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Research Article

On Quasi-Pseudometric Type Spaces

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We introduce the concept of a quasi-pseudometric type space and prove some fixed point theorems. Moreover, we connect this concept to the existing notion of quasi-cone metric space.

1. Introduction

Cone metric spaces were introduced in [1] and many fixed point results concerning mappings in such spaces have been established. In [2], Khamsi connected this concept with a generalised form of metric that he named *metric type*. Recently in [3], Shadda and Md Noorani discussed the newly introduced notion of quasi-cone metric spaces and proved some fixed point results of mappings on such spaces. Basically, cone metric spaces are defined by substituting, in the definition of a metric, the real line by a real Banach space that we endowed with a partial order. The fact that the introduced order is not linear does not allow us to always compare any two elements and then gives rise to a kind of duality in the definition of the induced topology, hence the convergence in such space. We introduce a quasi-pseudometric type structure and show that some proofs follow closely the classical proofs in the quasi-pseudometric case but generalize them.

2. Preliminaries

In this section, we recall some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a *quasi-pseudometric* on X if

(i) $d(x, x) = 0$ $\forall x \in X$,

(ii) $d(x, z) \leq d(x, y) + d(y, z)$ $\forall x, y, z \in X$.

Moreover, if $d(x, y) = 0 = d(y, x) \Rightarrow x = y$, then d is said to be a T_0-*quasi-pseudometric*. The latter condition is referred to as the T_0-condition.

Remark 2. (i) Let d be a quasi-pseudometric on X; then the map d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the *conjugate* of d. In the literature, d^{-1} is also denoted by d^t or $\overline{d}$.

(ii) It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, that is, $d^s(x, y) = \max\{d(x, y), d(y, x)\}$, defines a *metric* on X whenever d is a T_0-quasi-pseudometric.

Let (X, d) be a quasi-pseudometric space. Then for each $x \in X$ and $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \quad (1)$$

denotes the open ϵ-ball at x with respect to d. It should be noted that the collection

$$\{B_d(x, \epsilon) : x \in X, \epsilon > 0\} \quad (2)$$

yields a base for the topology $\tau(d)$ induced by d on X. In a similar manner, for each $x \in X$ and $\epsilon \geq 0$, we define

$$C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}, \quad (3)$$

known as the closed ϵ-ball at x with respect to d.

Also the collection

$$\{B_{d^{-1}}(x, \epsilon) : x \in X, \epsilon > 0\} \quad (4)$$

yields a base for the topology $\tau(d^{-1})$ induced by d^{-1} on X. The set $C_d(x,\epsilon)$ is $\tau(d^{-1})$-closed but not $\tau(d)$-closed in general.

The balls with respect to d are often called *forward balls* and the topology $\tau(d)$ is called *forward topology*, while the balls with respect to d^{-1} are often called *backward balls* and the topology $\tau(d^{-1})$ is called *backward topology*.

Definition 3. Let (X,d) be a quasi-pseudometric space. The convergence of a sequence (x_n) to x with respect to $\tau(d)$, called *d-convergence* or *left-convergence* and denoted by $x_n \xrightarrow{d} x$, is defined in the following way:

$$x_n \xrightarrow{d} x \Longleftrightarrow d(x,x_n) \longrightarrow 0. \tag{5}$$

Similarly, the convergence of a sequence (x_n) to x with respect to $\tau(d^{-1})$, called *d^{-1}-convergence* or *right-convergence* and denoted by $x_n \xrightarrow{d^{-1}} x$, is defined in the following way:

$$x_n \xrightarrow{d^{-1}} x \Longleftrightarrow d(x_n,x) \longrightarrow 0. \tag{6}$$

Finally, in a quasi-pseudometric space (X,d), we will say that a sequence (x_n) *d^s-converges* to x if it is both left and right convergent to x, and we denote it as $x_n \xrightarrow{d^s} x$ or $x_n \to x$ when there is no confusion. Hence

$$x_n \xrightarrow{d^s} x \Longleftrightarrow x_n \xrightarrow{d} x, \qquad x_n \xrightarrow{d^{-1}} x. \tag{7}$$

Definition 4. A sequence (x_n) in a quasi-pseudometric (X,d) is called

(a) *left K-Cauchy* with respect to d if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,k : n_0 \le k \le n \quad d(x_k,x_n) < \epsilon; \tag{8}$$

(b) *right K-Cauchy* with respect to d if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,k : n_0 \le k \le n \quad d(x_n,x_k) < \epsilon; \tag{9}$$

(c) *d^s-Cauchy* if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,k \ge n_0 \quad d(x_n,x_k) < \epsilon. \tag{10}$$

Remark 5. (i) A sequence is left K-Cauchy with respect to d if and only if it is right K-Cauchy with respect to d^{-1}.

