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Research Article

On Second Order Gap Balancing Numbers

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We consider the Diophantine equation $1^k + 2^k + \cdots + (x-1)^k = (x+2)^k + (x+3)^k + \cdots + (x+r)^k$ for some natural numbers x, k, and r, and we call $2x+1$ as kth order 2-gap balancing number. It was also proved that there are infinitely many first order 2-gap balancing numbers. In this paper, we show that the only second order 2-gap balancing number is 1.

1. Introduction

In [1], Finkelstein defined mth power numerical center x for y as solutions of the Diophantine equation:

$$1^m + 2^m + \cdots + x^m = x^m + (x+1)^m + (x+2)^m + \cdots + y^m. \tag{1}$$

For $m = 1$, it coincides with the notion of balancing numbers introduced by Behera and Panda [2]. Finkelstein conjectured that if $m > 1$ then there is no integer greater than 1 with mth power numerical center. Ingram in [3] proved Finkelstein's conjecture for $m = 5$. Further, in [4] Panda studied (1) slightly differently and called the solution x of (1) as mth order balancing number.

The concept of gap balancing numbers was introduced by Panda and Rout [5] in connection with the Diophantine equation:

$$1 + 2 + \cdots + (x-1) = (x+2) + (x+3) + \cdots + y. \tag{2}$$

They call $2x+1$ a 2-gap balancing number or g_2 balancing number for some y. Motivated by higher order balancing numbers [4] and 2-gap balancing numbers [5], we introduce higher order k-gap balancing number as follows.

Let k be the fixed odd positive integer. We call the positive integer x an mth order k-gap balancing number if

$$1^m + 2^m + \cdots + \left(x - \frac{k+1}{2}\right)^m = \left(x + \frac{k+1}{2}\right)^m + \left(x + \frac{k+3}{2}\right)^m + \cdots + y^m. \tag{3}$$

Equation (3) is equivalent to (1) when $k = 1$. Similarly for fixed even positive integer k, we call the positive integer $2x+1$ an mth order k-gap balancing number if

$$1^m + 2^m + \cdots + \left(x - \frac{k}{2}\right)^m = \left(x + \frac{k}{2} + 1\right)^m + \left(x + \frac{k+3}{2}\right)^m + \cdots + y^m. \tag{4}$$

In this paper, we prove the following theorem.

Theorem 1. *The only positive integer possessing second order 2-gap balancing number is 1.*

2. Background

Before we prove the main result of this paper, it is better to look into the special cases corresponding to $m = 1$. For $m = 1$ and $k = 2$ (4) is equivalent to (2). Further, (2) reduces to the Diophantine equation $2x^2 + 7 = y^2$ which again reduces to

the Pell's equation $8z^2 + 1 = w^2$, ensuring infinitude of the first order 2-gap balancing numbers. Now consider the case for $m = 1$ and k arbitrary. Like the previous case (4) reduces to the Pell like equations $2x^2 + 2k^2 - 1 = y^2$ for even k and $8x^2 + 2k^2 - 1 = y^2$ for odd k which also shows infinitude of solutions of first order k-gap balancing numbers.

To solve (4) for $m = 2$ and $k = 2$, we need the following results.

Theorem 2 (see [6]). *Let a, b, c be nonzero integers. Then the equation $ax^3 + by^3 = c$ has only finitely many solutions in integers (x, y).*

Theorem 3 (see [7]). *Let $K(\rho)$ be a cubic field over the field of rational numbers, and let $\alpha = A\rho^2 + B\rho + C$ be an integer in the ring $(1, \rho, \rho^2)$. Suppose $A \equiv B \equiv 0 \pmod{p^k}$, where p is an odd rational prime, and $(\alpha, p) = 1$. Further, suppose that $tA + sB \not\equiv 0 \pmod{p^{2k}}$, where t, s, and k are rational integers, and $k > 0$. Then if $\alpha^n = A_n\rho^2 + B_n\rho + C_n$, $tA_n + sB_n$ is never zero for any $n \neq 0$.*

Theorem 4 (see [8]). *The Diophantine equation $Ax^3 + By^3 = C$ ($C = 1$ or 3; $3 \nmid AB$ if $C = 3$; $A > B$; A, B positive integers) has at most one solution in nonzero integers (x, y). There is the unique exception for the equation $2x^3 + y^3 = 3$ which has exactly two integral solutions $(x, y) = (1, 1)$ and $(x, y) = (4, -5)$.*

Theorem 5 (see [9]). *If x and y are integers, then*

$$\begin{aligned} \left|x^3 - 2y^3\right| &\geqslant \max\{|x|, |y|\}^{0.53}, \\ \left|x^3 - 3y^3\right| &\geqslant \max\{|x|, |y|\}^{0.24}, \\ \left|x^3 - 6y^3\right| &\geqslant \max\{|x|, |y|\}^{0.65}, \end{aligned} \tag{5}$$

where the last inequality holds unless $|x| = 467$ and $|y| = 257$.

