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Research Article

Hermite-Hadamard and Simpson Type Inequalities for Differentiable *P*-GA-Functions

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The author introduces the concept of the P-GA-functions, gives Hermite-Hadamard's inequalities for P-GA-functions, and defines a new identity. By using this identity, the author obtains new estimates on generalization of Hadamard and Simpson type inequalities for P-GA-functions. Some applications to special means of real numbers are also given.

1. Introduction

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We recall that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be P-function on I or belong to the class $P(I)$ if it is nonnegative and

$$f(tx + (1-t)y) \le f(x) + f(y) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$. Note that $P(I)$ contain all nonnegative convex and quasiconvex functions [1].

The following inequalities are well known in the literature as Hermite-Hadamard inequality and Simpson inequality, respectively.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{3}$$

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\begin{aligned}\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.\end{aligned} \tag{4}$$

Definition 3 (see [2, 3]). A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f\left(x^t y^{1-t}\right) \le tf(x) + (1-t)f(y) \tag{5}$$

for all $x, y \in I$ and $t \in [0,1]$.

In recent years, many authors have studied errors estimations for Hermite-Hadamard and Simpson inequalities; for refinements, counterparts, and generalization concerning P-functions and GA-convex, see [4–11].

In this paper, the concept of the P-GA-function is introduced, Hermite-Hadamard's inequalities for P-GA-functions are established, and a new identity for differentiable functions is defined. By using this identity, the author obtains a generalization of Hadamard and Simpson type inequalities for P-GA-functions.

2. Main Results

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I; throughout this section we will take

$$I_f(\alpha,\lambda,a,b) = (1-\lambda) f\left(a^{1-\alpha}b^{\alpha}\right) + \lambda\left[\alpha f(a) + (1-\alpha) f(b)\right] - \frac{1}{\ln(b/a)}\int_a^b \frac{f(u)}{u}du, \quad (6)$$

where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$.

Definition 4. A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be P-GA-function (P-geometric-arithmetic function) on I if

$$f\left(x^t y^{1-t}\right) \le f(x) + f(y), \quad (7)$$

for any $x, y \in I$ and $t \in [0,1]$.

Proposition 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$. If f is P-function and nondecreasing, then f is P-GA-function on I.*

Proof. This follows from

$$f\left(x^t y^{1-t}\right) \le f(tx + (1-t)y) \le f(x) + f(y), \quad (8)$$

for all $x, y \in I$ and $t \in [0,1]$. □

Proposition 6. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$. If f is P-GA-function and nonincreasing, then f is P-function on I.*

Proof. The conclusion follows from

$$f(tx + (1-t)y) \le f\left(x^t y^{1-t}\right) \le f(x) + f(y) \quad (9)$$

for all $x, y \in I$ and $t \in [0,1]$, respectively. □

Hermite-Hadamard's inequalities can be represented for P-GA-functions as follows.

Theorem 7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ (f is integrable on $[a,b]$), where $a, b \in I$ with $a < b$. If f is a P-GA-function on $[a,b]$, then the following inequalities hold:*

$$f\left(\sqrt{ab}\right) \le \frac{2}{\ln(b/a)}\int_a^b \frac{f(u)}{u}du \le 2\left[f(a) + f(b)\right], \quad (10)$$

with $\alpha > 0$.

Proof. Since f is a P-GA-function on $[a,b]$, we have for all $x, y \in [a,b]$ (with $t = 1/2$ in inequality (7))

$$f\left(\sqrt{xy}\right) \le f(x) + f(y). \quad (11)$$

Choosing $x = a^t b^{1-t}$, $y = b^t a^{1-t}$, we get

$$f\left(\sqrt{ab}\right) \le f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right). \quad (12)$$

Integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$f\left(\sqrt{ab}\right) \le \int_0^1 f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right)dt = \frac{2}{\ln(b/a)}\int_a^b \frac{f(u)}{u}du, \quad (13)$$

and the first inequality is proved.