(ii) A sequence is d^s-Cauchy if and only if it is both left and right K-Cauchy.

Definition 6. A quasi-pseudometric space (X,d) is called *left-complete* provided that any left K-Cauchy sequence is d-convergent.

Definition 7. A quasi-pseudometric space (X,d) is called *right-complete* provided that any right K-Cauchy sequence is d-convergent.

Definition 8. A T_0-quasi-pseudometric space (X,d) is called *bicomplete* provided that the metric d^s on X is complete.

We now recall some known definitions, notations, and results concerning cones in Banach spaces.

Definition 9. Let E be a real Banach space with norm $\|\cdot\|$ and let P be a subset of E. Then P is called a cone if and only if

(1) P is closed and nonempty and $P \ne \{\theta\}$, where θ is the zero vector in E;

(2) for any $a,b \ge 0$, and $x,y \in P$, one has $ax+by \in P$;

(3) for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone P in a Banach space E, one defines on E a partial order $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P. \tag{11}$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \ne y$, while $x \ll y$ will stand for $y - x \in \operatorname{Int}(P)$ (where $\operatorname{Int}(P)$ designates the interior of P).

The cone P is called *normal* if there is a number $C > 0$, such that for all $x,y \in E$, one has

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \le C\|y\|. \tag{12}$$

The least positive number satisfying this inequality is called the *normal constant* of P. Therefore, one will then say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 10 (compare [3]). Let X be a nonempty set. Suppose the mapping $q : X \times X \to E$ satisfies

(q1) $\theta \preceq q(x,y)$ for all $x,y \in X$;

(q2) $q(x,y) = \theta = q(y,x)$ if and only if $x = y$;

(q3) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$.

Then, q is called a *quasi-cone metric* on X and (X,q) is called a *quasi-cone metric space*.

Definition 11 (compare [3]). A sequence in a quasi-cone metric space (X,q) is called

(a) *Q-Cauchy* or *bi-Cauchy* if, for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,m \ge n_0 \quad q(x_n,x_m) \ll c; \tag{13}$$

(b) *left (right) Cauchy* if, for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,m : n_0 \le m \le n \quad q(x_m,x_n) \ll c \quad (q(x_n,x_m) \ll c \text{ resp.}). \tag{14}$$

Remark 12. A sequence is Q-Cauchy if and only if it is both left and right Cauchy.

We also recall the following lemma, which we take from [4] and we give the proof as it is.

Lemma 13 (compare [4, Lemma 2]). *Let (X, q) be a cone metric space. Then for each $c \in E$, $c \gg \theta$, there exists $\sigma > 0$ such that $x \ll c$ whenever $\|x\| < \sigma$, $x \in E$.*

Proof. Since $c \gg \theta$, then $c \in \operatorname{Int}(P)$. Hence, find $\sigma > 0$ such that $\{x \in E : \|x - c\| < \sigma\} \subset \operatorname{Int}(P)$. Now if $\|x\| < \sigma$ then $\|(c-x)-c\| = \|-x\| = \|x\| < \sigma$ and hence $(c-x) \in \operatorname{Int}(P)$. □

Remark 14. Although the lemma is stated for a cone metric space, it remains valid for a quasi-cone metric space.

3. Some First Results

Definition 15. (1) In a quasi-cone metric space (X, q), one says that the sequence (x_n) *left-converges* to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that, for all $n > N$, $q(x_n, x) \ll c$.

(2) Similarly, in a quasi-cone metric space (X, q), one says that a sequence (x_n) *right- converges* to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that, for all $n > N$, $q(x, x_n) \ll c$.

(3) Finally, in a quasi-cone metric space (X, q), one says that the sequence (x_n) *converges* to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that, for all $n > N$, $q(x_n, x) \ll c$ and $q(x, x_n) \ll c$.

Definition 16. A quasi-cone metric space (X, q) is called

(1) *left-complete* (resp., *right-complete*) if every left Cauchy (resp., right Cauchy) sequence in X left (resp., right) converges,

(2) *bicomplete* if every Q-Cauchy sequence converges.

Remark 17. A quasi-cone metric space (X, q) is bicomplete if and only if it is left-complete and right-complete.

