Research Article

Iterative Algorithms for a Finite Family of Multivalued Quasi-Nonexpansive Mappings

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Let K be a nonempty closed and convex subset of a uniformly convex real Banach space E and let $T_1, \dots, T_m : K \rightarrow 2^K$ be m multivalued quasi-nonexpansive mappings. A new iterative algorithm is constructed and the corresponding sequence $\{x_n\}$ is proved to be an approximating fixed point sequence of each T_i ; that is, $\lim d(x_n, Tx_n) = 0$. Then, convergence theorems are proved under appropriate additional conditions. Our results extend and improve some important recent results (e.g., Abbas et al. (2011)).

1. Introduction

Let (X, d) be a metric space, K a nonempty subset of X , and $T : K \rightarrow 2^K$ a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in K : x \in Tx\}$.

For several years, the study of fixed point theory for *multivalued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, e.g., Brouwer [1], Kakutani [2], Nash [3, 4], Geanakoplos [5], Nadler Jr. [6], and Downing and Kirk [7]).

Interest in the study of fixed point theory for multivalued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as *Game Theory* and *Nonsmooth Differential Equations*.

Game Theory. Nash showed the existence of equilibria for noncooperative *static* games as a direct consequence of *multivalued* Brouwer or Kakutani fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multivalued mapping* whose fixed points coincide with the equilibrium points of the game. This, among other things, made Nash a recipient of Nobel Prize in Economic Sciences in 1994. However, it has been remarked that the applications of this theory to equilibrium are mostly *static*: they enhance understanding conditions under which

equilibrium may be achieved but do not indicate how to construct a process starting from a nonequilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by *iterative methods for fixed point of multivalued mappings*.

Nonsmooth Differential Equations. A large number of problems from mechanics and electrical engineering lead to differential inclusions and differential equations with discontinuous right-hand sides, for example, a dry friction force of some electronic devices. Below are two models:

$$\frac{du}{dt} = f(t, u), \quad \text{a.e. } t \in I := [-a, a], \quad u(0) = u_0, \quad (1)$$

where a and u_0 are fixed in $\mathbb{R}$. These types of differential equations do not have solutions in the classical sense. A generalized notion of solution is what is called a solution in the sense of Fillipov.

Consider the following *multivalued* initial value problem:

$$\begin{aligned} -\frac{d^2 u}{dt^2} &\in u - \frac{1}{4} - \frac{1}{4} \text{sign}(u - 1) \quad \text{on } \Omega = (0, \pi); \\ u(0) &= 0; \\ u(\pi) &= 0. \end{aligned} \quad (2)$$

Under some conditions, the solutions set of (1) and (2) coincides with the fixed point set of some multivalued mappings.

Let D be a nonempty subset of a normed linear space E . The set D is called *proximal* (see, e.g., [8–10]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf \{\|x - y\| : y \in D\} = d(x, D), \quad (3)$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed, and convex subset of a real Hilbert space is proximal. We denote by $CB(D)$ the families of nonempty closed and bounded subsets of D , $K(D)$ the families of nonempty compact subsets of D , and $C(D)$ the families of nonempty compact convex subsets of D . The *Hausdorff metric* on $CB(K)$ is defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \quad (4)$$

for all $A, B \in CB(K)$. A multivalued mapping $T : D(T) \subseteq E \rightarrow CB(E)$ is called *Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L \|x - y\| \quad \forall x, y \in D(T). \quad (5)$$

When $L \in (0, 1)$ in (5), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$. Finally, A multivalued mapping $T : K \rightarrow CB(K)$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

Several papers deal with the problem of *approximating* fixed points of *multivalued nonexpansive* mappings (see, e.g., [8–12] and the references therein) and their generalizations (see, e.g., [13–15]).

On the other hand, Abbas et al. [11] introduced a new one-step iterative process for approximating a common fixed point of two multivalued *nonexpansive mappings* in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let $S, T : K \rightarrow CB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \end{aligned} \quad (6)$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ are such that

$$\begin{aligned} \|y_n - p\| &\leq d(p, Sx_n); \\ \|z_n - p\| &\leq d(p, Tx_n) \end{aligned} \quad (7)$$

and $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$.

