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Research Article

On Connected m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-Residual Graphs

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We define m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs in which HPK is a hyperplane complete graph. We extend P. Erdös, F. Harary, and M. Klawe's definition of plane complete residual graph to hyperplane and obtain the hyperplane complete residual graph. Further, we obtain the minimum order of $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs. In addition, we obtain a unique minimal $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and a unique minimal m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs.

1. Introduction

A graph G is said to be F-residual graph [1], if for every vertex v in G, the graph obtained from G by removing the closed neighborhood of v is isomorphic [2–6] to F. We inductively define a multiply F-residual graph by saying that G is $m - F$-residual graph, if the removal of the closed neighborhood of any vertex of G results in an $(m-1)-F$-residual graph, where of course a $1-F$-residual graph is simply said to be F-residual graph.

It is natural to ask what is the minimum number of vertices that an $m - K_n$-residual graph must contain? It is easy to prove that this number is $(m + 1)n$ and that the only $m-K_n$-residual graph with this number of vertex is $(m+1)K_n$. In [1], Erdös et al. show that a connected K_n-residual graph must have at least $2(n + 1)$ points, if $n \neq 2$. Furthermore, the Cartesian product $G \cong K_{n+1} \times K_2$ is the only such graph with $2(n + 1)$ points for $n \neq 2, 3, 4$. They complete the result by determining all connected K_n-residual graphs of minimal order for $n = 2, 3, 4$.

In [1], the following conjectures were stated.

Conjecture 1 (see [1]). *If $n \neq 2$, then every connected $m - K_n$-residual graph has at least* $\min\{2n(m + 1), (n + m)(m + 1)\}$ *vertices.*

Conjecture 2 (see [1]). *For n large, there is a unique smallest connected $m - K_n$-residual graph.*

Theorem 3 (see [1]). (1) *If G is an F-residual graph, then for any vertex u in G, the degree $d(u) = v(G) - v(F) - 1$.*

(2) *Every $m - K_n$-residual graph has at least $(m + 1)n$ vertices, and $(m + 1)K_n$ is the only $m - K_n$-residual graph with $(m + 1)n$ vertices.*

(3) *Every connected K_n-residual graph has at least $2n + 2$ vertices, if $n \neq 2$.*

(4) *If $n \neq 2$, then $G \cong K_{n+1} \times K_2$ is a connected K_n-residual graph of minimum order, and, except for $n = 3$ and $n = 4$, it is the only such graph.*

We know that these supporting results are summarized in residual graph [7–16]. In this paper, we will study the important properties of hyperplane complete graph in relain to the minimum order of hyperplane complete graph and the minimum graph of hyperplane complete graph. We will investigate Erdös, Harary, and Klawe's residual graph and obtain the minimum order of $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs. And we also show a unique minimal connected $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs of order $(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t$

and a unique minimal connected m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs of order $(n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

In general, we follow the notation in [1]. In particular, $v(G)$ is the number of vertices in a graph G, $N(u)$ is the closed neighborhood of the vertex $u \in V(G)$, and $N^*(u) = N(u) \cup \{u\}$ are called as neighborhood and closed neighborhood of u in G.

2. On Connected $HPK(n_1, n_2, n_3, n_4)[K_t]$-Residual Graphs

Definition 4. A graph G is a 3-dimensional hyperplane complete graph, if for every $x \in V(G)$, $V(G) = V_1 \times V_2 \times V_3 \times V_4$, $x = \{(x_1, x_2, x_3, x_4) \mid x_i \in V_i, i = 1, 2, 3, 4\}$, $|V_i| = n_i$, $i = 1, 2, 3, 4$. Two vertices $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are adjacent to each other, if and only if $x \neq y$ and $x_k = y_k$ for some $k = 1, 2, 3, 4$. For convenience, it writes $HPK(n_1, n_2, n_3, n_4)$ instead of 3-dimensional hyperplane complete graphs.

Definition 5. Let $F = HPK(n_1, n_2, n_3, n_4)$, a graph G is $HPK(n_1, n_2, n_3, n_4)$-residual graph, if for every $u \in G$, it is $G - N^*(u) \cong F$. A multiply $HPK(n_1, n_2, n_3, n_4)$-residual graph is m-$HPK(n_1, n_2, n_3, n_4)$-residual graph, if the removal of the closed neighborhood of any vertex of G results in an $(m-1)$-$HPK(n_1, n_2, n_3, n_4)$-residual graph.

By Definitions 4 and 5, let $G = HPK(n_1, n_2, n_3, n_4)$ and $n = v(G) = n_1 n_2 n_3 n_4$, if $n_i = 1$ for some i, then G is indeed a complete graph K_n. In fact, $HPK(m, l) = K_m \times K_l$, and $HPK(l)$ is an empty graph with n vertices.

Definition 6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs. In the composition [17], G_1 and G_2 are recorded as $G = G_1[G_2]$, $G_1[G_2] = V_1 \times V_2$, and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ in G, if and only if $u_1 = v_1$, $u_2 \neq v_2$, u_2 is adjacent to v_2 in G_2, but u_1 is adjacent to v_1 in G_1.

By Definitions 4, 5, and Theorem 3, we have the following.