For the proof of the second inequality in (10) we first note that if f is a P-GA-function, then, for $t \in [0,1]$, it yields

$$f\left(a^t b^{1-t}\right) \le f(a) + f(b), \qquad f\left(b^t a^{1-t}\right) \le f(a) + f(b). \quad (14)$$

By adding side to side these inequalities and taking square root we have

$$f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right) \le 2\left[f(a) + f(b)\right], \quad (15)$$

and, integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{2}{\ln(b/a)}\int_a^b \frac{f(u)}{u}du \le 2\left[f(a) + f(b)\right]. \quad (16)$$

The proof is completed. □

In order to prove our main results we need the following identity.

Lemma 8. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$, and $\alpha > 0$ one has*

$$I_f(\alpha,\lambda,a,b) = \left(\ln\frac{b}{a}\right)\left\{a\alpha^2\int_0^1 (t-\lambda)\left(\frac{b}{a}\right)^{\alpha t} f'\left(a^{1-\alpha t}b^{\alpha t}\right)dt - b(1-\alpha)^2\int_0^1 (t-\lambda)\left(\frac{a}{b}\right)^{(1-\alpha)t} \times f'\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right)dt\right\}. \quad (17)$$

Proof. By integration by parts and changing the variable, we can state

$$\begin{aligned} a\left(\ln\frac{b}{a}\right)\alpha^2\int_0^1 (t-\lambda)\left(\frac{b}{a}\right)^{\alpha t} f'\left(a^{1-\alpha t}b^{\alpha t}\right)dt &= \alpha\int_0^1 (t-\lambda)\,df\left(a^{1-\alpha t}b^{\alpha t}\right) \\ &= \alpha(t-\lambda) f\left(a^{1-\alpha t}b^{\alpha t}\right)\Big|_0^1 - \alpha\int_0^1 f\left(a^{1-\alpha t}b^{\alpha t}\right)dt \\ &= \alpha(1-\lambda) f\left(a^{1-\alpha}b^{\alpha}\right) + \alpha\lambda f(a) \\ &\quad - \frac{1}{\ln(b/a)}\int_a^{a^{1-\alpha}b^{\alpha}} \frac{f(u)}{u}du, \end{aligned} \quad (18)$$

and similarly we get

$$
\begin{aligned}
& -b\left(\ln\frac{b}{a}\right)(1-\alpha)^2\int_0^1 (t-\lambda)\left(\frac{a}{b}\right)^{(1-\alpha)t} \\
&\qquad\qquad \times f'\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right)dt \\
&= (1-\alpha)\int_0^1 (t-\lambda)\,df\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right) \\
&= (1-\alpha)(t-\lambda)f\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right)\Big|_0^1 \\
&\quad -(1-\alpha)\int_0^1 f\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right)dt \\
&= (1-\alpha)(1-\lambda)f\left(a^{1-\alpha}b^{\alpha}\right)+(1-\alpha)\lambda f(b) \\
&\quad -\frac{1}{\ln(b/a)}\int_{a^{1-\alpha}b^{\alpha}}^{b}\frac{f(u)}{u}du.
\end{aligned} \tag{19}
$$

Adding the resulting identities we obtain the desired result. □

Theorem 9. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is P-GA-function on $[a,b]$ for some fixed $q \geq 1$, $\alpha, \lambda \in [0,1]$, then the following inequality holds:*

$$
\begin{aligned}
& \left|I_f(\alpha,\lambda,a,b)\right| \\
&\leq \left(\ln\frac{b}{a}\right)\left(\lambda^2-\lambda+\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q} \\
&\quad \times\left\{a\alpha^2 C^{1/q}\left(\lambda,\left(\frac{b}{a}\right)^{\alpha q}\right)\right. \\
&\qquad \left. +b(1-\alpha)^2 C^{1/q}\left(\lambda,\left(\frac{a}{b}\right)^{(1-\alpha)q}\right)\right\},
\end{aligned} \tag{20}
$$

where

$$
C(\lambda,u)=\frac{1}{\ln^2 u}\left[(u-\lambda(1+u))\ln u+2u^{\lambda}-u-1\right]. \tag{21}
$$

Proof. Since $|f'|^q$ is P-GA-function on $[a,b]$, for all $t \in [0,1]$,

$$
\begin{aligned}
\left|f'\left(a^{1-\alpha t}b^{\alpha t}\right)\right|^q &\leq \left|f'(a)\right|^q+\left|f'(b)\right|^q, \\
\left|f'\left(a^{(1-\alpha)t}b^{1-(1-\alpha)t}\right)\right|^q &\leq \left|f'(a)\right|^q+\left|f'(b)\right|^q.
\end{aligned} \tag{22}
$$

