article

Mixed Problem with an Integral Two-Space-Variables Condition for a Class of Hyperbolic Equations

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This paper is devoted to the proof of the existence and uniqueness of the classical solution of mixed problems which combine Neumann condition and integral two-space-variables condition for a class of hyperbolic equations. The proof is based on a priori estimate "energy inequality" and the density of the range of the operator generated by the problem considered.

1. Introduction

The integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow, and population dynamics.

Cannon was the first who drew attention to these problems with an integral one-space-variable condition [1], and their importance has been pointed out by Samarskii [2]. The existence and uniqueness of the classical solution of mixed problem combining a Dirichlet and integral condition for the equation of heat demonstrated by cannon [1] using the potential method.

Always using the potential method, Kamynin established in [3] the existence and uniqueness of the classical solution of a similar problem with a more general representation.

Subsequently, more works related to these problems with an integral one-space-variable have been published, among them, we cite the work of Benouar and Yurchuk [4], Cannon and Van Der Hoek [5, 6], Cannon-Esteva-Van Der Hoek [7], Ionkin [8], Jumarhon and McKee [9], Kartynnik [10], Lin [11], Shi [12] and Yurchuk [13]. In these works, mixed problems related to one-dimensional parabolic equations of second order combining a local condition and an integral condition was discussed. Also, by referring to the articles of Bouziani [14–16] and Bouziani and Benouar [17–19], the authors have studied mixed problems with integral conditions for some partial differential equations, specially hyperbolic equation with integral condition which has been investigated in Bouziani [20].

The present paper is devoted to the study of problems with a boundary integral two-space-variables condition for second-order hyperbolic equation.

2. Setting of the Problem

In the rectangle $\Omega = (0,1) \times (0,T)$, with $T < \infty$, we consider the hyperbolic equation:

$$\mathscr{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \tag{1}$$

where the coefficient $a(x,t)$ is a real-valued function belonging to $C^1(\overline{\Omega})$ such that

$$0 < c_0 \le a(x,t) \le c_1, \quad \frac{\partial a(x,t)}{\partial t} \le c_2. \tag{2}$$

in the rest of the paper, k, c_i, $i = 1, \dots, 12$, denote strictly positive constants.

we adjoin to (1) the initial conditions

$$\begin{aligned} \ell_1 u &= u(x,0) = \varphi(x), \quad x \in (0,1), \\ \ell_2 u &= \frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in (0,1), \end{aligned} \tag{3}$$

the Neumann condition

$$\frac{\partial u}{\partial x}(0,t)=0, \tag{4}$$

and the integral condition

$$\int_0^\alpha u(x,t)\,dx+\int_\beta^1 u(x,t)\,dx=0,\quad \alpha>0,\ \beta>0,\quad \alpha+\beta=1,\ t\in(0,T), \tag{5}$$

where φ and ψ are known functions.

We will assume that the function φ and f satisfy a compatibility conditions with (5), that is,

$$\int_0^\alpha \varphi(x)\,dx+\int_\beta^1 \varphi(x)\,dx=0, \tag{6}$$

$$\int_0^\alpha f(x,t)\,dx+\int_\beta^1 f(x,t)\,dx=0. \tag{7}$$

The presence of integral terms in boundary conditions can, in general, greatly complicate the application of standard functional or numerical techniques specially the integral two-space-variables condition. In order to avoid this difficulty, we introduce a technique to transfer this problem to another classically less complicated one which does not contain integral conditions. For that, we establish the following lemma.