Definition 18. Let (X, q) be a quasi-cone metric space. A function $f : X \to X$ is said to be *lipschitzian* if there exists some $\kappa \in \mathbb{R}$ such that

$$q(f(x), f(y)) \preceq \kappa q(x, y), \quad \forall x, y \in X. \tag{15}$$

The smallest constant which satisfies the above inequality is called the *lipschitizian constant* of f and is denoted by $\operatorname{Lip}(f)$. In particular f is said to be *contractive* if $\operatorname{Lip}(f) \in [0,1)$ and *expansive* if $\operatorname{Lip}(f) = 1$.

Lemma 19. *Let (X, d) be a quasi-pseudometric space. If a sequence (x_n) d^s-converges to x, then it is d^s-Cauchy.*

Proof. Since (x_n) d^s-converges to x, for every $\epsilon > 0$, there exist N_1 such that $d(x, x_k) < \epsilon/2$ for any $k \geq N_1$ and N_2 such that $d(x_m, x) < \epsilon/2$ for any $m \geq N_2$. Hence for any $n, p \geq \max\{N_1, N_2\}$, $d(x_n, x_p) \leq d(x_n, x) + d(x, x_p) < \epsilon$. □

Lemma 20. *Let (X, q) be a quasi-cone metric space and P a K-normal cone. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $q(x_n, x) \to \theta$ $(n \to \infty)$ and $q(x, x_n) \to \theta$ $(n \to \infty)$.*

Proof. Suppose (x_n) converges to x. For every real $\epsilon > 0$, choose $c \in E$ with $\theta \ll c$ and $K\|c\| < \epsilon$. Then there exists $N > 0$ such that for all $n > N$ $q(x_n, x) \ll c$ and $q(x, x_n) \ll c$. This implies that when $n > N$, $\|q(x_n, x)\| \leq K\|c\| < \epsilon$ and $\|q(x, x_n)\| \leq K\|c\| < \epsilon$. This means that $q(x_n, x) \to \theta$ and $q(x, x_n) \to \theta$.

Conversely, suppose that $q(x_n, x) \to \theta$ $(n \to \infty)$ and $q(x, x_n) \to \theta$ $(n \to \infty)$. For any $c \in E$ with $\theta \ll c$, there is $\sigma > 0$ such that $\|x\| < \sigma$ implies that $x \ll c$. For this σ, there exist N_1 and N_2 such that $\|q(x_n, x)\| < \sigma$ for any $n > N_1$ and $\|q(x, x_n)\| < \sigma$ for any $n > N_2$. Hence, for $n > \max\{N_1, N_2\}$, $c - q(x_n, x) \in \operatorname{Int}(P)$ and $c - q(x, x_n) \in \operatorname{Int}(P)$. Therefore (x_n) converges to x. □

Remark 21. In fact, a sequence (x_n) left-converges (resp., right-converges) to x if and only if $q(x_n, x) \to \theta$ (resp., $q(x, x_n) \to \theta$) $(n \to \infty)$.

Lemma 22. *Let (X, q) be a quasi-cone metric space and let (x_n) be a sequence in X. If (x_n) converges to x, then (x_n) is a bi-Cauchy sequence.*

Proof. For any $c \in E$ with $\theta \ll c$, there exists $N > 0$ such that, for all $m, n > N$, $q(x_n, x) \ll c/2$ and $q(x, x_m) \ll c/2$. Hence

$$q(x_n, x_m) \preceq q(x_n, x) + q(x, x_m) \ll c. \tag{16}$$

Therefore, (x_n) is a bi-Cauchy sequence. □

Lemma 23. *Let (X, q) be a quasi-cone metric space, P a K-normal cone, and (x_n) a sequence in X. Then (x_n) is a bi-Cauchy sequence if and only if $q(x_n, x_m) \to \theta$ as $n, m \to \infty$.*

Proof. Suppose that (x_n) is a bi-Cauchy sequence. For every real $\epsilon > 0$, choose $c \in E$ with $\theta \ll c$ and $K\|c\| < \epsilon$. Then there exists N such that, for all $n, m > N$, $q(x_n, x_m) \ll c$. Therefore, whenever $n, m > N$, $\|q(x_n, x_m)\| \leq K\|c\| < \epsilon$. This means that $q(x_n, x_m) \to \theta$ as $n, m \to \infty$.