The following lemma is a consequence of the definition of Hausdorff metric, as remarked by Nadler Jr. [6].

Lemma 1. *Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma. \quad (8)$$

Following the work of Abbas et al. [11], Rashwan and Altuqi [16] introduced a new scheme for approximation of a common fixed point of three multivalued nonexpansive mappings in uniformly convex Banach space. Let $T, S, R : K \rightarrow CB(K)$ be three multivalued nonexpansive mappings. They employed the following iterative process:

$$x_1 \in K, \quad (9)$$

$$x_{n+1} = a_n y_n + b_n z_n + c_n w_n, \quad n \geq 1,$$

where $y_n \in Tx_n$, $z_n \in Sx_n$, and $w_n \in Rx_n$ are such that

$$\begin{aligned} \|y_n - y_{n+1}\| &\leq H(Tx_n, Tx_{n+1}) + \eta_n; \\ \|z_n - z_{n+1}\| &\leq H(Sx_n, Sx_{n+1}) + \eta_n; \\ \|w_n - w_{n+1}\| &\leq H(Rx_n, Rx_{n+1}) + \eta_n \end{aligned} \quad (10)$$

and $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$.

Remark 2. Note that if y_n , z_n , and w_n are known, then the existence of y_{n+1} , z_{n+1} , and w_{n+1} satisfying (10) is guaranteed by Lemma 1.

Before we state the result of Rashwan and Altuqi [16], we need the following definition.

The mappings $T, S, R : K \rightarrow CB(K)$ are said to satisfy condition (C) if $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Sx$, or $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Rx$, or $d(x, y) \leq d(z, y)$, for $z \in Rx$ and $y \in Sx$.

Let $F = F(T) \cap F(S) \cap F(R)$ be the set of all common fixed points of the mappings T, S , and R .

Theorem RA (Rashwan and Altuqi [16]). *Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E . Let $T, S, R : K \rightarrow CB(K)$ be multivalued nonexpansive mappings satisfying condition (C) and $\{x_n\}$ the sequence defined by (9) and (10). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then*

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 = \lim_{n \rightarrow \infty} d(x_n, Rx_n). \quad (11)$$

Recently, Bunyawat and Suantai [17] introduced a one-step iterative scheme for finding a common fixed point of a finite family of multivalued quasi-nonexpansive mappings in a uniformly convex real Banach space. They proposed the following algorithm: let K be a nonempty closed and convex subset of a uniformly convex real Banach space E and $T_1, T_2, \dots, T_m : K \rightarrow CB(K)$ multivalued quasi-nonexpansive mappings. Let $\{x_n\}$ be the sequence defined iteratively from $x_1 \in K$ by

$$x_{n+1} = a_{n,0} x_{n,0} + a_{n,1} x_{n,1} + \dots + a_{n,m} x_{n,m}, \quad n \geq 1, \quad (12)$$

where the sequence $\{a_{n,i}\} \subset [0, 1]$ satisfies $\sum_{i=0}^m a_{n,i} = 1 \quad \forall n \geq 1$ and $x_{n,i} \in T_i x_n$ with $d(p, x_{n,i}) = d(p, T_i x_n)$.

Then, they proved the following theorem.

Theorem BS (Bunyawat and Suantai [17]). *Let E be a real Banach space and K a closed convex subset of X .*

Let $\{T_i : i = 1, 2, \dots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $CB(K)$ with $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then the sequence x_n defined by (12) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, m\}$ if and only if $\liminf d(x_n, F) = 0$.

It is our purpose in this paper to construct a new iterative algorithm and prove strong convergence theorems for approximating a common fixed point of a finite family of multivalued quasi-nonexpansive mappings in uniformly convex real Banach spaces. The class of mappings used in our theorems is much more larger than that of multivalued nonexpansive mappings. Our theorems generalize and extend those of Abbas et al. [11], Rashwan and Altwqi [16], and Bunyawat and Suantai [17] and many other important results.