Lemma 1. *Problem* (1)–(5) *is equivalent to the following problem* (*PR*):

$$(PR)=\begin{cases}\mathscr{L}u=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial u}{\partial x}\right)=f(x,t),\\ 0<c_0\le a(x,t)\le c_1,\quad \dfrac{\partial a(x,t)}{\partial t}\le c_2,\\ \ell_1 u=u(x,0)=\varphi(x),\quad x\in(0,1),\\ \ell_2 u=\dfrac{\partial u}{\partial t}(x,0)=\psi(x),\quad x\in(0,1),\\ \dfrac{\partial u}{\partial x}(0,t)=0,\qquad \dfrac{\partial u}{\partial x}(\alpha,t)=0,\\ \dfrac{\partial u}{\partial x}(\beta,t)=0,\qquad \dfrac{\partial u}{\partial x}(1,t)=0.\end{cases} \tag{8}$$

Proof. Let $u(x,t)$ be a solution of (1)–(5), we prove that

$$\frac{\partial u}{\partial x}(0,t)=0,\qquad \frac{\partial u}{\partial x}(\alpha,t)=0,\qquad \frac{\partial u}{\partial x}(\beta,t)=0,\qquad \frac{\partial u}{\partial x}(1,t)=0. \tag{9}$$

So, by integrating (1) with respect to x over $(0,\alpha)$ and $(\beta,1)$ and taking into account (6) and (7), we obtain

$$\int_0^\alpha \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)dx+\int_\beta^1 \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)dx=0, \tag{10}$$

and we get

$$a(\alpha,t)\frac{\partial u}{\partial x}(\alpha,t)-a(0,t)\frac{\partial u}{\partial x}(0,t)+a(1,t)\frac{\partial u}{\partial x}(1,t)-a(\beta,t)\frac{\partial u}{\partial x}(\beta,t)=0. \tag{11}$$

The integral condition (5) has nothing to do with the coefficient $a(x,t)$. Then, (11) imposes

$$\frac{\partial u}{\partial x}(\alpha,t)=\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=\frac{\partial u}{\partial x}(\beta,t)=0. \tag{12}$$

Let now $u(x,t)$ be a solution of (*PR*), then we are bound to prove that

$$\int_0^\alpha u(x,t)\,dx+\int_\beta^1 u(x,t)\,dx=0. \tag{13}$$

So, by integrating (1) with respect to x over $(0,\alpha)$ and taking into account that

$$\frac{\partial u}{\partial x}(0,t)=0,\qquad \frac{\partial u}{\partial x}(\alpha,t)=0, \tag{14}$$

we obtain

$$\frac{\partial^2}{\partial t^2}\int_0^\alpha u(x,t)\,dx=\int_0^\alpha f(x,t)\,dx, \tag{15}$$

and by integrating (1) with respect to x over $(\beta,1)$ and taking in consideration

$$\frac{\partial u}{\partial x}(\beta,t)=0,\qquad \frac{\partial u}{\partial x}(1,t)=0, \tag{16}$$

we obtain

$$\frac{\partial^2}{\partial t^2}\int_\beta^1 u(x,t)\,dx=\int_\beta^1 f(x,t)\,dx. \tag{17}$$

By combining the two preceding (15) and (17) and taking into account (7), we get

$$\int_0^\alpha u(x,t)\,dx+\int_\beta^1 u(x,t)\,dx=0. \tag{18}$$

□

3. A Priori Estimate and Its Consequences

In this paper, we prove the existence and the uniqueness of the solution of the problem (1)–(5) and of the operator equation

$$Lu=\mathscr{F}, \tag{19}$$

where $L=(\mathscr{L},\ell_1,\ell_2)$ with domain of difinition B consisting of functions $u\in L^2(\Omega)$ such that $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2\in L^2(\Omega)$, and u satisfies conditions (3) and (4); the operator L is considered from B to F, where B is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\|u\|_B^2=\int_0^1\left(\frac{\partial u}{\partial t}\right)^2dx+\int_0^1\left(\frac{\partial u}{\partial x}\right)^2dx, \tag{20}$$

and F is the Hilbert space consisting of all elements $\mathscr{F} = (f, \varphi, \psi)$ for which the norm

$$\|\mathscr{F}\|_F^2 = \int_0^1 f^2 dx\, dt + \int_0^1 \psi^2(x)\, dx + \int_0^1 \left(\frac{\partial \varphi}{\partial x}\right)^2 dx \quad (21)$$

is finite.