Hence, using Lemma 8 and power mean inequality, we get

$$
\begin{aligned}
& \left|I_f(\alpha,\lambda,a,b)\right| \\
&\leq \left(\ln\frac{b}{a}\right)\left(\int_0^1|t-\lambda|\,dt\right)^{1-1/q} \\
&\quad \times\left\{a\alpha^2\left(\int_0^1|t-\lambda|\left(\frac{b}{a}\right)^{\alpha q t}\right.\right. \\
&\qquad\qquad \left.\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{1/q}+b(1-\alpha)^2 \\
&\qquad \times\left(\int_0^1|t-\lambda|\left(\frac{a}{b}\right)^{(1-\alpha)qt}\right. \\
&\qquad\qquad \left.\left.\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{1/q}\right\} \\
&\leq \left(\ln\frac{b}{a}\right)\left(\lambda^2-\lambda+\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q} \\
&\quad \times\left\{a\alpha^2\left(\int_0^1|t-\lambda|\left(\frac{b}{a}\right)^{\alpha q t}dt\right)^{1/q}+b(1-\alpha)^2\right. \\
&\qquad \left.\times\left(\int_0^1|t-\lambda|\left(\frac{a}{b}\right)^{(1-\alpha)qt}dt\right)^{1/q}\right\} \\
&\leq \left(\ln\frac{b}{a}\right)\left(\lambda^2-\lambda+\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q} \\
&\quad \times\left\{a\alpha^2 C^{1/q}\left(\lambda,\left(\frac{b}{a}\right)^{\alpha q}\right)\right. \\
&\qquad \left. +b(1-\alpha)^2 C^{1/q}\left(\lambda,\left(\frac{a}{b}\right)^{(1-\alpha)q}\right)\right\},
\end{aligned} \tag{23}
$$

which completes the proof. □

Corollary 10. *Under the assumptions of Theorem 9 with $q = 1$, inequality* (20) *reduced to the following inequality:*

$$
\begin{aligned}
& \left|I_f(\alpha,\lambda,a,b)\right| \\
&\leq \left(\ln\frac{b}{a}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \\
&\quad \times\left\{a\alpha^2 C\left(\lambda,\left(\frac{b}{a}\right)^{\alpha}\right)+b(1-\alpha)^2 C\left(\lambda,\left(\frac{a}{b}\right)^{(1-\alpha)}\right)\right\}.
\end{aligned} \tag{24}
$$

Corollary 11. *Under the assumptions of Theorem 9 with* $\alpha = 1/2$, *inequality* (20) *reduced to the following inequality:*

$$\begin{aligned}&\left|(1-\lambda) f\left(\sqrt{ab}\right)+\lambda\left[\frac{f(a)+f(b)}{2}\right]\right.\\&\quad\left.-\frac{1}{\ln(b/a)}\int_a^b\frac{f(u)}{u}du\right|\\&\quad\le\frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\lambda^2-\lambda+\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{aC^{1/q}\left(\lambda,\left(\frac{b}{a}\right)^{q/2}\right)+bC^{1/q}\left(\lambda,\left(\frac{a}{b}\right)^{q/2}\right)\right\}.\end{aligned}\tag{25}$$

In particular, for $\lambda = 0$, *we get*

$$\begin{aligned}&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln(b/a)}\int_a^b\frac{f(u)}{u}du\right|\\&\quad\le\frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{aC^{1/q}\left(0,\left(\frac{b}{a}\right)^{q/2}\right)+bC^{1/q}\left(0,\left(\frac{a}{b}\right)^{q/2}\right)\right\}.\end{aligned}\tag{26}$$

For $\lambda = 1$, *we get*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln(b/a)}\int_a^b\frac{f(u)}{u}du\right|\\&\quad\le\frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1}{2}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{aC^{1/q}\left(1,\left(\frac{b}{a}\right)^{q/2}\right)+bC^{1/q}\left(1,\left(\frac{a}{b}\right)^{q/2}\right)\right\},\end{aligned}\tag{27}$$

and, for $\lambda = 1/3$,

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\sqrt{ab}\right)\right]-\frac{1}{\ln(b/a)}\int_a^b\frac{f(u)}{u}du\right|\\&\quad\le\frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{5}{18}\right)^{1-1/q}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{aC^{1/q}\left(\frac{1}{3},\left(\frac{b}{a}\right)^{q/2}\right)+bC^{1/q}\left(\frac{1}{3},\left(\frac{a}{b}\right)^{q/2}\right)\right\}.\end{aligned}\tag{28}$$

