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Research Article

On Hermite-Hadamard Type Inequalities for Riemann-Liouville Fractional Integrals via Two Kinds of Convexity

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We obtain some Hermite-Hadamard type inequalities for products of two m-convex functions via Riemann-Liouville integrals. The analogous results for (α, m)-convex functions are also established.

1. Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been extensively obtained by a number of authors (e.g., [1–7]).

In [8], Toader defined the concept of m-convexity as follows.

Definition 1 (see [8]). The function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in (0, 1]$, if for every $x, y \in [0, b]$ and $t \in (0, 1)$ one has

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{2}$$

In [3], Dragomir and Toader proved the following inequality of Hermite-Hadamard type for m-convex functions.

Theorem 2 (see [3]). *Let $f : [0, \infty) \to \mathbb{R}$ be a m-convex function with $m \in (0, 1]$; if $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then one has the following inequality:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx \\ \leq \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{f(b)+mf(a/m)}{2}\right\}.\end{aligned} \tag{3}$$

The notion of m-convexity has been further generalized in [9] as it is stated in the following definition.

Definition 3 (see [9]). The function $f : [0, b] \to \mathbb{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ one has

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y). \tag{4}$$

In [10], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

Theorem 4 (see [10]). *Let f and g be real-valued, nonnegative, and convex functions on $[a,b]$. Then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &\le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),\\ 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\ &\quad+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),\end{aligned}\tag{5}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Some Hermite-Hadamard type inequalities for products of two m-convex and (α,m)-convex functions are established in [11].

Theorem 5 (see [11]). *Let $f,g:[0,\infty)\to[0,\infty)$ be functions such that $fg\in L_1[a,b]$, where $0\le a<b<\infty$. If f is m_1-convex and g is m_2-convex on $[a,b]$ for some fixed $m_1,m_2\in(0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\le\min\{M_1,M_2\},\tag{6}$$

where

$$\begin{aligned}M_1&=\frac{1}{3}\left[f(a)\,g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\\&\quad+\frac{1}{6}\left[m_2f(a)\,g\left(\frac{b}{m_2}\right)+m_1f\left(\frac{b}{m_1}\right)g(a)\right],\\M_2&=\frac{1}{3}\left[f(b)\,g(b)+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\\&\quad+\frac{1}{6}\left[m_2f(b)\,g\left(\frac{a}{m_2}\right)+m_1f\left(\frac{a}{m_1}\right)g(b)\right].\end{aligned}\tag{7}$$

Theorem 6 (see [11]). *Let $f,g:[0,\infty)\to[0,\infty)$ be functions such that $fg\in L_1[a,b]$, where $0\le a<b<\infty$. If f is (α_1,m_1) convex and g is (α_2,m_2)-convex on $[a,b]$ for some fixed $\alpha_1,m_1,\alpha_2,m_2\in(0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\le\min\{N_1,N_2\},\tag{8}$$

where

$$\begin{aligned}N_1&=\frac{f(a)\,g(a)}{\alpha_1+\alpha_2+1}\\&\quad+m_2\left[\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_1+\alpha_2}\right]f(a)\,g\left(\frac{b}{m_2}\right)\\&\quad+m_1\left[\frac{1}{1+\alpha_2}-\frac{1}{1+\alpha_1+\alpha_2}\right]f\left(\frac{b}{m_1}\right)g(a)\\&\quad+m_1m_2\left(1-\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_2}+\frac{1}{1+\alpha_1+\alpha_2}\right)\\&\quad\times f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),\\N_2&=\frac{f(b)\,g(b)}{\alpha_1+\alpha_2+1}\\&\quad+m_2\left[\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_1+\alpha_2}\right]f(b)\,g\left(\frac{a}{m_2}\right)\\&\quad+m_1\left[\frac{1}{1+\alpha_2}-\frac{1}{1+\alpha_1+\alpha_2}\right]f\left(\frac{a}{m_1}\right)g(b)\\&\quad+m_1m_2\left(1-\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_2}+\frac{1}{1+\alpha_1+\alpha_2}\right)\\&\quad\times f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).\end{aligned}\tag{9}$$

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also proposed by many authors. In [12], Pachpatte established some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is obtained by Kirmaci et al. in [13]. In [14], Sarikaya et al. presented some integral inequalities for two h-convex functions.