Theorem 2. *For any function $u \in B$, we have the inequality*

$$\|u\|_B \le c\|Lu\|_F, \quad (22)$$

where c is a positive constant independent of u.

Proof. Multiplying the (1) by the following Mu:

$$Mu = \begin{cases} Mu_1 = 2(1-x)\dfrac{\partial u}{\partial t}, & 0 \le x \le \alpha, \\ Mu_2 = 2\dfrac{\partial u}{\partial t}, & \alpha \le x \le \beta, \\ Mu_3 = 2(2-x)\dfrac{\partial u}{\partial t}, & \beta \le x \le 1 \end{cases} \quad (23)$$

and integrating over Ω^τ, where $\Omega^\tau = (0,1) \times (0,\tau)$,

(1) by integrating over $\Omega_\alpha^\tau = \Omega_\alpha = (0,\alpha) \times (0,\tau)$. Consequently,

$$\begin{aligned} &\int_{\Omega_\alpha} \mathscr{L}u \cdot Mu_1 dx\, dt \\ &\quad = 2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx\, dt \\ &\qquad - 2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) dx\, dt \\ &\quad = 2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t} f\, dx\, dt. \end{aligned} \quad (24)$$

Employing integration by parts in (24), and taking into account the boundary conditions in (PR), we obtain

$$\begin{aligned} &2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx\, dt \\ &\quad = \int_0^\alpha (1-x)\left(\frac{\partial u}{\partial t}\right)^2 dx - \int_0^\alpha (1-x)\psi^2(x)\, dx, \\ &\qquad - 2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) dx\, dt \\ &\quad = -\int_0^\tau \left(2(1-x)a(x,t)\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)\Bigg|_{x=0}^{x=\alpha} dt \\ &\qquad + 2\int_{\Omega_\alpha} (1-x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} dx\, dt \\ &\qquad - 2\int_{\Omega_\alpha} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\, dt \\ &\quad = -\int_0^\alpha (1-x)a(x,0)\left(\frac{\partial \varphi}{\partial x}\right)^2 dx \\ &\qquad + \int_0^\alpha (1-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\qquad - \int_{\Omega_\alpha} (1-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx\, dt \\ &\qquad - 2\int_{\Omega_\alpha} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\, dt. \end{aligned} \quad (25)$$

Using the Cauchy's inequality, we get

$$2\int_{\Omega_\alpha} (1-x)\frac{\partial u}{\partial t} f\, dx\, dt \le \int_{\Omega_\alpha} f^2 dx\, dt + \int_{\Omega_\alpha} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt. \quad (26)$$

Substituting (25) and (26) into (24), we obtain

$$\begin{aligned} &\int_0^\alpha (1-x)\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^\alpha (1-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le \int_{\Omega_\alpha} f^2 dx\, dt + \int_{\Omega_\alpha} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt \\ &\qquad + \int_0^\alpha (1-x)a(x,0)\left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_0^\alpha (1-x)\psi^2(x)\, dx \\ &\qquad + \int_{\Omega_\alpha} (1-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx\, dt \\ &\qquad + 2\int_{\Omega_\alpha} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\, dt. \end{aligned} \quad (27)$$