Since we are dealing with Diophantine equations of degree three, we need to discuss cubic field $K(\theta)$, where $\theta^3 = 2$ (see [10]). The necessary information for our problem is as follows.

(1) The integers of $K(\theta)$ are of the form $\alpha = A + B\theta + C\theta^2$, where A, B, and C are rational integers.

(2) The ring of integers of $K(\theta)$ is a unique factorization domain.

(3) By Dirichlet's theorem on units, there is only one fundamental unit of the field, which we designate by ϵ_0 of $K(\theta)$, with $0 < \epsilon_0 < 1$, is given by

$$\epsilon_0 = -1 + \theta. \tag{6}$$

All the units of the field are given by $\pm\epsilon_0^m$, where m is any rational integer. Any such power of ϵ_0 is of the form $a+b\theta+c\theta^2$, where a, b, and c are rational integers. Norm of $\alpha = A + B\theta + C\theta^2$ is given by $N(\alpha) = A^3 + 2B^3 + 4C^3 - 6ABC$. All units of norm 1 in $K(\theta)$ is given by ϵ_0^m.

First, we have to find the number of equivalence classes of associated primes of norm 3, 5, and 71.

Since $x^3 - 2 \equiv (x-2)^3 \pmod 3$, 3 is a perfect cube in $K(\theta)$, apart from unit factors. So

$$3 = (\theta + 1)^3 (\theta - 1), \tag{7}$$

and 3 is the cube of a prime of norm 3 times an unit factor. Hence, there is only one equivalence class of associated primes of norm 3 in $K(\theta)$, as any integer of norm 3 in $K(\theta)$ must divide 3, apart from unit factors and there is only one such integer.

Furthermore, 5 is a rational prime of the form $3r + 2$. So 5 can split into two primes in $K(\theta)$. That is

$$5 = \left(-3 + 2\theta^2\right)\left(9 + 8\theta + 6\theta^2\right), \tag{8}$$

where the norm of first factor is 5 and norm of second factor is 25. Hence there is only one equivalence class of associated primes of norm 5 in $K(\theta)$, as any integer in $K(\theta)$ with norm 5 must divide 5, and apart from unit factors, there is only one such integer.

Lastly, since 71 is a rational prime of the form $3r + 2$, it splits into two primes in $K(\theta)$:

$$71 = (5 - 3\theta)\left(25 + 15\theta + 9\theta^2\right). \tag{9}$$

Thus the norm of first factor of 71 is 71 and the norm of second factor is 5041. Hence there is only one equivalence class of associated primes of norm 71 in $K(\theta)$.

3. Proof of Theorem 1

Let x be a second order 2-gap balancing number. Equation (4) with $m = 2$ and $k = 2$, simplifies to

$$\begin{aligned} &\frac{(x-1)\,x\,(2x-1)}{6} \\ &\quad = \frac{y(y+1)(2y+1)}{6} - \frac{(x+1)(x+2)(2x+3)}{6}. \end{aligned} \tag{10}$$

Simplification of the above equation gives

$$2(2x+1)^3 + 11(2x+1) = (2y+1)^3 - (2y+1). \tag{11}$$

Setting $A = -(2x+1)$ and $B = (2y+1)$, we get

$$-2\left[A^3 + 11A\right] = B^3 - B. \tag{12}$$

We shall now prove several lemmas which together imply that the only solution of (12) subject to the conditions

$$A \text{ is negative and odd}, \qquad B \text{ is positive and odd} \tag{13}$$

is $(A, B) = (-3, 5)$.

Lemma 6. *All the integral solutions of* (12) *satisfying the condition* (13) *correspond to the integral solutions of the equations:*

$$2u^3 + v^3 = 5, 15, 25, 71, 75, 213, 355, 1065, 1775, \text{ and } 5325. \tag{14}$$

Proof. Let (A, B) be any integral solution of (12) subject to the conditions (13). Let $\gcd(A, B) = d$. Letting $A = du$ and $B = dv$ and substituting in (12), we get

$$d^2\left(v^3 + 2u^3\right) = v - 22u, \tag{15}$$

where

$$\left.\begin{array}{l} \gcd(u, v) = 1, \\ u \text{ is negative and odd}, \quad v \text{ is positive and odd} \end{array}\right\}. \tag{16}$$