2. Preliminaries

Lemma 3 (Chang et al. [18]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) = \{x \in E : \|x\| \leq r\}$. Then, for any given subset $\{u_1, u_2, \dots, u_N\} \subset B_r(0)$ and for any positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ with $\lambda_1 + \lambda_2 + \dots + \lambda_N = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow [0, +\infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \dots, N\}$ with $i < j$,*

$$\left\| \sum_{k=1}^N \lambda_k u_k \right\|^2 \leq \sum_{k=1}^N \lambda_k \|u_k\|^2 - \lambda_i \lambda_j g(\|u_i - u_j\|). \quad (13)$$

3. Main Results

In this paper, we propose the following iterative algorithm.

Let $m \geq 1$ and K a nonempty closed convex subset of a uniformly convex real Banach space E . Let $T_1, \dots, T_m : K \rightarrow CB(K)$ be multivalued quasi-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined iteratively as follows:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= \lambda_0 x_n + \lambda_1 u_n^1 + \dots + \lambda_m u_n^m, \end{aligned} \quad (14)$$

where $u_n^i \in T_i x_n$, $i = 1, \dots, m$, $\lambda_i \in (0, 1)$, $i = 0, 1, \dots, m$, with $\lambda_0 + \lambda_1 + \dots + \lambda_m = 1$. In the sequel, we will write $F := \bigcap_{i=1}^m F(T_i)$ for the set of common fixed points of the mappings T_i , $i = 1, \dots, m$.

Lemma 4. *Let K be a nonempty, closed, and convex subset of a real Banach space E . For $m \geq 1$, let $\{T_i : i = 1, 2, \dots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $CB(K)$ with $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and such that $T_i(p) = \{p\} \forall p \in F$ and $i = 1, 2, \dots, m$. Let $\{x_n\}$ be the sequence defined by (14). Then, for all $p \in F$, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.*

Proof. Let $p \in F$. By (14) and the quasi-nonexpansiveness of T_i we have

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \lambda_0 (x_n - p) + \sum_{i=1}^m \lambda_i (u_n^i - p) \right\| \\ &\leq \left\| \lambda_0 (x_n - p) \right\| + \sum_{i=1}^m \lambda_i \|u_n^i - p\| \\ &\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^m \lambda_i H(T_i x_n, T_i(p)) \\ &\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^m \lambda_i \|x_n - p\| = \|x_n - p\|. \end{aligned} \quad (15)$$

Equation (15) implies that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing. Since it is bounded from below by 0, we conclude that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. $\square$

Next, we prove the following theorem.

Theorem 5. *Let K be a nonempty, closed, and convex subset of a real uniformly convex Banach space E . For $i = 1, \dots, m$ let $T_i : K \rightarrow CB(K)$ be a multivalued quasi-nonexpansive mapping. Suppose that $F \neq \emptyset$ and that $T_i(p) = \{p\}$ for all $p \in F$. Let $\{x_n\}$ be the sequence defined by (14). Then, for all $i = 1, \dots, m$,*

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0. \quad (16)$$

Proof. Using Lemma 4 and the quasi-nonexpansiveness of T_i , there exists some positive real r such that

$$\|u_n^i - p\| \leq \|x_n - p\| \leq r. \quad (17)$$

This implies that, for each i , $1 \leq i \leq m$,

$$\|u_n^i - x_n\| \leq 2r. \quad (18)$$

Next, let $1 \leq i \leq m$; using inequality (17) and Lemma 3, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \left\| \lambda_0 (x_n - p) + \sum_{k=1}^m \lambda_k (u_n^k - p) \right\|^2 \\ &\leq \lambda_0 \|x_n - p\|^2 + \sum_{k=1}^m \lambda_k \|u_n^k - p\|^2 - \lambda_0 \lambda_i g(\|u_n^i - x_n\|) \\ &\leq \lambda_0 \|x_n - p\|^2 + \sum_{k=1}^m \lambda_k \|x_n - p\|^2 - \lambda_0 \lambda_i g(\|u_n^i - x_n\|) \\ &= \|x_n - p\|^2 - \lambda_0 \lambda_i g(\|u_n^i - x_n\|). \end{aligned} \quad (19)$$