Then, using the Cauchy's inequality and according to conditions (2), we get

$$\begin{aligned} &\beta\int_0^\alpha \left(\frac{\partial u}{\partial t}\right)^2 dx + \beta c_0 \int_0^\alpha \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le \int_{\Omega_\alpha} f^2 dx\, dt + (c_1+1)\int_{\Omega_\alpha} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt + \int_0^\alpha \psi^2(x)\, dx \\ &\qquad + (c_1+c_2)\int_{\Omega_\alpha} \left(\frac{\partial u}{\partial x}\right)^2 dx\, dt + c_1\int_0^\alpha \left(\frac{\partial \varphi}{\partial x}\right)^2 dx, \end{aligned} \quad (28)$$

so, we obtain

$$\begin{aligned}&\int_0^\alpha \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^\alpha \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le c_3 \left(\int_{\Omega_\alpha} f^2 dx\, dt + \int_0^\alpha \psi^2(x)\, dx \right. \\ &\qquad \left. + \int_0^\alpha \left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_{\Omega_\alpha} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt \right) \\ &\quad + c_4 \int_{\Omega_\alpha} \left(\frac{\partial u}{\partial x}\right)^2 dx\, dt,\end{aligned} \tag{29}$$

where

$$\begin{aligned}c_3 &= \frac{c_1 + 1}{\min(\beta, \beta c_0)}, \\ c_4 &= \frac{c_1 + c_2}{\min(\beta, \beta c_0)}.\end{aligned} \tag{30}$$

By virtue of Lemma 7.1 in [21] and by using it twice, we find

$$\begin{aligned}&\int_0^\alpha \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^\alpha \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le c_5 \left(\int_{\Omega_\alpha} f^2 dx\, dt + \int_0^\alpha \psi^2(x)\, dx + \int_0^\alpha \left(\frac{\partial \varphi}{\partial x}\right)^2 dx \right),\end{aligned} \tag{31}$$

where

$$c_5 = c_3 \exp(c_4 T) \exp(c_3 T \exp(c_4 T)). \tag{32}$$

(2) by integrating over $\Omega^\tau_{\alpha,\beta} = \Omega_{\alpha,\beta} = (\alpha, \beta) \times (0, \tau)$. Consequently,

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}} \mathscr{L}u \cdot Mu_2 dx\, dt \\ &\quad = 2 \int_{\Omega_{\alpha,\beta}} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx\, dt \\ &\qquad - 2 \int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial t} dx\, dt \\ &\quad = 2 \int_{\Omega_{\alpha,\beta}} f \frac{\partial u}{\partial t} dx\, dt.\end{aligned} \tag{33}$$

Employing integration by parts in (33) and taking into account the boundary conditions in (PR), we obtain

$$\begin{aligned}&2 \int_{\Omega_{\alpha,\beta}} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx\, dt \\ &\quad = \int_\alpha^\beta \left(\frac{\partial u}{\partial t}\right)^2 dx - \int_\alpha^\beta \psi^2(x)\, dx, \\ &\quad - 2 \int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial u}{\partial x}\right) \left(\frac{\partial u}{\partial t}\right) dx\, dt \\ &\quad = - \int_0^\tau \left(2a(x,t) \frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right)\Big|_{x=\alpha}^{x=\beta} dt \\ &\qquad + 2 \int_{\Omega_{\alpha,\beta}} a(x,t) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx\, dt \\ &\quad = - \int_\alpha^\beta a(x,0) \left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_\alpha^\beta a(x,\tau) \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\qquad - \int_{\Omega_{\alpha,\beta}} \frac{\partial a(x,t)}{\partial t} \left(\frac{\partial u}{\partial x}\right)^2 dx\, dt.\end{aligned} \tag{34}$$

By virtue of the Cauchy's inequality, we obtain

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}} f \frac{\partial u}{\partial t} dx\, dt \\ &\quad \le 2 \int_{\Omega_{\alpha,\beta}} f^2 dx\, dt + \frac{1}{2} \int_{\Omega_{\alpha,\beta}} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt.\end{aligned} \tag{35}$$