Theorem 12. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|^q$ *is P-GA-function on* $[a,b]$ *for some fixed* $q > 1$, $\alpha, \lambda \in [0,1]$, *then the following inequality holds:*

$$\begin{aligned}&\left|I_f(\alpha,\lambda,a,b)\right|\\&\quad\le\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\left[\lambda^{p+1}+(1-\lambda)^{p+1}\right]\right)^{1/p}\\&\qquad\times\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{a^{1-\alpha}\alpha^2L_{\alpha q-1}^{\alpha-1/q}(a,b)+b^{\alpha}(1-\alpha)^2L_{(1-\alpha)q-1}^{1-\alpha-1/q}(a,b)\right\},\end{aligned}\tag{29}$$

where $(1/p)+(1/q)=1$ *and* $L_n(a,b)$ *is n-logarithmic mean defined with* $L_n(a,b) := ((b^{n+1}-a^{n+1})/((n+1)(b-a)))^{1/n}$, $n \in \mathbb{R}\setminus\{-1,0\}$.

Proof. Since $|f'|^q$ is P-GA-function on $[a,b]$ and using Lemma 8 and Hölder inequality, we get

$$\begin{aligned}&\left|I_f(\alpha,\lambda,a,b)\right|\\&\quad\le\left(\ln\frac{b}{a}\right)\left(\int_0^1|t-\lambda|^p dt\right)^{1/p}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{a\alpha^2\left(\int_0^1\left(\frac{b}{a}\right)^{\alpha qt}dt\right)^{1/q}\right.\\&\qquad\qquad\left.+b(1-\alpha)^2\left(\int_0^1\left(\frac{a}{b}\right)^{(1-\alpha)qt}dt\right)^{1/q}\right\}\\&\quad\le\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\left[\lambda^{p+1}+(1-\lambda)^{p+1}\right]\right)^{1/p}\\&\qquad\times\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\\&\qquad\times\left\{a^{1-\alpha}\alpha^2L_{\alpha q-1}^{\alpha-1/q}(a,b)+b^{\alpha}(1-\alpha)^2L_{(1-\alpha)q-1}^{1-\alpha-1/q}(a,b)\right\}.\end{aligned}\tag{30}$$

Here it is seen by simple computation that

$$\begin{aligned}\int_0^1|t-\lambda|^p dt&=\frac{1}{p+1}\left[\lambda^{p+1}+(1-\lambda)^{p+1}\right],\\\int_0^1\left(\frac{b}{a}\right)^{\alpha qt}dt&=\frac{L_{\alpha q-1}^{\alpha q-1}(a,b)}{a^{\alpha q}},\\\int_0^1\left(\frac{a}{b}\right)^{(1-\alpha)qt}dt&=\frac{L_{(1-\alpha)q-1}^{(1-\alpha)q-1}(a,b)}{b^{(1-\alpha)q}}.\end{aligned}\tag{31}$$

Hence, the proof is completed. □

Corollary 13. *Under the assumptions of Theorem 12 with $\alpha = 1/2$, inequality (29) reduced to the following inequality:*

$$
\begin{aligned}
&(1-\lambda) f\left(\sqrt{ab}\right) + \lambda\left[\frac{f(a)+f(b)}{2}\right] \\
&\quad - \frac{1}{\ln(b/a)} \int_a^b \frac{f(u)}{u} du \\
&\quad \le \frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\left[\lambda^{p+1} + (1-\lambda)^{p+1}\right]\right)^{1/p} \\
&\qquad \times \left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{1/q} \times \left(\sqrt{a}+\sqrt{b}\right) L_{q/2-1}^{(q-2)/2q}(a,b).
\end{aligned} \tag{32}
$$