Lemma 7. *If F is a connected noncomplete graph, then there is no disconnected F-residual graph.*

Proof. Suppose that G is an F-residual graph, if G is disconnected, let $G = G_1 \cup G_2$, where G_1 is a connected component of the G. Then, for any $v \in G_1$, so

$$\begin{aligned} G - N^*(v) &= (G_1 \cup G_2) - N^*_{G_1}(v) \\ &= G_2 \cup \left(G_1 - N^*_{G_1}(v)\right) \cong F. \end{aligned} \tag{1}$$

Since F is connected, so $V(G_1) - N^*_{G_1}(v) = \phi$ and $G_2 \cong F$. Thus, G_2 is connected, and G_2 is not a complete graph; hence, there are two vertices u and w in G_2, and u is nonadjacent to w, let

$$G - N^*(u) = (G_1 \cup G_2) - N^*_{G_2}(v) = G_1 \cup \left(G_2 - N^*_{G_2}(u)\right). \tag{2}$$

Since $w \in G_2 - N^*_{G_2}(u)$. Hence, $G - N^*(u)$ is disconnected; however $G - N^*(u) \cong F$ is connected, which is a contraction. □

Theorem 8. *Assume that $HPK(n_1 + m), (n_2 + m), \ldots, (n_r + m)[K_t]$ is an m-$HPK(n_1, n_2, \ldots, n_r)[K_t]$-residual graph, particularly, $HPK(2, 2, \ldots, 2, n + 1)[K_t]$ is a K_{nt}-residual graph.*

Proof. The proof follows using Definitions 4, 5, 6, and $HPK(1, 1, \ldots, 1, n)[K_t] \cong K_{nt}$. □

Lemma 9. *Let $F = HPK(n_1, n_2, n_3, n_4)[K_t]$, $n_1, n_2, n_3, n_4 \geq 4$. x, y, z, w is pairwise nonadjacent in F, then $|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)| = 24t$.*

Proof. By Definitions 5 and 6 and let $V(F) = V_1 \times V_2 \times V_3 \times V_4 \times V_5$, $V_i = 1, 2, \ldots, n_i$, $i = 1, 2, 3, 4$, $V_5 = 1, 2, \ldots, t$. Let $x = (x_1, x_2, x_3, x_4, x_5)$, $y = (y_1, y_2, y_3, y_4, y_5)$, $z = (z_1, z_2, z_3, z_4, z_5)$, x, y, z is pairwise nonadjacent in F, if and only if $x \neq y$ and $x_i = y_i$ for some $i = 1, 2, 3, 4$, since x, y, z, w is pairwise nonadjacent, then $x_1 \neq y_1 \neq z_1 \neq w_1$, $x_2 \neq y_2 \neq z_2 \neq w_2$, $x_3 \neq y_3 \neq z_3 \neq w_3$, $x_4 \neq y_4 \neq z_4 \neq w_4$. And $n_1, n_2, n_3, n_4 \geq 4$, hence,

$$\begin{aligned} &N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w) \\ &\quad = \{(x_1, y_2, z_3, w_4, r), (x_1, y_2, w_3, z_4, r), \\ &\qquad (x_1, z_2, y_3, w_4, r), (x_1, z_2, w_3, y_4, r), \\ &\qquad (x_1, w_2, y_3, z_4, r), (x_1, w_2, z_3, y_4, r), \\ &\qquad (y_1, x_2, z_3, w_4, r), (y_1, x_2, w_3, z_4, r), \\ &\qquad (y_1, z_2, x_3, w_4, r), (y_1, z_2, w_3, x_4, r), \\ &\qquad (y_1, w_2, z_3, x_4, r), (y_1, w_2, x_3, z_4, r), \\ &\qquad (z_1, x_2, y_3, w_4, r), (z_1, x_2, w_3, y_4, r), \\ &\qquad (z_1, y_2, x_3, w_4, r), (z_1, y_2, w_3, x_4, r), \\ &\qquad (z_1, w_2, x_3, y_4, r), (z_1, w_2, y_3, x_4, r), \\ &\qquad (w_1, x_2, y_3, z_4, r), (w_1, x_2, z_3, y_4, r), \\ &\qquad (w_1, y_2, x_3, z_4, r), (w_1, y_2, z_3, x_4, r), \\ &\qquad (w_1, z_2, x_3, y_4, r), (w_1, z_2, y_3, z_4, r) \mid r \in V_5\}, \end{aligned} \tag{3}$$

then $|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)| = 24t$. □

Lemma 10. *Assume that G is an F-residual graph, $F = HPK(n_1, n_2, n_3, n_4)[K_t]$, n_1, n_2, n_3, $n_4 \geq 4$, x, y is pairwise nonadjacent in G, then $|N^*(x) \cap N^*(y)| = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - 2n_1 n_2 n_3 n_4 + (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)$.*

Proof. The proof follows using Definitions 4, 5, and 6. □

Lemma 11. *Assume that G is an F-residual graph, $F = HPK(n_1, n_2, n_3, n_4)[K_t]$, n_1, n_2, n_3, $n_4 \geq 4$, for any $u \in V(G)$, then $d(u) \geq [(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1 n_2 n_3 n_4]t - 1$.*