Substituting (34) and (35) into (33) and according to conditions (2), we get

$$\begin{aligned}&\int_\alpha^\beta \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_\alpha^\beta \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le c_6 \left(\int_{\Omega_{\alpha,\beta}} f^2 dx\, dt + \int_\alpha^\beta \psi^2(x)\, dx \right. \\ &\qquad \left. + \int_\alpha^\beta \left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_{\Omega_{\alpha,\beta}} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt \right) \\ &\quad + c_7 \int_{\Omega_{\alpha,\beta}} \left(\frac{\partial u}{\partial x}\right)^2 dx\, dt,\end{aligned} \tag{36}$$

where

$$\begin{aligned}c_6 &= \frac{c_2}{\min(1, c_0)}, \\ c_7 &= \frac{\max(2, c_1)}{\min(1, c_0)}.\end{aligned} \tag{37}$$

By virtue of Lemma 7.1 in [21] and by using it twice, we find

$$\int_\alpha^\beta \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_\alpha^\beta \left(\frac{\partial u}{\partial x}\right)^2 dx \le c_8 \left(\int_{\Omega_{\alpha,\beta}} f^2 dx\,dt + \int_\alpha^\beta \psi^2(x)\,dx + \int_\alpha^\beta \left(\frac{\partial \varphi}{\partial x}\right)^2 dx\right), \tag{38}$$

where

$$c_8 = c_6 \exp(c_7 T) \exp(c_6 T \exp(c_7 T)). \tag{39}$$

(3) by integrating over $\Omega_\beta^\tau = \Omega_\beta = (\beta, 1) \times (0, \tau)$. Consequently,

$$\begin{aligned}\int_{\Omega_\beta} \mathscr{L}u \cdot Mu_3 dx\,dt &= 2\int_{\Omega_\beta} (2-x) \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx\,dt \\ &\quad - 2\int_{\Omega_\beta} (2-x) \frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) dx\,dt \\ &= 2\int_{\Omega_\beta} (2-x)\frac{\partial u}{\partial t} f\,dx\,dt.\end{aligned} \tag{40}$$

Employing integration by parts in (40) and taking into account the boundary conditions in (PR), we obtain

$$\begin{aligned}&2\int_{\Omega_\beta} (2-x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx\,dt \\ &\quad = \int_\beta^1 (2-x)\left(\frac{\partial u}{\partial t}\right)^2 dx - \int_\beta^1 (2-x)\psi^2(x)\,dx, \\ &-2\int_{\Omega_\beta} (2-x)\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) dx\,dt \\ &\quad = -2\int_0^\tau \left((2-x)\,a(x,t)\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)\Big|_{x=0}^{x=\alpha} dt \\ &\quad + 2\int_{\Omega_\beta} (2-x)\,a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} dx\,dt \\ &\quad - 2\int_{\Omega_\beta} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\,dt \\ &\quad = -\int_\beta^1 (2-x)\,a(x,0)\left(\frac{\partial \varphi}{\partial x}\right)^2 dx \\ &\quad + \int_\beta^1 (2-x)\,a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad - \int_{\Omega_\beta} (2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt \\ &\quad - 2\int_{\Omega_\beta} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\,dt.\end{aligned} \tag{41}$$

Using the Cauchy's inequality, we get

$$\begin{aligned}&2\int_{\Omega_\alpha} (2-x)\frac{\partial u}{\partial t} f\,dx\,dt \\ &\quad \le 2\int_{\Omega_\alpha} f^2 dx\,dt + 2\int_{\Omega_\alpha}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt.\end{aligned} \tag{42}$$

Substituting (41) and (42) into (40), we obtain

$$\begin{aligned}&\int_\beta^1 (2-x)\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_\beta^1 (2-x)\,a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le 2\int_{\Omega_\beta} f^2 dx\,dt + 2\int_{\Omega_\beta}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt \\ &\quad + \int_\beta^1 (2-x)\,a(x,0)\left(\frac{\partial \varphi}{\partial x}\right)^2 dx + \int_\beta^1 (2-x)\,\psi^2(x)\,dx \\ &\quad + \int_{\Omega_\beta} (2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt \\ &\quad + 2\int_{\Omega_\beta} a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx\,dt.\end{aligned} \tag{43}$$