Therefore

$$\lambda_0 \lambda_i g(\|u_n^i - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \quad (20)$$

This implies that

$$\lim_{n \rightarrow \infty} g(\|u_n^i - x_n\|) = 0. \quad (21)$$

It follows that

$$\lim_{n \rightarrow \infty} \|u_n^i - x_n\| = 0. \quad (22)$$

Since $u_n^i \in T_i x_n$, we have

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0 \quad \forall i = 1, \dots, m. \quad (23)$$

This completes the proof. $\square$

We now approximate common fixed points of $\{T_1, T_2, \dots, T_m\}$ through strong convergence of the sequence $\{x_n\}$ defined by (14). We start with the following definition.

Definition 6. A family $\{T_1, \dots, T_m : K \rightarrow CB(K)\}$ is said to satisfy *Condition (I*)*, if there exists a strictly increasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and $i_0, 1 \leq i_0 \leq m$, such that

$$d(x, T_{i_0} x) \geq f(d(x, F)) \quad \forall x \in K. \quad (24)$$

Theorem 7. Let K be a nonempty, closed, and convex subset of a uniformly convex real Banach space E . For $m \geq 1$, let $T_1, T_2, \dots, T_m : K \rightarrow CB(K)$ be a multivalued quasi-nonexpansive mapping. Assume that $\{T_1, \dots, T_m\}$ satisfies condition (I*). If $F \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i, i = 1, \dots, m\}$.

Proof. From Theorem 5, we have $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, m$. Using the fact that $\{T_1, \dots, T_m\}$ satisfies condition (I*), it follows that there exists some i_0 such that $\lim_{n \rightarrow \infty} f(d(x_n, F(T_{i_0}))) = 0$. Thus there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T_{i_0})$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j} \quad \forall j \in \mathbb{N}. \quad (25)$$

By setting p_j in place of p and following the same arguments as in the proof of Lemma 4, we obtain from inequality (15) that

$$\|x_{n_{j+1}} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}. \quad (26)$$

We now show that $\{p_j\}$ is a Cauchy sequence in K . Notice that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}. \end{aligned} \quad (27)$$

This shows that $\{p_j\}$ is a Cauchy sequence in K and thus converges strongly to some $p \in K$. Using the fact that T_i is quasi-nonexpansive and $p_j \rightarrow p$, we have

$$d(p_j, T_i(p)) \leq H(T_i(p)_j, T_i(p)) \leq \|p_j - p\|, \quad (28)$$

so that $d(p, T_i(p)) = 0$ and thus $p \in T_i(p)$. Therefore, $p \in F(T_i)$ and $\{x_{n_j}\}$ converges strongly to p . Setting p_j in place of p in the proof of Lemma 4, it follows from inequality (15) that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p . This completes the proof. $\square$

Definition 8. A mapping $T : K \rightarrow CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, T x_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p \in K$. One notes that if K is compact, then every multivalued mapping $T : K \rightarrow CB(K)$ is hemicompact.

Theorem 9. Let K be a nonempty, closed, and convex subset of a uniformly convex real Banach space E . For $i = 1, \dots, m$ let $T_i : K \rightarrow CB(K)$ be a multivalued continuous and quasi-nonexpansive mapping. Assume that T_{i_0} is hemicompact for some i_0 . If $F \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ defined by (14) converges strongly to a common fixed point of $\{T_i, i = 1, \dots, m\}$.