Conversely, suppose that $q(x_n, x_m) \to \theta$ as $n, m \to \infty$. For any $c \in E$ with $\theta \ll c$, there is $\sigma > 0$ such that $\|x\| < \sigma$ implies that $x \ll c$. For this σ, there exist N such that $\|q(x_n, x_m)\| < \sigma$ for any $n, m > N_1$. Hence $c - q(x_n, x_m) \in \operatorname{Int}(P)$. Therefore (x_n) is a bi-Cauchy sequence. □

4. First Fixed Points Results

Theorem 24. *Let (X, q) be a bicomplete quasi-cone metric space and P a K-normal cone. Suppose that a mapping $T : X \to X$ satisfies the contractive condition*

$$q(Tx, Ty) \preceq kq(x, y) \quad \forall x, y \in X, \tag{17}$$

where $k \in [0, 1)$. Then T has a unique fixed point. Moreover for any $x \in X$, the orbit $\{T^n x, n \geq 0\}$ converges to the fixed point.

Proof. Take an arbitrary $x_0 \in X$ and denote $x_n = T^n x_0$. Then

$$\begin{aligned} q(x_n, x_{n+1}) &= q(Tx_{n-1}, Tx_n) \preceq kq(x_{n-1}, x_n) \\ &\preceq k^2 q(x_{n-2}, x_{n-1}) \preceq \cdots \preceq k^n q(x_0, x_1). \end{aligned} \tag{18}$$

Similarly,

$$q(x_{n+1}, x_n) \preceq k^n q(x_1, x_0). \quad (19)$$

So for $n < m$,

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{k^n}{1-k} q(x_0, x_1). \end{aligned} \quad (20)$$

It entails that $\|q(x_n, x_m)\| \le K(k^n/(1-k))\|q(x_0, x_1)\| \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Similarly for $n > m$

$$q(x_n, x_m) \preceq \frac{k^n}{1-k} q(x_1, x_0). \quad (21)$$

It entails that $\|q(x_n, x_m)\| \le K(k^n/(1-k))\|q(x_1, x_0)\| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence (x_n) is a bi-Cauchy sequence. Since (X, q) is bicomplete, there exists $x^* \in X$ such that (x_n) converges to x^*.

Moreover since

$$\begin{aligned} q(Tx^*, x^*) &\preceq q(Tx^*, Tx_n) + q(Tx_n, x^*) \\ &\preceq kq(x^*, x_n) + q(x_{n+1}, x^*), \\ q(x^*, Tx^*) &\preceq q(x^*, Tx_n) + q(Tx_n, Tx^*) \\ &\preceq q(x^*, x_{n+1}) + kq(x_n, x^*), \end{aligned} \quad (22)$$

we have that

$$\begin{aligned} \|q(Tx^*, x^*)\| &\le K(k\|q(x^*, x_n)\| + \|q(x_{n+1}, x^*)\|) \longrightarrow 0, \\ \|q(x^*, Tx^*)\| &\le K(k\|q(x_n, x^*)\| + \|q(x^*, x_{n+1})\|) \longrightarrow 0. \end{aligned} \quad (23)$$

Hence $\|q(Tx^*, x^*)\| = 0 = \|q(x^*, Tx^*)\|$. This implies, using property (q2), that $Tx^* = x^*$. So x^* is a fixed point.

If z^* is another fixed point of T, then

$$\begin{aligned} q(x^*, z^*) &= q(Tx^*, Tz^*) \preceq kq(x^*, z^*), \\ q(z^*, x^*) &= q(Tz^*, Tx^*) \preceq kq(z^*, x^*). \end{aligned} \quad (24)$$

Hence, $\|q(x^*, z^*)\| = 0 = \|q(z^*, x^*)\|$ and $x^* = z^*$. Therefore the fixed point is unique. □

Corollary 25. *Let (X, q) be a bicomplete quasi-cone metric space and P a K-normal cone. For $c \in E$ with $0 \ll c$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X : q(x_0, x) \preceq c\}$. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition*

$$q(Tx, Ty) \preceq kq(x, y), \quad \forall x, y \in B(x_0, c), \quad (25)$$

where $k \in [0, 1)$ is a constant and $q(x_0, Tx_0) \preceq (1-k)c$. Then T has a unique fixed point in $B(x_0, c)$.

Proof. We only need to prove that $B(x_0, c)$ is bicomplete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$.

Suppose (x_n) is a bi-Cauchy sequence in $B(x_0, c)$. Then (x_n) is also a bi-Cauchy sequence in X. By the bicompleteness of X, there is $x \in X$ such that (x_n) converges to x. We have

$$q(x_0, x) \preceq q(x_0, x_n) + q(x_n, x) \preceq q(x_n, x) + c. \quad (26)$$

Since (x_n) converges to x, $q(x_n, x) \longrightarrow \theta$. Hence $q(x_0, x) \preceq c$ and $x \in B(x_0, c)$. Therefore, $B(x_0, c)$ is bicomplete.