It is remarkable that Sarikaya et al. [15] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 7 (see [15]). *Let $f:[a,b]\to R$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\\&\le\frac{f(a)+f(b)}{2},\end{aligned}\tag{10}$$

with $\alpha>0$.

We remark that the symbols $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\ge0$ with $a\ge0$ which are defined by

$$J_{a^+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\quad x>a,\tag{11}$$

$$J_{b^-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\quad x<b,\tag{12}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$.

In this paper, we obtain some new Hermite-Hadamard type inequalities for products of two m-convex functions via Riemann-Liouville integrals. The analogous results for (α, m)-convex functions are also given.

2. Inequalities for Products of Two Functions for Riemann-Liouville Fractional Integrals

Theorem 8. *Let $f, g : [0, \infty) \to [0, \infty)$, $0 \le a < b$, be functions such that $fg \in L_1[a, b]$. If f is m_1-convex and g is m_2-convex on $[a, b]$ with $m_1, m_2 \in (0, 1]$, then one has*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b) g(b) \\
&\quad \le \frac{f(a) g(a)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(a) g\left(\frac{b}{m_2}\right) \\
&\qquad + \frac{m_1}{(\alpha+1)(\alpha+2)} g(a) f\left(\frac{b}{m_1}\right) \\
&\qquad + \frac{2m_1 m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right), \\
&\frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a) g(a) \\
&\quad \le \frac{f(b) g(b)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(b) g\left(\frac{a}{m_2}\right) \\
&\qquad + \frac{m_1}{(\alpha+1)(\alpha+2)} g(b) f\left(\frac{a}{m_1}\right) \\
&\qquad + \frac{2m_1 m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).
\end{aligned} \tag{13}$$

Proof. Since f is m_1-convex and g is m_2-convex on $[a, b]$, then for $t \in [0, 1]$ we get

$$\begin{aligned}
f(ta + (1-t)b) &\le tf(a) + m_1(1-t) f\left(\frac{b}{m_1}\right), \\
g(ta + (1-t)b) &\le tg(a) + m_2(1-t) g\left(\frac{b}{m_2}\right).
\end{aligned} \tag{14}$$

From (14), we get

$$\begin{aligned}
&f(ta + (1-t)b) g(ta + (1-t)b) \\
&\quad \le t^2 f(a) g(a) + m_2 f(a) g\left(\frac{b}{m_2}\right) t(1-t) \\
&\qquad + m_1 g(a) f\left(\frac{b}{m_1}\right) t(1-t) \\
&\qquad + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) (1-t)^2.
\end{aligned} \tag{15}$$

Multiplying both sides of the above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f(ta + (1-t)b) g(ta + (1-t)b)\, dt \\
&\quad = \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u) g(u) \frac{du}{a-b} \\
&\quad = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b) g(b) \\
&\quad \le f(a) g(a) \int_0^1 t^{\alpha+1} dt + m_2 f(a) g\left(\frac{b}{m_2}\right) \int_0^1 t^\alpha (1-t)\, dt \\
&\qquad + m_1 g(a) f\left(\frac{b}{m_1}\right) \int_0^1 t^\alpha (1-t)\, dt \\
&\qquad + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \int_0^1 t^{\alpha-1} (1-t)^2 dt \\
&\quad = \frac{f(a) g(a)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(a) g\left(\frac{b}{m_2}\right) \\
&\qquad + \frac{m_1}{(\alpha+1)(\alpha+2)} g(a) f\left(\frac{b}{m_1}\right) \\
&\qquad + \frac{2m_1 m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right).
\end{aligned} \tag{16}$$