Proof. From Theorem 5, we have that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$. Since T_{i_0} is hemicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p$ for some $p \in K$. Since for each $i = 1, 2, \dots, m$, T_i is continuous, we have $d(x_{n_j}, T_i x_{n_j}) \rightarrow d(p, T_i(p))$. Therefore, $d(p, T_i(p)) = 0$ and so $p \in F(T_i)$ for all $i = 1, 2, \dots, m$. Setting $x^* = p$ in the proof of Theorem 5, it follows from inequality (15) that the sequence $\{\|x_n - p\|\}$ is decreasing and bounded from below. Therefore, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p . This completes the proof. $\square$

The following result gives a necessary and sufficient condition for the strong convergence of the sequence in (14) to a common fixed point of a finite family of multivalued nonexpansive maps $T_i, i = 1, 2, \dots, m$.

Theorem 10. Let K be a nonempty, closed, and convex subset of a real Banach space E . For $i = 1, \dots, m$ let $T_i : K \rightarrow CB(K)$ be a multivalued nonexpansive mapping. Assume that $F \neq \emptyset$ and that $T_i(p) = \{p\}$ for all $p \in F$. Then the sequence $\{x_n\}$ defined by (14) converges strongly to a common fixed point of $\{T_i, i = 1, \dots, m\}$ if and only if $\liminf d(x_n, F) = 0$.

Proof. The necessity is obvious. Conversely, suppose that $\liminf d(x_n, F) = 0$. Let $p \in F$. By (15), we have $\|x_{n+1} - p\| \leq \|x_n - p\|$. This gives $d(x_{n+1}, F) \leq d(x_n, F)$. Hence, $\lim d(x_n, F)$ exists. By hypothesis, $\liminf d(x_n, F) = 0$ so we must have $\lim d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in K . Let $\epsilon > 0$ be given and since $\lim d(x_n, F) = 0$, there exists $N \geq 0$ such that, for all $n \geq N$, we have

$$d(x_n, F) < \frac{\epsilon}{4}. \quad (29)$$

In particular, $\inf\{\|x_N - p\| : p \in F\} < \epsilon/4$, so there exists $p \in F$ such that

$$\|x_N - p\| < \frac{\epsilon}{2}. \quad (30)$$

Now for $m, n \geq N$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_N - p\| \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned} \quad (31)$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and therefore, it converges in K . Let $\lim x_n = x^*$. Now, for each $i = 1, 2, 3, \dots, m$, we have

$$\begin{aligned} d(x^*, T_i x^*) &\leq d(x^*, x_n) + d(x_n, T_i x_n) + H(T_i x_n, T_i x^*) \\ &\leq d(p, x_n) + d(x_n, T_i x_n) + d(x_n, p). \end{aligned} \quad (32)$$

By letting $n \rightarrow +\infty$, it follows that $d(x^*, T_i x^*) = 0$, $i = 1, 2, 3, \dots, m$, which implies that $x^* \in T_i x^*$. Consequently, $x^* \in F$. $\square$

Corollary 11 (Abbas et al. [11]). *Let E be a real Banach space and D a nonempty, closed, and convex subset of E . Let T, S be multivalued nonexpansive mappings of D into $K(D)$ such that $F(T) \cap F(S) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$. Let $\{x_n\}$ be a sequence defined iteratively by*

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \quad n \geq 1, \end{aligned} \quad (33)$$

where $y_n \in Tx_n$, $z_n \in Sx_n$ are such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ whenever p is a fixed point of any one of the mappings T and S . Then, $\{x_n\}$ converges strongly to a common fixed point of $F(T) \cap F(S)$ if and only if $\liminf d(x_n, F) = 0$.

Corollary 12 (Bunyawat and Suantai [17]). *Let E be a real Banach space and K a closed convex subset of X . Let $\{T_i : i = 1, 2, \dots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then the sequence x_n defined by (12) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, m\}$ if and only if $\liminf d(x_n, F) = 0$.*

Remark 13. The recursion formula (14) used in our theorems is easier to use than the recursion formula (6) of Abbas et al., the one of Rashwan and Altwqi (9), and the one of Bunyawat and Suantai (12) in the following sense: in our algorithm, $u_n^i \in T_i x_n$ for $i = 1, \dots, m$ and do not have to satisfy the restrictive conditions: (7) in the recursion formula (6), (10) in the recursion formulas (9), and similar restrictive conditions in the recursion formula (12).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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