For every $x \in B(x_0, c)$,

$$\begin{aligned} q(x_0, Tx) &\preceq q(x_0, Tx_0) + q(Tx_0, Tx) \\ &\preceq (1-k)c + kq(x_0, x) \preceq (1-k)c + kc = c. \end{aligned} \quad (27)$$

Hence $Tx \in B(x_0, c)$. □

Remark 26. A weaker version of this corollary is actually sufficient. Indeed, it is enough to consider (X, q) as a left-complete quasi-cone metric space with the same assumption. In this case, we would just have to prove that $B(x_0, c)$ is left-complete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$.

Corollary 27. *Let (X, q) be a bicomplete quasi-cone metric space and P a K-normal cone. Suppose a mapping $T : X \longrightarrow X$ satisfies for some positive integer n,*

$$q(T^n x, T^n y) \preceq kq(x, y), \quad \forall x, y \in X, \quad (28)$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. From Theorem 24, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$, so Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is unique. □

Theorem 28. *Let (X, q) be a quasi-cone metric space over the Banach space E with the K-normal cone P. The mapping $Q : X \times X \longrightarrow [0, \infty)$ defined by $Q(x, y) = \|q(x, y)\|$ satisfies the following properties:*

(Q1) $Q(x, x) = 0$ for any $x \in X$;

(Q2) $Q(x, y) \le K(Q(x, z_1) + Q(z_1, z_2) + \cdots + Q(z_n, y))$, for any points $x, y, z_i \in X$, $i = 1, 2, \ldots, n$.

Proof. The proof of (Q1) is immediate by property (q2) of the quasi-cone metric. In order to prove (Q2), consider $x, y, z_1, \ldots, z_n$ as points in X. Using property (q3), we get

$$q(x, y) \preceq (q(x, z_1) + q(z_1, z_2) + \cdots + q(z_n, y)). \quad (29)$$

Since P is K-normal

$$\|q(x, y)\| \le K(\|q(x, z_1) + q(z_1, z_2) + \cdots + q(z_n, y)\|), \quad (30)$$

which implies that

$$\|q(x, y)\| \le K(\|q(x, z_1)\| + \|q(z_1, z_2)\| + \cdots + \|q(z_n, y)\|). \quad (31)$$

This completes the proof. □

We are therefore led to the following definition.

Definition 29. Let X be a nonempty set, and let the function $D : X \times X \to [0, \infty)$ satisfy the following properties:

(D1) $D(x, x) = 0$ for any $x \in X$;

(D2) $D(x, y) \leq \alpha(D(x, z_1) + D(z_1, z_2) + \cdots + D(x_n, y))$ for any points $x, y, z_i \in X$, $i = 1, 2, \ldots, n$ and some constant $\alpha > 0$.

Then (X, D, α) is called a quasi-pseudometric type space. Moreover, if $D(x, y) = 0 = D(y, x) \Rightarrow x = y$, then D is said to be a T_0-*quasi-pseudometric type space*. The latter condition is referred to as the T_0-condition.

Remark 30. (i) Let D be a quasi-pseudometric type on X; then the map D^{-1} defined by $D^{-1}(x, y) = D(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric type on X, called the conjugate of D. We will also denote D^{-1} by D^t or $\overline{D}$.

(ii) It is easy to verify that the function D^s defined by $D^s := D \vee D^{-1}$, that is, $D^s(x, y) = \max\{D(x, y), D(y, x)\}$, defines a *metric-type* (see [2]) on X whenever D is a T_0-quasi-pseudometric type.

(iii) If we substitute the property (D1) by the following property,

(D3) $D(x, y) = 0 \Leftrightarrow x = y$,

we obtain a T_0-quasi-pseudometric type space directly. For instance, this could be done if the map D is obtained from quasi-cone metric.

The concepts of *left K-Cauchy*, *right K-Cauchy*, D^s-*Cauchy,* and *convergence* for a quasi-pseudometric type space are defined in a similar way as defined for a quasi-pseudometric space. Moreover, for $\alpha = 1$, we recover the classical quasi-pseudometric; hence quasi-pseudometric type generalizes quasi-pseudometric.

Definition 31. A quasi-pseudometric type space (X, D, α) is called *left-complete* provided that any left K-Cauchy sequence is D-convergent.