Let

$$2u^3 + v^3 = c. \tag{17}$$

Then $v - 22u = cd^2$ or equivalently,

$$v = 22u + cd^2. \tag{18}$$

Substituting (18) in (17) we get

$$d^6c^3 + 66uc^2d^4 + 1452u^2d^2c + 10650u^3 = c \tag{19}$$

and using (17) and (19), we obtain

$$d^6c^3 + 66uc^2d^4 + 1452u^2d^2c + 5324c = 5325v^3. \tag{20}$$

Therfore $c \mid 5325v^3$, hence $c \mid 5325$ as $\gcd(c, v) = 1$. So the possible values of c are 1, 3, 5, 15, 25, 71, 75, 213, 355, 1065, 1775, and 5325. Thus, solving (12) subject to (13) can be reduced to solving the set of equations:

$$2u^3 + v^3 = 1, 3, 5, 15, 25, 71, 75, 213, 355, 1065, 1775, \text{ and } 5325. \tag{21}$$

However, the equation $2u^3 + v^3 = 1, 3$ has solutions $(u, v) = (1, -1), (1, 1), (4, -5)$ by Theorem 4 which violates the condition in (16). Hence Lemma 6 follows. □

Lemma 7. *The equation $2u^3 + v^3 = 5$ is impossible in rational integers (u, v).*

Proof. We consider the integer $-3 + 2\theta^2$ whose norm is 5 and any other integer of norm 5 in $K(\theta)$ must be of the form $(-3 + 2\theta^2)(\epsilon_0^m)$, as all primes of norm 5 in $K(\theta)$ are associated. Let $\epsilon_0^m = a_m + b_m\theta + c_m\theta^2$ and

$$X_m + Y_m\theta + Z_m\theta^2 = \left(-3 + 2\theta^2\right)\left(a_m + b_m\theta + c_m\theta^2\right). \tag{22}$$

Hence

$$Z_m = -3c_m + 2a_m. \tag{23}$$

We seek all integers of $K(\theta)$ with norm 5 of the form $a + b\theta$. Hence Z_m must be zero, and thus the congruence $Z_m \equiv 0 \pmod{t}$ must be solvable for every modulus t. We shall show that

$$Z_m \not\equiv 0 \pmod{7}. \tag{24}$$

Therefore we only need to check $Z_i \not\equiv 0 \pmod 7$ for $i = 0, 1, \ldots, 6$. Using (23), we have the values of Z_0 to Z_6 modulo 7 as 2, −2, −1, 4, 3, −1, and −2 and none of these is zero. This completes the proof of Lemma 7. □

Lemma 8. *The only integral solution of the equation $2u^3 + v^3 = 71$ is $(u, v) = (-3, 5)$.*

Proof. We seek all the integers of $K(\theta)$ which are of the form $a + b\theta$. Since all primes of norm 71 in $K(\theta)$ are associated, any such prime must be an associate of $5 - 3\theta$. Let $\epsilon_0^m = a_m + b_m\theta + c_m\theta^2$ be a unit of $K(\theta)$. Our requirement is that the coefficient of θ^2 in

$$(5 - 3\theta)\left(a_m + b_m\theta + c_m\theta^2\right) \tag{25}$$

be zero. This gives

$$5c_m - 3b_m = 0 \tag{26}$$

We claim that (26) is impossible for $m \neq 0$. Now let

$$x_m + y_m\theta + z_m\theta^2 = (5 - 3\theta)\left(\epsilon_0^m\right). \tag{27}$$

Since

$$\epsilon_0^3 + 3\epsilon_0^2 + 3\epsilon_0 - 1 = 0, \tag{28}$$

we have $\epsilon_0^{m+3} + 3\epsilon_0^{m+2} + 3\epsilon_0^{m+1} - \epsilon_0^m = 0$, from which it follows that

$$z_{m+3} + 3z_{m+2} + 3z_{m+1} - z_m = 0. \tag{29}$$

Since from (27)

$$x_m + y_m\theta + z_m\theta^2 = (5 - 3\theta)\left(a_m + b_m\theta + c_m\theta^2\right), \tag{30}$$

we have $z_m = 5c_m - 3b_m$ by (26). Also from (29)

$$z_{m+3} \equiv z_m \pmod 3. \tag{31}$$

Now $\epsilon_0^3 = 1 + 3\theta - 3\theta^2$ which shows that

$$\begin{aligned} b_3 &\equiv c_3 \equiv 0 \pmod 3, \\ z_3 &\equiv 5c_3 - 3b_3 \equiv 3 \pmod 9. \end{aligned} \tag{32}$$