In particular, for $\lambda = 0$, we get

$$
\begin{aligned}
&\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln(b/a)} \int_a^b \frac{f(u)}{u} du\right| \\
&\quad \le \frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\right)^{1/p} \left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{1/q} \\
&\qquad \times \left(\sqrt{a}+\sqrt{b}\right) L_{q/2-1}^{(q-2)/2q}(a,b).
\end{aligned} \tag{33}
$$

For $\lambda = 1$, we get

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(u)}{u} du\right| \\
&\quad \le \frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\right)^{1/p} \left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{1/q} \\
&\qquad \times \left(\sqrt{a}+\sqrt{b}\right) L_{q/2-1}^{(q-2)/2q}(a,b),
\end{aligned} \tag{34}
$$

and, for $\lambda = 1/3$, we get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\sqrt{ab}\right)\right] - \frac{1}{\ln(b/a)} \int_a^b \frac{f(u)}{u} du\right| \\
&\quad \le \frac{1}{4}\left(\ln\frac{b}{a}\right)\left(\frac{1+2^{p+1}}{(p+1)3^{p+1}}\right)^{1/p} \left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{1/q} \\
&\qquad \times \left(\sqrt{a}+\sqrt{b}\right) L_{q/2-1}^{(q-2)/2q}(a,b).
\end{aligned} \tag{35}
$$

3. Application to Special Means

Let us recall the following special means of two nonnegative numbers a, b with $b > a$:

(1) the arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \tag{36}$$

(2) the weighted arithmetic mean

$$A_\alpha = A_\alpha(a,b) := \alpha a + (1-\alpha) b, \quad \alpha \in [0,1], \tag{37}$$

(3) the geometric mean

$$G = G(a,b) := \frac{a+b}{2}, \tag{38}$$

(4) the weighted geometric mean

$$G_\alpha = G_\alpha(a,b) := a^\alpha b^{1-\alpha}, \quad \alpha \in [0,1], \tag{39}$$

(5) the logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \tag{40}$$

(6) the n-logarithmic mean

$$L_n = L_n(a,b) := \left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{1/n}, \quad n \in \mathbb{R} \setminus \{-1, 0\}. \tag{41}$$

Proposition 14. *For $b > a > 0$, $n \in \mathbb{N}$, $n \ge 2$, and $q \ge 1$, one has*

$$
\begin{aligned}
&\left|(1-\lambda) G_{1-\alpha}^n(a,b) + \lambda A_\alpha(a^n, b^n) - L_n^n(a,b)\right| \\
&\quad \le n\left(\ln\frac{b}{a}\right)\left(\lambda^2 - \lambda + \frac{1}{2}\right)^{1-1/q} \left(a^{(n-1)q} + b^{(n-1)q}\right)^{1/q} \\
&\qquad \times \left\{a\alpha^2 C^{1/q}\left(\lambda, \left(\frac{b}{a}\right)^{\alpha q}\right)\right. \\
&\qquad\quad \left. + b(1-\alpha)^2 C^{1/q}\left(\lambda, \left(\frac{a}{b}\right)^{(1-\alpha)q}\right)\right\},
\end{aligned} \tag{42}
$$

where C is defined as in (21).

Proof. Let $f(x) = x^n$, $x > 0$, $n \ge 2$, and $q \ge 1$. □

Proposition 15. *For $b > a > 0$, $n \in \mathbb{N}$, $n \ge 2$, and $q > 1$, one has*

$$
\begin{aligned}
&\left|(1-\lambda) G_{1-\alpha}^n(a,b) + \lambda A_\alpha(a^n, b^n) - L_n^n(a,b)\right| \\
&\quad \le n\left(\ln\frac{b}{a}\right)\left(\frac{1}{p+1}\left[\lambda^{p+1} + (1-\lambda)^{p+1}\right]\right)^{1/p} \\
&\qquad \times \left(a^{(n-1)q} + b^{(n-1)q}\right)^{1/q} \\
&\qquad \times \left\{a^{1-\alpha}\alpha^2 L_{\alpha q-1}^{\alpha-1/q}(a,b) + b^\alpha (1-\alpha)^2 L_{(1-\alpha)q-1}^{1-\alpha-1/q}(a,b)\right\}.
\end{aligned} \tag{43}
$$

Proof. Let $f(x) = x^n$, $x > 0$, $n \ge 2$, and $q > 1$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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