Definition 32. A T_0-quasi-pseudometric type space (X, D, α) is called *bicomplete* provided that the metric type space (X, D^s) is complete.

Definition 33. Let (X, D, α) be a quasi-pseudometric type space. A function $f : X \to X$ is called *lipschitzian* if there exists some $\lambda \geq 0$ such that

$$D(fx, fy) \leq \lambda D(x, y) \quad \forall x, y \in X. \tag{32}$$

The smallest constant λ will be denoted by $\mathrm{Lip}(f)$.

Definition 34. Let (X, D, α) be a quasi-pseudometric type space. A function $f : X \to X$ is called *D-sequentially continuous* if, for any D-convergent sequence (x_n) with $x_n \xrightarrow{D} x$, the sequence (fx_n) D-converges to fx; that is, $(fx_n) \xrightarrow{D} fx$.

5. Some Fixed Point Results

In [2], Khamsi proved the following.

Theorem 35. *Let (X, d) be a complete metric type space. Let $T : (X, d) \to (X, d)$ be a map such that T^n is lipschitzian for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$. Then T has a unique fixed point $\omega \in X$. Moreover for any $x \in X$, the orbit $\{T^n x, n \geq 0\}$ converges to ω.*

We state here an analogue of Khamsi's theorem.

Theorem 36. *Let (X, D, α) be a bicomplete quasi-pseudometric type. Let $T : (X, D, \alpha) \to (X, D, \alpha)$ be a map such that T^n is lipschitzian for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$. Then T has a unique fixed point $\omega \in X$. Moreover for any $x \in X$, the orbit $\{T^n x, n \geq 0\}$ converges to ω.*

Proof. We just have to prove that $T : (X, D^s) \to (X, D^s)$ is a map such that T^n is lipschitzian for all $n \geq 0$.

Indeed, since $T : (X, D, \alpha) \to (X, D, \alpha)$ is a map such that T^n is lipschitzian for all $n \geq 0$, then

$$D(T^n x, T^n y) \leq \mathrm{Lip}(T^n) D(x, y) \quad \forall x, y \in X. \tag{33}$$

Since for any $x, y \in X$, we have

$$D^{-1}(T^n x, T^n y) = D(T^n y, T^n x) \leq \mathrm{Lip}(T^n) D(y, x) \quad \forall n \geq 0, \tag{34}$$

that is,

$$D^{-1}(T^n x, T^n y) \leq \mathrm{Lip}(T^n) D^{-1}(x, y), \tag{35}$$

we see that $T : (X, D^{-1}, \alpha) \to (X, D^{-1}, \alpha)$ is a map such that T^n is lipschitzian for all $n \geq 0$.

Therefore

$$\begin{gathered} D(T^n x, T^n y) \leq \mathrm{Lip}(T^n) D(x, y) \leq \mathrm{Lip}(T^n) D^s(x, y), \\ D^{-1}(T^n x, T^n y) \leq \mathrm{Lip}(T^n) D^{-1}(x, y) \leq \mathrm{Lip}(T^n) D^s(x, y), \end{gathered} \tag{36}$$

for all $x, y \in X$ and for all $n \geq 0$. Hence

$$D^s(T^n x, T^n y) \leq \mathrm{Lip}(T^n) D^s(x, y), \tag{37}$$

for all $x, y \in X$ and for all $n \geq 0$, so, $T : (X, D^s) \to (X, D^s)$ is a map such that T^n is lipschitzian for all $n \geq 0$.

By assumption, (X, D, α) is bicomplete; hence (X, D^s) is complete. Therefore, by Theorem 35, T has a unique fixed point $\omega \in X$ and for any $x \in X$, the orbit $\{T^n x, n \geq 0\}$ converges to ω. □

The connection between a quasi-cone metric space and a quasi-pseudometric type space is given by the following corollary.

Corollary 37. *Let (X, q) be a bicomplete quasi-cone metric space over the Banach space E with the K-normal cone P. Consider $Q : X \times X \to [0; \infty)$ defined by $Q(x, y) = \|q(x, y)\|$. Let $T : X \to X$ be a contraction with constant $0 < \kappa < 1$. Then*

$$Q(T^n x, T^n y) \leq K\kappa^n Q(x, y), \tag{38}$$

for any $x, y \in X$ and $n \geq 0$. Hence $Lip(T^n) \leq K\kappa^n$, for any $n \geq 0$. Therefore $\sum_{n=0}^{\infty} Lip(T^n)$ is convergent, which implies that T has a fixed point ω and any orbit converges to ω.