Then, using the Cauchy's ε-inequality and according to conditions (2), we get

$$\begin{aligned}&\int_\beta^1 \left(\frac{\partial u}{\partial t}\right)^2 dx + c_0\int_\beta^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &\quad \le 2\int_{\Omega_\beta} f^2 dx\,dt + (c_1+2)\int_{\Omega_\beta}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt \\ &\quad + 2\int_\beta^1 \psi^2(x)\,dx + (c_1+2c_2)\int_{\Omega_\beta}\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt \\ &\quad + 2c_1\int_\beta^1 \left(\frac{\partial \varphi}{\partial x}\right)^2 dx,\end{aligned} \tag{44}$$

so, we obtain

$$\begin{aligned}\int_\beta^1\left(\frac{\partial u}{\partial t}\right)^2dx+\int_\beta^1\left(\frac{\partial u}{\partial x}\right)^2dx\\ \le c_9\left(\int_{\Omega_\beta}f^2dx\,dt+\int_\beta^1\psi^2(x)\,dx\right.\\ \left.+\int_\beta^1\left(\frac{\partial\varphi}{\partial x}\right)^2dx+\int_{\Omega_\beta}\left(\frac{\partial u}{\partial t}\right)^2dx\,dt\right)\\ +c_{10}\int_{\Omega_\beta}\left(\frac{\partial u}{\partial x}\right)^2dx\,dt,\end{aligned}\tag{45}$$

where

$$\begin{aligned}c_9&=\frac{2c_1+2}{\min(1,c_0)},\\ c_{10}&=\frac{c_1+2c_2}{\min(1,c_0)}.\end{aligned}\tag{46}$$

By virtue of Lemma 7.1 in [21] and by using it twice, we find

$$\begin{aligned}\int_\beta^1\left(\frac{\partial u}{\partial t}\right)^2dx+\int_\beta^1\left(\frac{\partial u}{\partial x}\right)^2dx\\ \le c_{11}\left(\int_{\Omega_\beta}f^2dx\,dt+\int_\beta^1\psi^2(x)\,dx+\int_\beta^1\left(\frac{\partial\varphi}{\partial x}\right)^2dx\right),\end{aligned}\tag{47}$$

where

$$c_{11}=c_9\exp(c_{10}T)\exp(c_9T\exp(c_{10}T)).\tag{48}$$

Combining inequalities (31), (38), and (47), we obtain

$$\begin{aligned}\int_0^1\left(\frac{\partial u}{\partial t}\right)^2dx+\int_0^1\left(\frac{\partial u}{\partial x}\right)^2dx\\ \le c_{12}\left(\int_\Omega f^2dx\,dt+\int_0^1\psi^2(x)\,dx+\int_0^1\left(\frac{\partial\varphi}{\partial x}\right)^2dx\right),\end{aligned}\tag{49}$$

where

$$c_{12}=\max(c_5,c_8,c_{11}).\tag{50}$$

The right-hand side of (49) is independent of τ; hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c=(c_{12})^{1/2}$. □

Corollary 3. *A solution of the problem* (1)–(5) *is unique if it exists and depends continuously on* $\mathcal{F}\in F$.

4. Solvability of the Problem

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u\in B$ and for arbitrary $\mathcal{F}=(f,\phi,\psi)\in F$.

Theorem 4. *Suppose the conditions of Theorem 2 are satisfied. Then, the problem* (1)–(5) *admits a unique solution* $u=L^{-1}\mathcal{F}$.

Proof. First, we prove that $R(L)$ is dense in F for the special case where $D(L)\equiv B$ is reduced to $D_0(L)$, where $D_0(L)=\{u,u\in D(L):\ell_1u=0,\ \ell_2u=0\}$. □

Proposition 5. *Let the conditions of Theorem 4 be satisfied. If for* $\omega\in L^2(\Omega)$ *and for all* $u\in D_0(L)$, *we have*

$$\int_\Omega\mathcal{L}u\cdot\omega\,dx\,dt=0\tag{51}$$

then, ω *vanishes almost everywhere in* Ω.