Proof. For any $v \in V(G)$, and v is nonadjacent to u in G, let $F = G - N^*(v) \cong HPK(n_1, n_2, n_3, n_4)[K_t]$, then $u \in V(F)$, by Definitions 6, then

$$\begin{aligned} \left|V(F) - N_F^*(u)\right| &= (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t, \\ \left|N_F^*(u)\right| &= n_1 n_2 n_3 n_4 t - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t, \end{aligned} \tag{4}$$

hence, $d_F(u) = n_1 n_2 n_3 n_4 t - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t - 1$. Since v, u, w is pairwise nonadjacent in G, by Lemma 10, then $|N^*(u) \cap N^*(v) \cap N^*(w)| \geq 6(n_1 + n_2 + n_3 + n_4)t$, so

$$\begin{aligned} d(u) &= d_F(u) + \left|N^*(u) \cap N^*(v)\right| \\ &\geq [n_1 n_2 n_3 n_4 - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1) \\ &\quad + (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - 2n_1 n_2 n_3 n_4 \\ &\quad + (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)]\, t - 1 \\ &= [(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1 n_2 n_3 n_4]\, t - 1. \end{aligned} \tag{5}$$

This completes the proof. □

Theorem 12. *Let G is $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$, then $v(G) \geq (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t$.*

Proof. For any $u \in G$, let $G - N^*(u) = F \cong HPK(n_1, n_2, n_3, n_4)[K_t]$, by Theorem 3 and Lemma 11, then

$$\begin{aligned} v(G) &= d(u) + 1 + v(F) \\ &\geq [(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1 n_2 n_3 n_4]\, t \\ &\quad + n_1 n_2 n_3 n_4 t = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)\, t. \end{aligned} \tag{6}$$

□

Theorem 13. *Let G is $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$ and $v(G) = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t$, then $G \cong HPK(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1)[K_t]$.*

In order to proof Theorem 13, we have the following Lemma.

Lemma 14. *Assume that $F = HPK(n_1, n_2, n_3, n_4)[K_t]$, $n_1, n_2, n_3, n_4 \geq 4$. x, y, z, w is pairwise nonadjacent in F, let $x = (x_1, x_2, x_3, x_4, x_5)$, $y = (y_1, y_2, y_3, y_4, y_5)$, $z = (z_1, z_2, z_3, z_4, z_5)$, $w = (w_1, w_2, w_3, w_4, w_5)$, then*

(1) *$N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^*(X) = \{(w_1, x_2, y_3, z_4, r \mid r \in V_5)\}$, and X contains x_2, y_3, z_4 54 set intersection, and*

$$\begin{aligned} &N^*(X) \\ &\quad = N(x_1, x_2, x_3, y_4, r) \cap N(x_1, x_2, z_3, x_4, r) \\ &\qquad \cap N(x_1, x_2, z_3, y_4, r) \\ &\qquad \cap N(y_1, x_2, x_3, x_4, r) \cap N(y_1, x_2, x_3, y_4, r) \\ &\qquad \cap N(y_1, x_2, z_3, x_4, r) \cap N(y_1, x_2, z_3, y_4, r) \\ &\qquad \cap N(z_1, x_2, x_3, x_4, r) \cap N(z_1, x_2, x_3, y_4, r) \\ &\qquad \cap N(z_1, x_2, z_3, x_4, r) \cap N(z_1, x_2, z_3, y_4, r) \\ &\qquad \cap N(x_1, y_2, y_3, x_4, r) \cap N(x_1, y_2, y_3, y_4, r) \\ &\qquad \cap N(x_1, z_2, y_3, x_4, r) \cap N(x_1, z_2, y_3, y_4, r) \\ &\qquad \cap N(y_1, y_2, y_3, x_4, r) \cap N(y_1, z_2, y_3, x_4, r) \\ &\qquad \cap N(y_1, z_2, y_3, y_4, r) \cap N(z_1, y_2, y_3, x_4, r) \\ &\qquad \cap N(z_1, y_2, y_3, y_4, r) \cap N(z_1, z_2, y_3, x_4, r) \\ &\qquad \cap N(z_1, z_2, y_3, y_4, r) \cap N(x_1, y_2, x_3, z_4, r) \\ &\qquad \cap N(x_1, y_2, z_3, z_4, r) \cap N(x_1, z_2, x_3, z_4, r) \\ &\qquad \cap N(x_1, z_2, z_3, z_4, r) \cap N(y_1, y_2, x_3, z_4, r) \\ &\qquad \cap N(y_1, y_2, z_3, z_4, r) \cap N(y_1, z_2, x_3, z_4, r) \\ &\qquad \cap N(y_1, y_2, z_3, z_4, r) \cap N(z_1, y_2, x_3, z_4, r) \\ &\qquad \cap N(z_1, y_2, z_3, z_4, r) \cap N(z_1, z_2, x_3, z_4, r) \\ &\qquad \cap N(x_1, x_2, y_3, x_4, r) \cap N(x_1, x_2, y_3, y_4, r) \\ &\qquad \cap N(y_1, x_2, y_3, x_4, r) \cap N(y_1, x_2, y_3, y_4, r) \\ &\qquad \cap N(z_1, x_2, y_3, x_4, r) \cap N(z_1, x_2, y_3, y_4, r) \\ &\qquad \cap N(x_1, x_2, x_3, z_4, r) \cap N(x_1, x_2, z_3, z_4, r) \\ &\qquad \cap N(y_1, x_2, x_3, z_4, r) \cap N(y_1, x_2, z_3, z_4, r) \\ &\qquad \cap N(z_1, x_2, x_3, z_4, r) \cap N(z_1, x_2, z_3, z_4, r) \\ &\qquad \cap N(x_1, y_2, y_3, z_4, r) \cap N(x_1, z_2, y_3, z_4, r) \\ &\qquad \cap N(y_1, y_2, y_3, z_4, r) \cap N(y_1, z_2, y_3, z_4, r) \\ &\qquad \cap N(z_1, y_2, y_3, z_4, r) \cap N(z_1, z_2, y_3, z_4, r) \\ &\qquad \cap N(x_1, x_2, y_3, z_4, r) \cap N(y_1, x_2, y_3, z_4, r) \\ &\qquad \cap N(z_1, x_2, y_3, z_4, r), \end{aligned} \tag{7}$$