Proof. It is enough to prove that the metric type space (X, Q^s) is complete. Let (x_n) be a Q^s-Cauchy sequence. Therefore $\lim_{n,m\to\infty} Q^s(x_n, x_m) = 0$, which implies that the sequence (x_n) is bi-Cauchy in (X, q). Since (X, q) is bicomplete, there exists $x^* \in X$ such that $q(x_n, x^*) \to \theta$ and $q(x^*, x_n) \to \theta$. Hence $x_n \xrightarrow{Q^s} x^*$.

Moreover, since T is a contraction with constant κ, we have that

$$q(T^n x, T^n y) \leq \kappa q(T^{n-1}x, T^{n-1}y) \leq \cdots \leq \kappa^n q(x, y) \quad \text{for any } x, y \in X, \quad n \geq 0. \tag{39}$$

Hence $\mathrm{Lip}(T^n) \leq K\kappa^n$, for any $n \geq 0$. ☐

6. More Fixed Point Results

We begin with the following lemmas.

Lemma 38. *Let (y_n) be a sequence in a quasi-pseudometric type space (X, D, α) such that*

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n), \tag{40}$$

for some $\lambda > 0$ with $\lambda < 1/\alpha$. Then (y_n) is left K-Cauchy.

Proof. Let $m < n \in \mathbb{N}$. From the condition (Qb) in the definition of a quasi-pseudometric type, we can write

$$\begin{aligned} D(y_m, y_n) &\leq \alpha(D(y_m, y_{m+1}) + D(y_{m+1}, y_n)) \\ &\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) \\ &\quad + D(y_{m+2}, y_n) \\ &\ \ \vdots \\ &\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) + \cdots \\ &\quad + \alpha^{n-m-1} D(y_{n-2}, y_{n-1}) + \alpha^{n-m} D(y_{n-1}, y_n). \end{aligned} \tag{41}$$

From (40) and $\lambda < 1/\alpha$, the above becomes

$$\begin{aligned} D(y_m, y_n) &\leq (\alpha\lambda^m + \alpha^2\lambda^{m+1} + \cdots + \alpha^{n-m+1}\lambda^{n-1}) D(y_0, y_1) \\ &\leq \alpha\lambda^m (1 + \alpha\lambda + \cdots + (\alpha\lambda)^{n-1}) D(y_0, y_1) \\ &\leq \frac{\alpha\lambda^m}{1 - \alpha\lambda} D(y_0, y_1) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{42}$$

It follows that (y_n) is left K-Cauchy. ☐

Similarly, we have the following.

Lemma 39. *Let (y_n) be a sequence in a quasi-pseudometric type space (X, D, α) such that*

$$D^{-1}(y_n, y_{n+1}) \leq \lambda D^{-1}(y_{n-1}, y_n), \tag{43}$$

for some $\lambda > 0$ with $\lambda < 1/\alpha$. Then (y_n) is right K-Cauchy.

Theorem 40. *Let (X, D, α) be a Hausdorff left-complete T_0-quasi-pseudometric type space, and let $f : X \to X$ be a D-sequentially continuous function such that for some $\lambda > 0$ with $\lambda/(1 - \lambda) < 1/\alpha$,*

$$D(fx, fy) \leq \lambda(D(x, fx) + D(y, fy)) \quad \forall x, y, z \in X. \tag{44}$$

Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $(f^n(x_0))$ D-converges to z.

Proof. Take an arbitrary $x_0 \in X$ and denote $y_n = f^n(x_0)$. Then

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fy_{n-1}, fy_n) \\ &\leq \lambda(D(y_{n-1}, fy_{n-1}) + D(y_n, fy_n)) \\ &\leq \lambda(D(y_{n-1}, y_n) + D(y_n, y_{n+1})), \end{aligned} \tag{45}$$

which implies that

$$D(y_n, y_{n+1}) \leq \frac{\lambda}{1 - \lambda} D(y_{n-1}, y_n). \tag{46}$$

Hence, since $\lambda/(1 - \lambda) < 1/\alpha$, by Lemma 38 we have that (y_n) is left K-Cauchy and since (X, D, α) is left-complete and f D-sequentially continuous, there exists y^* such that $y_n \xrightarrow{D} y^*$ and $y_{n+1} \xrightarrow{D} fy^*$. Since X is Hauforff, the limit is unique, hence $y^* = fy^*$.