Proof. The scalar product of F is defined by

$$\begin{aligned}(Lu,\omega)_F=\int_\Omega\mathcal{L}u\cdot\omega\,dx\,dt+\int_0^1\left(\frac{\partial\ell_1u}{\partial x}\right)\left(\frac{\partial\omega_0}{\partial x}\right)dx\\ +\int_0^1(\ell_2u)(\omega_1)\,dx.\end{aligned}\tag{52}$$

the equality (51) can be written as follows:

$$\int_\Omega\frac{\partial^2u}{\partial t^2}\cdot\omega\,dx\,dt=\int_\Omega\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\cdot\omega\,dx\,dt.\tag{53}$$

If we put

$$u=\mathbb{J}_t\left(e^{kt}z\right)=\int_0^te^{k\tau}z(x,\tau)\,d\tau,\tag{54}$$

where k is a constant such that $kc_0-c_2\ge0$ and $z,\partial z/\partial x$, $(\partial/\partial x)(a(\partial\mathbb{J}_t(e^{kt}z)/\partial x))\in L^2(\Omega)$, then, u satisfies the boundary conditions in (PR). As a result of (53), we obtain the equality

$$\int_\Omega\frac{\partial\left(e^{kt}z\right)}{\partial t}\omega\,dx\,dt=\int_\Omega\frac{\partial}{\partial x}\left(a\frac{\partial\mathbb{J}_t\left(e^{\tau}z\right)}{\partial x}\right)\cdot\omega\,dx\,dt.\tag{55}$$

The left-hand side of (55) shows that the mapping

$$L^2(\Omega)\ni z\longrightarrow\int_0^T\frac{\partial}{\partial x}\left(a\frac{\partial\mathbb{J}_t\left(e^{\tau}z\right)}{\partial x}\right)\cdot\omega\,dt\tag{56}$$

is a continuous linear functional of z. From the right-hand side of (55), It is true if the function ω has the following properties:

$$\begin{aligned}&\frac{\partial\omega}{\partial x},\mathbb{J}_t^\star(a\omega),\mathbb{J}_t^\star\left(a\frac{\partial\omega}{\partial x}\right),\frac{\partial}{\partial x}\mathbb{J}_t^\star(a\omega),\\ &\qquad\frac{\partial}{\partial x}\mathbb{J}_t^\star\left(a\frac{\partial\omega}{\partial x}\right)\in L^2(\Omega),\\ &\omega(x,0)=0,\qquad\omega(0,t)=0,\\ &\omega(1,t)=0;\qquad\mathbb{J}_t^\star(a\omega)\big|_{x=1}=0.\end{aligned}\tag{57}$$

In terms of the given function ω and from the equality (55), we give the function ω in terms of z as follows:

$$\omega(x,t)=z(x,t),\tag{58}$$

and z satisfies the same conditions of the function ω.

Replacing ω in (55) by its representation (58), we obtain

$$\int_\Omega \frac{\partial\left(e^{kt}z\right)}{\partial t} z\,dx\,dt = \int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial\left(\int_0^t e^{k\tau}z(x,\tau)\,d\tau\right)}{\partial x}\right) z\,dx\,dt. \tag{59}$$