(2)

$$N(w_1, w_2, w_3, w_4, r) \cap N(x_1, x_2, x_3, x_4, r)$$

$$\begin{aligned}
&\cap N(w_1, y_2, y_3, y_4, r) \cap N(y_1, w_2, y_3, y_4, r) \\
&\cap N(w_1, z_2, z_3, z_4, r) \cap N(z_1, w_2, z_3, z_4, r) \\
&- N(y_1, y_2, y_3, y_4, r) - N(z_1, z_2, z_3, z_4, r) \\
&= \{(w_1, w_2, x_3, x_4, r) \mid r \in V_5\},
\end{aligned} \tag{8}$$

(3)

$$\begin{aligned}
M = {} & N(w_1, w_2, w_3, w_4, r) \cap N(x_1, x_2, x_3, x_4, r) \\
&\cap N(w_1, y_2, y_3, y_4, r) \cap N(y_1, w_2, y_3, y_4, r) \\
&\cap N(y_1, y_2, w_3, y_4, r) \cap N(w_1, z_2, z_3, z_4, r) \\
&\cap N(z_1, w_2, z_3, z_4, r) \cap N(z_1, z_2, w_3, z_4, r) \\
&- N(y_1, y_2, y_3, y_4, r) - N(z_1, z_2, z_3, z_4, r) \\
&= \{(w_1, w_2, w_3, x_4, r) \mid r \in V_5\}.
\end{aligned} \tag{9}$$

Proof. Part (1) follows from Definition 6 and Lemma 9.

Similarly,

$$\begin{aligned}
&N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^*(Y) \\
&\quad = \{(x_1, w_2, y_3, z_4, r \mid r \in V_5)\}, \\
&N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^*(Z) \\
&\quad = \{(x_1, y_2, w_3, z_4, r \mid r \in V_5)\}, \\
&N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^*(W) \\
&\quad = \{(x_1, y_2, z_3, w_4, r \mid r \in V_5)\},
\end{aligned} \tag{10}$$

where $N^*(Y)$ contains x_1, y_3, z_4 54 set intersection, $N^*(Z)$ contains x_1, y_2, z_4 54 set intersection, and $N^*(W)$ contains x_1, y_2, z_3 54 set intersection.

(2) Let $v = (v_1, v_2, v_3, v_4, v_5) \in M$, by Definition 6, and v is not adjacent y, z, hence, $v_i \neq y_i \neq z_i, i = 1, 2, 3, 4$, and $v = (v_1, v_2, v_3, v_4, v_5)$ is adjacent to (w_1, y_2, y_3, y_4, r), (y_1, w_2, y_3, y_4, r), (w_1, z_2, z_3, z_4, r), (z_1, w_2, z_3, z_4, r), hence $v_1 = w_1$, $v_2 = w_2$, because v is adjacent to x, then $x_1 \neq w_1 = v_1$, $x_2 \neq w_2 = v_2$, hence, $v_3 = x_3$, $v_4 = x_4$, for any $v_5 \in V_5$.

Part (3) follows from the proof of part (2). □

Proof of Theorem 13. We are now ready to prove the following quantitative version of our theorem, we have four cases as follows.

Case 1. For any $u \in V(G)$, let $F_1 = G - N^*(u) = HPK(n_1, n_2, n_3, n_4)[K_t]$, by Definition 6, $V(F_1) = \{(x_1, x_2, x_3, x_4, x_5) \mid 1 \leq x_i \leq n_i, i = 1, 2, 3, 4; 1 \leq x_5 \leq t\}$, $x = (x_1, x_2, x_3, x_4, x_5)$ is adjacent to $y = (y_1, y_2, y_3, y_4, y_5)$ in F_1, if and only if $x \neq y$ and for any $x_i = y_i$, $i = 1, 2, 3, 4$.