For uniqueness, assume by contradiction that there exists another fixed point z^*. Then

$$\begin{aligned} D(y^*, z^*) &= D(fy^*, fz^*) \leq \lambda(D(y^*, fy^*) + D(z^*, fz^*)) \\ &= 0, \\ D(z^*, y^*) &= D(fz^*, fy^*) \leq \lambda(D(z^*, fz^*) + D(y^*, fy^*)) \\ &= 0. \end{aligned} \tag{47}$$

Hence $D(y^*, z^*) = 0 = D(z^*, y^*)$ and using the T_0-condition, we conclude that $y^* = z^*$. ☐

Theorem 41. *Let (X, D, α) be a Hausdorff left-complete T_0-quasi-pseudometric type space, and let $f : X \to X$ be a D-sequentially continuous function such that for some $\lambda > 0$ with $\lambda\alpha^2/(1 - \lambda\alpha) < 1$,*

$$D(fx, fy) \leq \lambda(D(fx, y) + D(x, fy)) \quad \forall x, y \in X. \tag{48}$$

Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $(f^n(x_0))$ D-converges to z.

Proof. Take an arbitrary $x_0 \in X$ and denote $y_n = f^n(x_0)$. Then

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fy_{n-1}, fy_n) \\ &\leq \lambda (D(fy_{n-1}, y_n) + D(y_{n-1}, fy_n)) \\ &\leq \lambda D(y_{n-1}, y_{n+1}) \\ &\leq \lambda\alpha (D(y_{n-1}, y_n) + D(y_n, y_{n+1})), \end{aligned} \tag{49}$$

which implies that

$$D(y_n, y_{n+1}) \leq \frac{\lambda\alpha}{1-\lambda\alpha} D(y_{n-1}, y_n). \tag{50}$$

Hence, since $\lambda\alpha^2/(1-\lambda\alpha) < 1/\alpha$, by Lemma 38, we have that (y_n) is left K-Cauchy and since (X, D, α) is left-complete and f D-sequentially continuous, there exists y^* such that $y_n \xrightarrow{D} y^*$ and $y_{n+1} \xrightarrow{D} fy^*$. Since X is Hauforff, the limit is unique, hence $y^* = fy^*$.

For uniqueness, assume by contradiction that there exists another fixed point z^*. Then

$$\begin{aligned} D(y^*, z^*) &= D(fy^*, fz^*) \leq \lambda (D(fy^*, z^*) + D(y^*, fz^*)), \\ D(z^*, y^*) &= D(fz^*, fy^*) \leq \lambda (D(fz^*, y^*) + D(z^*, fz^*)). \end{aligned} \tag{51}$$

Hence $D(y^*, z^*) = 0 = D(z^*, y^*)$ and using the T_0-condition, we conclude that $y^* = z^*$. □

Theorem 42. *Let (X, D, α) be a Hausdorff left-complete T_0-quasi-pseudometric type space and let $f : X \to X$ be a D-sequentially continuous function such that for some $\lambda > 0$ with $\lambda < 1/\alpha < 1/\alpha$ and any $\gamma > 0$,*

$$D(fx, fy) \leq \lambda D(x, y) + \gamma D(fx, y) \quad \forall x, y \in X. \tag{52}$$

Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $(f^n(x_0))$ D-converges to z.

Corollary 43. *Let (X, D, α) be a Hausdorff left-complete T_0-quasi-pseudometric type space and let $f : X \to X$ be a D-sequentially continuous function such that for some $\lambda_1, \lambda_3, \lambda_4, \lambda_5 > 0$ with $(\lambda_1 + \lambda_3 + \alpha\lambda_5)/(1 - \lambda_4 - \alpha\lambda_5) < 1/\alpha$ and any $\lambda_2 > 0$*

$$\begin{aligned} D(fx, fy) \leq{}& \lambda_1 D(x, y) + \lambda_2 D(fx, y) + \lambda_3 D(x, fx) \\ &+ \lambda_4 D(y, fy) + \lambda_5 D(x, fy), \end{aligned} \tag{53}$$

for all $x, y \in X$. Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $(f^n(x_0))$ D-converges to z.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[2] M. A. Khamsi, "Remarks on cone metric spaces and fixed point theorems of contractive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 315398, 7 pages, 2010.

[3] F. Shadda and M. S. Md Noorani, "Fixed point results in quasi-cone metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 303626, 7 pages, 2013.

[4] D. Turkoglu and M. Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings," *Acta Mathematica Sinica*, vol. 26, no. 3, pp. 489–496, 2010.