Integrating by parts the right-hand side of (59) with respect to x and t and by taking the conditions of the function z yields

$$\begin{aligned}\int_\Omega \frac{\partial\left(e^{kt}z\right)}{\partial t} z\,dx\,dt &= \int_0^T \left(a(x,t)\frac{\partial \mathbb{I}_t\left(e^{k\tau}z\right)}{\partial x}\right)(z)\Bigg|_{x=0}^{x=1} dt \\ &= -\int_\Omega a(x,t)\frac{\partial\left(\mathbb{I}_t\left(e^{k\tau}z\right)\right)}{\partial x}\frac{\partial z}{\partial x}dx\,dt \\ &= -\frac{1}{2}\int_0^1 e^{-kt}a(x,t)\left(\frac{\partial \mathbb{I}_t\left(e^{k\tau}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T} dx \\ &\quad -\frac{1}{2}\int_\Omega e^{-kt}\left(ka(x,t)-\frac{\partial a(x,t)}{\partial t}\right) \\ &\quad \times\left(\frac{\partial \mathbb{I}_t\left(e^{k\tau}z\right)}{\partial x}\right)^2 dx\,dt.\end{aligned} \tag{60}$$

According to condition (2), we obtain

$$\int_\Omega \frac{\partial\left(e^{kt}z\right)}{\partial t} z\,dx\,dt \le -\frac{1}{2}\left(kc_0-c_2\right)\int_\Omega e^{-kt}\left(\frac{\partial \mathbb{I}_t\left(e^{k\tau}z\right)}{\partial x}\right)^2 dx\,dt \le 0. \tag{61}$$

Integrating by parts the left-hand side of (61) with respect to t, and by taking the conditions of the function z, we obtain

$$\begin{aligned}\int_\Omega \frac{\partial\left(e^{kt}z\right)}{\partial t} z\,dx\,dt &= \int_\Omega\left(e^{kt}\frac{\partial z}{\partial t}z + ke^{kt}z^2\right)dx\,dt \\ &= \frac{1}{2}\int_0^1 e^{kt}z(x,T)^2dx \\ &\quad + \frac{3}{2}\int_\Omega ke^{kt}z^2dx\,dt.\end{aligned} \tag{62}$$

Then, by (61) and (62) we obtain

$$\frac{1}{2}\int_0^1 e^{kt}(z(x,T))^2dx + \frac{3}{2}\int_\Omega ke^{kt}z^2dx\,dt \le 0. \tag{63}$$

And thus $z = 0$ in Ω, hence $\omega = 0$ in Ω. This proves Proposition 5. □

We return to the proof of Theorem 4. We have already noted that it is sufficient to prove that the set $R(L)$ dense in F.

Suppose that for some $W = (\omega, \omega_0, \omega_1) \in R(L)^\perp$ and for all $u \in D(L) \equiv B$ it holds

$$\begin{aligned}(Lu,\omega)_F &= \int_\Omega \mathscr{L}u\cdot\omega\,dx\,dt \\ &\quad + \int_0^1\left(\frac{\partial \ell_1 u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right)dx \\ &\quad + \int_0^1 (\ell_2 u)(\omega_1)\,dx = 0.\end{aligned} \tag{64}$$

Then, we must prove that $W = 0$. Putting $u \in D_0(L)$ in (64), we have

$$\int_\Omega \mathscr{L}u\cdot\omega\,dx\,dt = 0, \quad u \in D_0(L). \tag{65}$$

Hence, Proposition 5 implies that $\omega = 0$. Thus, (64) takes the form

$$\begin{aligned}&\int_0^1\left(\frac{\partial \ell_1 u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right)dx \\ &\quad + \int_0^1 (\ell_2 u)(\omega_1)\,dx = 0, \quad u \in D(L).\end{aligned} \tag{66}$$

Since the ranges of the trace operators ℓ_1 and ℓ_2 are independents and the ranges of values ℓ_1 and ℓ_2 are everywhere dense in the Hilbert space F with the norm

$$\left(\int_0^1\left[\left(\frac{\partial \ell_1 u}{\partial x}\right)^2 + (\ell_2 u)^2\right]dx\right)^{1/2}, \tag{67}$$

then the equality (66) implies that $\omega_0 = 0$, $\omega_1 = 0$ (we recall satisfies a compatibility conditions). Hence $W = 0$ implies $(\overline{R(L)} = F)$. Therefore, the proof of Theorem 4 is complete.

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