Case 2. Let $F_2 = G - N^*(n_1, n_2, n_3, n_4, 1) = HPK(n_1, n_2, n_3, n_4)[K_t]$, by Lemma 14, using $V(F_1) \cap V(F_2)$ in the name of the vertex of $V(F_2) - V(F_1)$ in the naming of vertices. First, prove that $u = (0, 0, 0, 0, 1)$, for any $x, y \in V(F_1) \cap V(F_2)$, $x = (x_1, x_2, x_3, x_4, 1)$, $y = (y_1, y_2, y_3, y_4, 1)$, $x_i \neq y_i$, $i = 1, 2, 3, 4$. For convenience, $N(x_1, x_2, x_3, x_4, x_5)$ is the closed neighborhood of a vertex $(x_1, x_2, x_3, x_4, x_5)$, $N^*(x_1, x_2, x_3, x_4, x_5) = N(x_1, x_2, x_3, x_4, x_5) \cup (x_1, x_2, x_3, x_4, x_5)$, by Lemma 14, let

$$\begin{aligned}
&N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(X) \\
&\quad = \{(0, x_2, y_3, z_4, r) \mid 1 \leq r \leq t\}, \\
&N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(Y) \\
&\quad = \{(x_1, 0, y_3, z_4, r) \mid 1 \leq r \leq t\}, \\
&N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(Z) \\
&\quad = \{(x_1, y_2, 0, z_4, r) \mid 1 \leq r \leq t\}, \\
&N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(W) \\
&\quad = \{(x_1, y_2, z_3, 0, r) \mid 1 \leq r \leq t\},
\end{aligned} \tag{11}$$

$$\begin{aligned}
&N(0,0,0,0,1) \cap N(x_1, x_2, x_3, x_4, 1) \\
&\quad \cap N(0, y_2, y_3, y_4, 1) \cap N(y_1, 0, y_3, y_4, 1) \\
&\quad \cap N(0, z_2, z_3, z_4, 1) \cap N(z_1, 0, z_3, z_4, 1) \\
&\quad - N(y_1, y_2, y_3, y_4, 1) - N(z_1, z_2, z_3, z_4, 1) \\
&\quad = \{(0, 0, x_3, x_4, r) \mid 1 \leq r \leq t\}, \\
&N(0,0,0,0,1) \cap N(x_1, x_2, x_3, x_4, 1) \\
&\quad \cap N(y_1, 0, y_3, y_4, 1) \cap N(y_1, y_2, 0, y_4, 1) \\
&\quad \cap N(z_1, 0, z_3, z_4, 1) \cap N(z_1, z_2, 0, z_4, 1) \\
&\quad - N(y_1, y_2, y_3, y_4, 1) - N(z_1, z_2, z_3, z_4, 1) \\
&\quad = \{(x_1, 0, 0, x_4, r) \mid 1 \leq r \leq t\}, \\
&N(0,0,0,0,1) \cap N(x_1, x_2, x_3, x_4, 1) \\
&\quad \cap N(y_1, y_2, 0, y_4, 1) \cap N(y_1, y_2, y_3, 0, 1) \\
&\quad \cap N(z_1, z_2, 0, z_4, 1) \cap N(z_1, z_2, z_3, 0, 1) \\
&\quad - N(y_1, y_2, y_3, y_4, 1) - N(z_1, z_2, z_3, z_4, 1) \\
&\quad = \{(x_1, x_2, 0, 0, r) \mid 1 \leq r \leq t\}, \\
&N(0,0,0,0,1) \cap N(x_1, x_2, x_3, x_4, 1) \\
&\quad \cap N(0, y_2, y_3, y_4, 1) \cap N(y_1, y_2, 0, y_4, 1) \\
&\quad \cap N(0, z_2, z_3, z_4, 1) \cap N(z_1, z_2, 0, z_4, 1) \\
&\quad - N(y_1, y_2, y_3, y_4, 1) - N(z_1, z_2, z_3, z_4, 1) \\
&\quad = \{(0, x_2, 0, x_4, r) \mid 1 \leq r \leq t\}, \\
&N(0,0,0,0,1) \cap N(x_1, x_2, x_3, x_4, 1) \\
&\quad \cap N(0, y_2, y_3, y_4, 1) \cap N(y_1, y_2, y_3, 0, 1)
\end{aligned}$$

$$\begin{aligned}
&\cap N(0,z_2,z_3,z_4,1)\cap N(z_1,z_2,z_3,0,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(0,x_2,x_3,0,r) \mid 1\le r\le t\},\\
&N(0,0,0,0,1)\cap N(x_1,x_2,x_3,x_4,1)\\
&\quad \cap N(y_1,0,y_3,y_4,1)\cap N(y_1,y_2,y_3,0,1)\\
&\quad \cap N(z_1,0,z_3,z_4,1)\cap N(z_1,z_2,z_4,0,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(x_1,0,x_3,0,r)\ 1\le r\le t\},
\end{aligned} \tag{12}$$