Analogously, we obtain

$$\begin{aligned}
&f((1-t)a + tb) g((1-t)a + tb) \\
&\quad \le t^2 f(b) g(b) + m_2 f(b) g\left(\frac{a}{m_2}\right) t(1-t) \\
&\qquad + m_1 g(b) f\left(\frac{a}{m_1}\right) t(1-t) \\
&\qquad + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) (1-t)^2.
\end{aligned} \tag{17}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f((1-t)a + tb) g((1-t)a + tb)\, dt \\
&\quad = \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v) g(v) \frac{dv}{b-a} \\
&\quad = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a) g(a)
\end{aligned}$$

$$\begin{aligned}
&\leq f(b)\,g(b)\int_0^1 t^{\alpha+1}dt + m_2 f(b)\,g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha}(1-t)\,dt \\
&\quad + m_1 g(b)\,f\left(\frac{a}{m_1}\right)\int_0^1 t^{\alpha}(1-t)\,dt \\
&\quad + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha-1}(1-t)^2 dt \\
&= \frac{f(b)\,g(b)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(b)\,g\left(\frac{a}{m_2}\right) \\
&\quad + \frac{m_1}{(\alpha+1)(\alpha+2)} g(b)\,f\left(\frac{a}{m_1}\right) \\
&\quad + \frac{2m_1 m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right),
\end{aligned} \tag{18}$$

which completes the proof. □

Corollary 9. *With assumptions in Theorem 8, if $\alpha = 1$, one gets*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
&\quad \leq \frac{1}{3}\left[f(a)\,g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right] \\
&\qquad + \frac{1}{6}\left[m_2 f(a)\,g\left(\frac{b}{m_2}\right) + m_1 f\left(\frac{b}{m_1}\right) g(a)\right], \\
&\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
&\quad \leq \frac{1}{3}\left[f(b)\,g(b) + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\right] \\
&\qquad + \frac{1}{6}\left[m_2 f(b)\,g\left(\frac{a}{m_2}\right) + m_1 f\left(\frac{a}{m_1}\right) g(b)\right],
\end{aligned} \tag{19}$$

which is just the result in Theorem 5.

Corollary 10. *With assumptions in Theorem 8, one gets*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)\,g(b) + J_{b^-}^{\alpha} f(a)\,g(a)\right] \\
&\quad \leq \frac{1}{2(\alpha+2)}\left[f(a)\,g(a) + f(b)\,g(b)\right] \\
&\qquad + \frac{m_2}{2(\alpha+1)(\alpha+2)}\left[f(b)\,g\left(\frac{a}{m_2}\right) + f(a)\,g\left(\frac{b}{m_2}\right)\right] \\
&\qquad + \frac{m_1}{2(\alpha+1)(\alpha+2)}\left[g(a)\,f\left(\frac{b}{m_1}\right) + g(b)\,f\left(\frac{a}{m_1}\right)\right] \\
&\qquad + \frac{m_1 m_2}{\alpha(\alpha+1)(\alpha+2)} \\
&\qquad \times \left[f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) + f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\right].
\end{aligned} \tag{20}$$

Corollary 11. *With assumptions in Theorem 8, if one chooses $g : [a,b] \to R$ as $g(x) = 1$ and $m_2 = 1$ for all $x \in [a,b]$, one has*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^{\alpha} f(b) \leq \frac{f(a)}{\alpha+1} + \frac{m_1}{\alpha(\alpha+1)} f\left(\frac{b}{m_1}\right), \\
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{b^-}^{\alpha} f(a) \leq \frac{f(b)}{\alpha+1} + \frac{m_1}{\alpha(\alpha+1)} f\left(\frac{a}{m_1}\right), \\
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\
&\quad \leq \frac{f(b)+f(a)}{\alpha+1} + \frac{m_1}{\alpha(\alpha+1)}\left[f\left(\frac{b}{m_1}\right) + f\left(\frac{a}{m_1}\right)\right].
\end{aligned} \tag{21}$$