Hence by Theorem 3, z_m is never zero for any $m \neq 0$ which completes the proof of the lemma. □

Lemma 9. *The only integral solution of the equation $2u^3 + v^3 = 25$ is $(u, v) = (-1, 3)$.*

Proof. In this case we seek all integers of the form $a + b\theta$ with norm 25. Here we employ the same method of proof as in Lemma 8. Observe that

$$25 = (3 - \theta)\left(9 + 3\theta + \theta^2\right). \tag{33}$$

We need to show that the coefficient of θ^2 in

$$(3 - \theta)\left(a_m + b_m\theta + c_m\theta^2\right) \tag{34}$$

is zero. That is

$$3c_m - b_m = 0 \tag{35}$$

we claim that (35) is never zero. As in previous lemma

$$\begin{aligned} b_3 &\equiv c_3 \equiv 0 \pmod 3, \\ z_3 &\equiv 3c_3 - b_3 \equiv 3 \pmod 9 \end{aligned} \tag{36}$$

Hence by Theorem 3, z_m is never zero for any $m \neq 0$. □

Lemma 10. *The equation $2u^3 + v^3 = 15$ is impossible in rational integers (u, v).*

Proof. We consider the integer $1 - 3\theta + 2\theta^2$ whose norm is 15 and any other integer of norm 15 in $K(\theta)$ must be of the form $(1 - 3\theta + 2\theta^2)(\epsilon_0^m)$, as all primes of norm 5 in $K(\theta)$ are associated and also all primes of norm 3. Let $\epsilon_0^m = a_m + b_m\theta + c_m\theta^2$ and

$$X_m + Y_m\theta + Z_m\theta^2 = \left(1 - 3\theta + 2\theta^2\right)\left(a_m + b_m\theta + c_m\theta^2\right). \tag{37}$$

Hence

$$Z_m = c_m - 3b_m + 2a_m. \tag{38}$$

We seek all integers of $K(\theta)$ with norm 15 of the form $a + b\theta$. Hence Z_m must be zero, and thus the congruence $Z_m \equiv 0 \pmod{t}$ must be solvable for every modulus t. We shall now show that

$$Z_m \not\equiv 0 \pmod{31}. \tag{39}$$

Also we have by manual verification $\epsilon_0^{10} \equiv -6 \pmod{31}$, $\epsilon_0^{20} \equiv 5 \pmod{31}$, and $\epsilon_0^{30} \equiv 1 \pmod{31}$. Therefore Z_m satisfies the conditions $Z_{m+10} \equiv -6Z_m \pmod{31}$, $Z_{m+20} \equiv 5Z_m \pmod{31}$ and $Z_{m+30} \equiv Z_m \pmod{31}$. Therefore we only need to check $Z_i \not\equiv 0 \pmod{31}$ for $i = 0, 1, \ldots, 9$. Using (31) and (38) we have the values of Z_0 to Z_9, modulo 31 is -5, 3, 10, -10, -2, 14, 16, 1, -6, and 19 and none of these is zero. This completes the proof of Lemma 10. □

Lemma 11. *The equation $2u^3+v^3 = 75$ is impossible in rational integers (u, v).*

Proof. We are interested to find all elements of the form $a + b\theta$ such that $N(a + b\theta) = 75$. The integer $11 - 6\theta - 2\theta^2$ has norm 75. Any other integer of norm 75 must be of this form $(11 - 6\theta - 2\theta^2)\epsilon_0^m$. Therefore

$$X_m + Y_m\theta + Z_m\theta^2 = \left(11 - 6\theta - 2\theta^2\right)\left(a_m + b_m\theta + c_m\theta^2\right). \tag{40}$$

Hence

$$Z_m = 11c_m - 6b_m - 2a_m. \tag{41}$$

The values of Z_0 to Z_9 modulo 31 are -2, -4, 21, 9, -1, -3, -10, -19, 25, 12 and none of these is zero. This completes the proof of Lemma 11. □

Lemma 12. *The equations $2u^3 + v^3 = 213, 355, 1065, 1775, 5325$ are impossible in rational integers (u, v).*