$$\begin{aligned}
&N(0,0,0,0,1)\cap N(x_1,x_2,x_3,x_4,1)\\
&\quad \cap N(0,y_2,y_3,y_4,1)\cap N(y_1,0,y_3,y_4,1)\\
&\quad \cap N(y_1,y_2,0,y_4,1)\cap N(0,z_2,z_3,z_4,1)\\
&\quad \cap N(z_1,0,z_3,z_4,1)\cap N(z_1,z_2,0,z_4,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(0,0,0,x_4,r) \mid 1\le r\le t\},\\
&N(0,0,0,0,1)\cap N(x_1,x_2,x_3,x_4,1)\\
&\quad \cap N(y_1,0,y_3,y_4,1)\cap N(y_1,y_2,0,y_4,1)\\
&\quad \cap N(y_1,y_2,y_3,0,1)\cap N(z_1,0,z_3,z_4,1)\\
&\quad \cap N(z_1,z_2,0,z_4,1)\cap N(z_1,z_2,z_3,0,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(x_1,0,0,0,r) \mid 1\le r\le t\},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&N(0,0,0,0,1)\cap N(x_1,x_2,x_3,x_4,1)\\
&\quad \cap N(0,y_2,y_3,y_4,1)\cap N(y_1,0,y_3,y_4,1)\\
&\quad \cap N(y_1,y_2,y_3,0,1)\cap N(0,z_2,z_3,z_4,1)\\
&\quad \cap N(z_1,0,z_3,z_4,1)\cap N(z_1,z_2,z_3,0,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(0,0,x_3,0,r) \mid 1\le r\le t\},\\
&N(0,0,0,0,1)\cap N(x_1,x_2,x_3,x_4,1)\\
&\quad \cap N(0,y_2,y_3,y_4,1)\cap N(y_1,y_2,0,y_4,1)\\
&\quad \cap N(y_1,y_2,y_3,0,1)\cap N(0,z_2,z_3,z_4,1)\\
&\quad \cap N(z_1,z_2,0,z_4,1)\cap N(z_1,z_2,z_3,0,1)\\
&\quad - N(y_1,y_2,y_3,y_4,1) - N(z_1,z_2,z_3,z_4,1)\\
&= \{(0,x_2,0,0,r) \mid 1\le r\le t\}.
\end{aligned}$$

Lastly let $\{v \in V(F_2) \mid N^*_{F_2}(v) = N^*_{F_2}(u)\} = \{(0,0,0,0,r) \mid 1 \le r \le t\}$, so far $V(F_2) - V(F_1)$ vertex already all named, the uniqueness of the name. By Lemma 14 and Definition 6, then $F_2 \cong HPK(n_1,n_2,n_3,n_4)[K_t]$, $V(F_2) = \{(x_1,x_2,x_3,x_4,x_5) \mid 0 \le x_i \le x_{n_i}, i = 1,2,3,4; 1 \le x_5 \le t\}$. Two vertices $x = (x_1,x_2,x_3,x_4,x_5)$ and $y = (y_1,y_2,y_3,x_4,x_5)$ are adjacent to each other, if and only if exist $x \ne y$ and $x_i = y_i$ for $i = 1,2,3,4$.

Case 3. Let $F_3 = G - N^*(1,1,1,1,1) \cong HPK(n_1,n_2,n_3,n_4)[K_t]$, to F_3 not named point naming, these point sets are composed of

$$\begin{aligned}
&V(F_3) - V(F_1) - V(F_2)\\
&\quad = V(F_3)\cap\overline{(V(F_1)\cup V(F_2))}\\
&\quad = V(F_3)\cap(N^*(0,0,0,0,1)\cap N^*(n_1,n_2,n_3,n_4,1)\\
&\qquad \cap N^*(m_1,m_2,m_3,m_4,1))\\
&\quad = N^*(0,0,0,0,1)\cap N^*(n_1,n_2,n_3,n_4,1)\\
&\qquad \cap N^*(m_1,m_2,m_3,m_4,1) - N^*(1,1,1,1,1),
\end{aligned} \tag{14}$$

for any $x = (x_1,x_2,x_3,n_4,1) \in V(F_1)\cap V(F_2)\cap V(F_3)$, $2 \le x_i \le n_{i-1}$, $i = 1,2,3,4$. By Lemma 14 and (11), we can name

$$\begin{aligned}
&\{(0,x_2,n_3,m_4,r),(x_1,0,n_3,m_4,r),(x_1,n_2,0,m_4,r),\\
&\quad (x_1,n_3,m_4,0,r) \mid 1\le r\le t\}.
\end{aligned} \tag{15}$$

By Lemma 14 and (12), we can name

$$\begin{aligned}
&\{(0,0,n_3,m_4,r),(0,n_2,m_3,0,r),(0,n_2,0,m_4,r),\\
&\quad (n_1,0,0,m_4,r),(n_2,m_2,0,0,r),\\
&\quad (n_1,0,m_3,0,r) \mid 1\le r\le t\}.
\end{aligned} \tag{16}$$

By Lemma 14 and (13), we can name

$$\begin{aligned}
&\{(0,0,0,m_4,r),(0,0,m_3,0,r),\\
&\quad (0,m_2,0,0,r),(m_1,0,0,0,r) \mid 1\le r\le t\}.
\end{aligned} \tag{17}$$

Similarly, we can name

$$\begin{aligned}
&\{(0,n_2,n_3,n_4,r),(n_1,0,n_3,n_4,r),\\
&\quad (n_1,n_2,0,n_4,r),(n_1,n_2,n_3,0,r) \mid 1\le r\le t\}.
\end{aligned} \tag{18}$$

For the $V(F_2)\cap V(F_3)$ point rename, by Lemma 14 and the uniqueness of the name get the name and in F_2 name is the same, then $V(F_3) = \{(x_1,x_2,x_3,x_4,x_5) \mid 0 \le x_i \le n_i, x_i \ne 1, i = 1,2,3,4; 1 \le x_4 \le t\}$, and two vertices $x = (x_1,x_2,x_3,x_4,x_5)$, $y = (y_1,y_2,y_3,y_4,y_5)$ are adjacent to each other in the F_3, if and only if $x \ne y$ and, for any $x_i = y_i$, for $i = 1,2,3,4$.