Theorem 12. *Let $f, g : [0,\infty) \to [0,\infty)$, $0 \leq a < b$, be functions such that $fg \in L_1[a,b]$. If f is (α_1, m_1)-convex and g is (α_2, m_2)-convex on $[a,b]$ with $(\alpha_1, m_1), (\alpha_2, m_2) \in (0,1]^2$, respectively, then one has*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^{\alpha} f(b)\,g(b) \\
&\quad \leq \frac{1}{\alpha_1+\alpha_2+\alpha} f(a)\,g(a) \\
&\qquad + \frac{\alpha_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)} m_2 f(a)\,g\left(\frac{b}{m_2}\right) \\
&\qquad + \frac{\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)} m_1 g(a)\,f\left(\frac{b}{m_1}\right) \\
&\qquad + \left(\frac{1}{\alpha} - \frac{1}{\alpha+\alpha_1} - \frac{1}{\alpha+\alpha_2} + \frac{1}{\alpha+\alpha_1+\alpha_2}\right) \\
&\qquad \times m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right), \\
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{b^-}^{\alpha} f(a)\,g(a) \\
&\quad \leq \frac{1}{\alpha_1+\alpha_2+\alpha} f(b)\,g(b) \\
&\qquad + \frac{\alpha_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)} m_2 f(b)\,g\left(\frac{a}{m_2}\right) \\
&\qquad + \frac{\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)} m_1 g(b)\,f\left(\frac{a}{m_1}\right)
\end{aligned}$$

$$+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right)$$
$$\times m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right). \tag{22}$$

Proof. Since f is (α_1, m_1)-convex and g is (α_2, m_2)-convex on $[a, b]$, then for $t \in [0, 1]$ we get

$$\begin{aligned} f(ta+(1-t)b) &\le t^{\alpha_1} f(a) + m_1\left(1-t^{\alpha_1}\right) f\left(\frac{b}{m_1}\right), \\ g(ta+(1-t)b) &\le t^{\alpha_2} g(a) + m_2\left(1-t^{\alpha_2}\right) g\left(\frac{b}{m_2}\right). \end{aligned} \tag{23}$$

From (23), we get

$$\begin{aligned} &f(ta+(1-t)b)\, g(ta+(1-t)b) \\ &\quad\le t^{\alpha_1+\alpha_2} f(a)\, g(a) + m_2 f(a)\, g\left(\frac{b}{m_2}\right) t^{\alpha_1}\left(1-t^{\alpha_2}\right) \\ &\qquad + m_1 g(a)\, f\left(\frac{b}{m_1}\right) t^{\alpha_2}\left(1-t^{\alpha_1}\right) \\ &\qquad + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right). \end{aligned} \tag{24}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g(ta+(1-t)b)\, dt \\ &\quad= \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\, g(u)\, \frac{du}{a-b} \\ &\quad= \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^{\alpha} f(b)\, g(b) \\ &\quad\le f(a)\, g(a) \int_0^1 t^{\alpha_1+\alpha_2+\alpha-1} dt \\ &\qquad + m_2 f(a)\, g\left(\frac{b}{m_2}\right) \int_0^1 t^{\alpha-1} t^{\alpha_1}\left(1-t^{\alpha_2}\right) dt \\ &\qquad + m_1 g(a)\, f\left(\frac{b}{m_1}\right) \int_0^1 t^{\alpha-1} t^{\alpha_2}\left(1-t^{\alpha_1}\right) dt \\ &\qquad + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \int_0^1 t^{\alpha-1}\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right) dt \\ &\quad= \frac{1}{\alpha_1+\alpha_2+\alpha} f(a)\, g(a) \\ &\qquad + \frac{\alpha_2 m_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)} f(a)\, g\left(\frac{b}{m_2}\right) \\ &\qquad + \frac{\alpha_1 m_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)} g(a)\, f\left(\frac{b}{m_1}\right) \\ &\qquad + \left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right) \\ &\qquad \times m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right). \end{aligned} \tag{25}$$