Case 4. Named G all unnamed point, this kind of composition is set $N^*(0,0,0,0,1) \cap N^*(n_1,n_2,n_3,n_4,1) \cap N^*(1,1,1,1,1)$. Let $F_4 = G - N^*(2,2,2,2,1) =$

$HPK(n_1, n_2, n_3, n_4)[K_t]$. Obviously, $(0,0,0,0,1), (n_1, n_2, n_3, n_4, 1), (1,1,1,1,1) \in V(F_4)$. By Lemmas 9 and 14, then

$$\begin{aligned} &\left|N^*_{F_4}(0,0,0,0,0,1) \cap N^*_{F_4}(n_1,n_2,n_3,n_4,1)\right. \\ &\quad \left.\cap N^*_{F_4}(1,1,1,1,1)\right| = 24t, \\ &N^*_{F_4}(0,0,0,0,1) \cap N^*_{F_4}(n_1,n_2,n_3,n_4,1) \\ &\quad \cap N^*_{F_4}(1,1,1,1,1) \subset V(F_4). \end{aligned} \tag{19}$$

As described in Case 3, in accordance with the Lemma 14, we name $V(F_2) \cap V(F_4), V(F_3) \cap V(F_4)$ the same as in F_2, F_3. Similarly, by Lemma 14 and (11), we can name

$$\begin{aligned} &(0,1,n_3,n_4,r), (0,n_2,1,n_4,r), (0,n_2,n_3,1,r), \\ &(1,0,n_3,n_4,r), (1,n_2,n_3,0,r), (1,n_2,0,n_4,r), \quad 1 \le r \le t. \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned} V(F_4) = \{&(x_1,x_2,x_3,x_4,x_5) \mid 0 \le x_i \le n_i, x_i \ne 2, \\ &i = 1,2,3,4, 1 \le x_4 \le t\}, \end{aligned} \tag{21}$$

two vertices $x = (x_1,x_2,x_3,x_4,x_5)$, $y = (y_1,y_2,y_3,y_4,y_5)$ are adjacent to each other in F_4, if and only if, exist $x \ne y$ and $x_i = y_i$ for $i = 1,2,3,4$.

So far, we have the G of each vertex named and $V(G) = \{(x_1,x_2,x_3,x_4,x_5) \mid 0 \le x_i \le n_i, i = 1,2,3,4, 1 \le x_4 \le t\})$. According to the naming rules, for any $x = (x_1,x_2,x_3,x_4,x_5)$, $y = (y_1,y_2,y_3,y_4,y_5)$, if x and y in the same $F_i, i = 1,2,3,4,5$, x is adjacent to y, if and only if $x \ne y$ and for any $x_i = y_i$, for $i = 1,2,3,4$. Otherwise, for any $z \in V(G)$, z is not adjacent x, y. Let $G_1 = G - N^*(z) \cong HPK(n_1,n_2,n_3,n_4)[K_t]$, then $x, y \in V(G_1)$, the uniqueness of the name also get x and y is adjacent to in G. So x is adjacent to y in G, if and only if $x \ne y$ and for any $x_i = y_i$, for $i = 1,2,3,4$.

With comprehensive discussed above, then $G \cong HPK(n_1+1, n_2+1, n_3+1, n_4+1)[K_t]$. The proof is complete. □

3. Connected m-$HPK(n_1,n_2,n_3,n_4)[K_t]$-Residual Graphs

Lemma 15. *Assume that graph G is m-$HPK(n_1,n_2,n_3,n_4)[K_t]$-residual graph, $n_1,n_2,n_3,n_4 \ge 4$, x,y,z, and w is pairwise nonadjacent in G, then $|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)| \ge 24t$.*

Proof. By Definitions 4, 5, 6, and $n_1,n_2,n_3,n_4 \ge 4$, we can turn out vertex $u_1,u_2,\ldots,u_{m+2},u_{m+3}$, $u_{i+1} \in G_i = G_{i-1} - N^*(u_i)$, $i = 1,2,\ldots,m+2,m+3$, let $G_0 = G$, $u_1 \in G_0$, for convenience, $u_1 \in V(G_0)$ instead of $u \in G$, let $u_1,u_2,\ldots,u_{m+2},u_{m+3}$ be pairwise nonadjacent. Let $x = u_1, y = u_2, z = u_3$, hence

$$\begin{aligned} F &= G - N^*(u_4) - \cdots - N^*(u_{m+2}) - N^*(u_{m+3}) \\ &\cong HPK(n_1,n_2,n_3,n_4)[K_t], \end{aligned} \tag{22}$$

x, y, z, w is pairwise nonadjacent in F, by Lemma 9, then

$$\begin{aligned} &\left|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)\right| \\ &\quad \ge N^*_F(x) \cap N^*_F(y) \cap N^*_F(z) \cap N^*_F(w) \ge 24t. \end{aligned} \tag{23}$$