Similarly, we have

$$\begin{aligned} &f((1-t)a+tb)\, g((1-t)a+tb) \\ &\quad\le t^{\alpha_1+\alpha_2} f(b)\, g(b) + m_2 f(b)\, g\left(\frac{a}{m_2}\right) t^{\alpha_1}\left(1-t^{\alpha_2}\right) \\ &\qquad + m_1 g(b)\, f\left(\frac{a}{m_1}\right) t^{\alpha_2}\left(1-t^{\alpha_1}\right) \\ &\qquad + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right), \end{aligned} \tag{26}$$

$$\begin{aligned} &\int_0^1 t^{\alpha-1} f((1-t)a+tb)\, g((1-t)a+tb)\, dt \\ &\quad= \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\, g(v)\, \frac{dv}{b-a} \\ &\quad= \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{b^-}^{\alpha} f(a)\, g(a) \\ &\quad\le f(b)\, g(b) \int_0^1 t^{\alpha_1+\alpha_2+\alpha-1} dt \\ &\qquad + m_2 f(b)\, g\left(\frac{a}{m_2}\right) \int_0^1 t^{\alpha-1} t^{\alpha_1}\left(1-t^{\alpha_2}\right) dt \\ &\qquad + m_1 g(b)\, f\left(\frac{a}{m_1}\right) \int_0^1 t^{\alpha-1} t^{\alpha_2}\left(1-t^{\alpha_1}\right) dt \\ &\qquad + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) \int_0^1 t^{\alpha-1}\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right) dt \\ &\quad= \frac{1}{\alpha_1+\alpha_2+\alpha} f(b)\, g(b) \\ &\qquad + \frac{\alpha_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)} m_2 f(b)\, g\left(\frac{a}{m_2}\right) \\ &\qquad + \frac{\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)} m_1 g(b)\, f\left(\frac{a}{m_1}\right) \end{aligned}$$

$$+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right)$$
$$\times m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right). \tag{27}$$

We get the desired result. □

Corollary 13. *With assumptions in Theorem 12, if $\alpha = 1$, then*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
&\quad\le \frac{f(a)\,g(a)}{\alpha_1+\alpha_2+1}+m_2\left[\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_1+\alpha_2}\right] \\
&\qquad\times f(a)\,g\left(\frac{b}{m_2}\right) \\
&\qquad+m_1\left[\frac{1}{1+\alpha_2}-\frac{1}{1+\alpha_1+\alpha_2}\right] f\left(\frac{b}{m_1}\right) g(a) \\
&\qquad+m_1 m_2\left(1-\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_2}+\frac{1}{1+\alpha_1+\alpha_2}\right) \\
&\qquad\times f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right), \\
&\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
&\quad\le \frac{f(b)\,g(b)}{\alpha_1+\alpha_2+1}+m_2\left[\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_1+\alpha_2}\right] \\
&\qquad\times f(b)\,g\left(\frac{a}{m_2}\right) \\
&\qquad+m_1\left[\frac{1}{1+\alpha_2}-\frac{1}{1+\alpha_1+\alpha_2}\right] f\left(\frac{a}{m_1}\right) g(b) \\
&\qquad+m_1 m_2\left(1-\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_2}+\frac{1}{1+\alpha_1+\alpha_2}\right) \\
&\qquad\times f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right),
\end{aligned} \tag{28}$$

which is just the result in Theorem 6.

Corollary 14. *With assumptions in Theorem 12, one gets*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)\,g(b)+J_{b^-}^\alpha f(a)\,g(a)\right] \\
&\quad\le \frac{1}{\alpha_1+\alpha_2+\alpha}\left[f(a)\,g(a)+f(b)\,g(b)\right] \\
&\qquad+\frac{\alpha_2 m_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)} \\
&\qquad\times\left[f(b)\,g\left(\frac{a}{m_2}\right)+f(a)\,g\left(\frac{b}{m_2}\right)\right] \\
&\qquad+\frac{\alpha_1 m_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)} \\
&\qquad\times\left[g(b)\,f\left(\frac{a}{m_1}\right)+g(a)\,f\left(\frac{b}{m_1}\right)\right] \\
&\qquad+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right) \\
&\qquad\times m_1 m_2\left[f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)+f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right].
\end{aligned} \tag{29}$$

3. Conclusion

In this paper, we obtain some new Hermite-Hadamard type inequalities for products of two m-convex functions via Riemann-Liouville integrals. The analogous results for (α, m)-convex functions are also established. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for products of two convex functions on the coordinates via Riemann-Liouville integrals.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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