□

Lemma 16. *Assume that graph G is m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \ge 4$, $u, v \in V(G)$ is pairwise nonadjacent in G, then $|N^*(u) \cap N^*(v)| \ge [(n_1+m+1)(n_2+m+1)(n_3+m+1)(n_4+m+1) - 2(n_1+m)(n_2+m)(n_3+m)(n_4+m) + (n_1+m-1)(n_2+m-1)(n_3+m-1)(n_4+m-1)]t$.*

Proof. For $m = 1$, by Lemma 10, we are done. Suppose that $m \ge 1$, it is true. For $m+1$, let G be a connected $(m+1)$-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, x, y, z is pairwise nonadjacent in G, for any $w \in G$, w is not adjacent x, y, z in G. Let $H = G - N^*(w)$, then x, y is pairwise nonadjacent in H, H is m-$HPK(n_1,n_2,n_3,n_4)[K_t]$-residual graph. By inductive hypothesis, $|N^*(u) \cap N^*(v)| \ge [(n_1+m+1)(n_2+m+1)(n_3+m+1)(n_4+m+1) - 2(n_1+m)(n_2+m)(n_3+m)(n_4+m) + (n_1+m-1)(n_2+m-1)(n_3+m-1)(n_4+m-1)]t$. □

Lemma 17. *Assume that graph G is m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \ge 4$, $u \in V(G)$, then $d(u) \ge [(n_1+m)(n_2+m)(n_3+m)(n_4+m) - (n_1+m-1)(n_2+m-1)(n_3+m-1)(n_4+m-1)]t - 1$.*

Proof. For $m = 1$, by Lemma 14, we are done. Suppose that $m \ge 1$, it is true. For $m+1$, let G be a connected $(m+1)$-$HPK(n_1,n_2,n_3,n_4)[K_t]$-residual graph, u,v,w is pairwise nonadjacent in G. Let $H = G - N^*(v)$, then $u \in H$, H is m-$HPK(n_1,n_2,n_3,n_4)[K_t]$-residual graph. By inductive hypothesis and by Lemma 16, then

$$\begin{aligned} &d(u) \\ &\quad = d_H(u) + \left|N^*(u) \cap N^*(v) \cap N^*(w)\right| \\ &\quad \ge [(n_1+m)(n_2+m)(n_3+m)(n_4+m) \\ &\qquad - (n_1+m-1)(n_2+m-1) \\ &\qquad \times (n_3+m-1)(n_4+m-1)]t - 1 \\ &\qquad + [(n_1+m+1)(n_2+m+1)(n_3+m+1)(n_4+m+1) \\ &\qquad - 2(n_1+m)(n_2+m)(n_3+m)(n_4+m) \\ &\qquad + (n_1+m-1)(n_2+m-1) \\ &\qquad \times (n_3+m-1)(n_4+m-1)]t \\ &\quad = [(n_1+m+1)(n_2+m+1)(n_3+m+1)(n_4+m+1) \\ &\qquad - (n_1+m)(n_2+m)(n_3+m)(n_4+m)]t - 1. \end{aligned} \tag{24}$$

□

Theorem 18. *Let G is m-$HPK(n_1,n_2,n_3,n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \ge 4$, then $v(G) \ge (n_1+m)(n_2+m)(n_3+m)(n_4+m)t$.*

Proof. For $m = 1$, by Theorem 12, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let G be a connected $(m + 1)$-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, for any $u \in G$. Let $H = G - N^*(u)$, H is m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph. By inductive hypothesis and by Lemma 17, then

$$\begin{aligned} v(H) &\geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t, \\ |N^*(u)| &\geq [(n_1 + m + 1)(n_2 + m + 1) \\ &\quad \times (n_3 + m + 1)(n_4 + m + 1) \\ &\quad - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)]t, \\ v(G) &\geq P(H)|N^*(u)| \\ &\geq (n_1 + m + 1)(n_2 + m + 1) \\ &\quad (n_3 + m + 1)(n_4 + m + 1)t. \end{aligned} \tag{25}$$

□

Theorem 19. *Let G is m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$, $v(G) = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$, then $G \cong HPK(n_1 + m, n_2 + m, n_3 + m, (n_4 + m))[K_t]$.*

Proof. For $m = 1$, by Theorem 12, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let G be a connected $(m+1)$-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$, $v(G) = (n_1+m+1)(n_2+m+1)(n_3+m+1)(n_4+m+1)t$, for any $u \in G$. Let $H = G - N^*(u)$, H is m-$HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graph. By inductive hypothesis and by Lemma 17, then

$$\begin{aligned} &v(H) \\ &\quad = v(G) - |N^*(u)| \\ &\quad \leq (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1)t \\ &\qquad - [(n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1) \\ &\qquad\quad - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)]t \\ &\quad = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t, \end{aligned} \tag{26}$$

by Theorem 18, $v(H) = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$. By inductive hypothesis, $H \cong HPK(n_1 + m, n_2 + m, n_3 + m, n_4 + m)[K_t]$. By Definition 6, let G be a connected $HPK(n_1+m, n_2+m, n_3 + m, , n_4 + m)[K_t]$-residual graph, $n_i + m \geq n_i \geq 4$, with conditions and Theorem 12, then $G \cong HPK(n_1 + m + 1, n_2 + m + 1, n_3 + m + 1, n_4 + m + 1)[K_t]$. □

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