Proof. Using the norm of 3, 5, and 71 and multiplicative property of the norm, the integers having norms 213, 355, 1065, 1775, and 5325 are $-11 + 3\theta + 5\theta^2$, $-27 + 9\theta + 10\theta^2$, $45+11\theta-37\theta^2$, $15-14\theta+3\theta^2$, and $-43+20\theta+12\theta^2$, respectively. Also we know that all primes of norm 3, 5, and 71 in $K(\theta)$ are associated. Therefore the integers whose norms are 213, 355, 1065, 1775, and 5325, respectively, can be represented by

$$\begin{aligned}
&X_m^1 + Y_m^1\theta + Z_m^1\theta^2 \\
&\quad = \left(-11 + 3\theta + 5\theta^2\right)\left(a_m^1 + b_m^1\theta + c_m^1\theta^2\right), \\
&X_m^2 + Y_m^2\theta + Z_m^2\theta^2 \\
&\quad = \left(-27 + 9\theta + 10\theta^2\right)\left(a_m^2 + b_m^2\theta + c_m^2\theta^2\right), \\
&X_m^3 + Y_m^3\theta + Z_m^3\theta^2 \\
&\quad = \left(45 + 11\theta - 37\theta^2\right)\left(a_m^3 + b_m^3\theta + c_m^3\theta^2\right), \\
&X_m^4 + Y_m^4\theta + Z_m^4\theta^2 \\
&\quad = \left(15 - 14\theta + 3\theta^2\right)\left(a_m^4 + b_m^4\theta + c_m^4\theta^2\right), \\
&X_m^5 + Y_m^5\theta + Z_m^5\theta^2 \\
&\quad = \left(-43 + 20\theta + 12\theta^2\right)\left(a_m^5 + b_m^5\theta + c_m^5\theta^2\right).
\end{aligned} \tag{42}$$

Hence,

$$\begin{gathered}
Z_m^1 = -11c_m^1 + 3b_m^1 + 5a_m^1, \qquad Z_m^2 = -27c_m^2 + 9b_m^2 + 10a_m^2, \\
Z_m^3 = 45c_m^3 + 11b_m^3 - 37a_m^3, \\
Z_m^4 = 15c_m^4 - 14b_m^4 + 3a_m^4, \qquad Z_m^5 = -43c_m^5 + 20b_m^5 + 12a_m^5.
\end{gathered} \tag{43}$$

But we seek all integers of the form $a + b\theta$. That means $Z_m^i \equiv 0 \pmod{t}$ for any modulus t and for $i = 1, 2, \ldots, 5$. We want to show that $Z_m^i \not\equiv 0 \pmod{t}$ for some t. Also Z_m^i satisfy the congruences $Z_{m+10}^i \equiv -6Z_m^i \pmod{31}$, $Z_{m+20}^i \equiv 5Z_m^i \pmod{31}$, and $Z_{m+30}^i \equiv Z_m^i \pmod{31}$. Using these congruences the values of Z_m^i modulo 31 are as follows. The values of Z_0^1 to Z_9^1 modulo 31 are 5, -2, -12, 16, -14, 13, 19, 2, 15, and -13, the values of Z_0^2 to Z_9^2 modulo 31 are 10, -1, -4, 21, -2, 20, 2, -6, 1, and -14, the values of Z_0^3 to Z_9^3 modulo 31 are 6, 17, -14, -15, 11, -2, 20, -12, 5, and 10, the values of Z_0^4 to Z_9^4 modulo 31 are 3, -17, -10, 9, 4, 7, 7, -7, 7, and 7, and the values of Z_0^5 to Z_9^5 modulo 31 are 12, 8, -3, 5, -2, 3, 2, -15, 17, and 2 and none of these is zero. This completes the proof of Lemma 12 □

Till now we get the solutions (u, v) of (14) satisfying the conditions (16). We need to find the solutions (A, B) of (12) for which the exact value of d must be calculated. It follows from Lemma 8 to Lemma 12 that the only integral solution of (14) is $(u, v) = (-1, 3), (-3, 5)$. In both the cases, both u and v are relatively prime and odd; u is negative and v is positive. Therefore from (15), we have

$$25d^2 = 25, \qquad 71d^2 = 71. \tag{44}$$

In either case, $d = 1$.

Thus, the only integral solution of (12) satisfying conditions (13) is $(A, B) = (-1, 3), (-3, 5)$. Hence the only integral

solution of (10) is $(x, y) = (1, 2)$. This completes the proof of Theorem 1.

Equation (14) is equivalent to $x^3 - 2y^3 = c$ by setting $y = -u$ and $x = v$. Theorem 5 give a lower bound for the absolute value of $x^3 - 2y^3$. This lower bound also immediately gives the proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] R. Finkelstein, "The house problem," *The American Mathematical Monthly*, vol. 72, pp. 1082–1088, 1965.

[2] A. Behera and G. K. Panda, "On the square roots of triangular numbers," *The Fibonacci Quarterly*, vol. 37, no. 2, pp. 98–105, 1999.

[3] P. Ingram, "On k-th power numerical centres," *Comptes Rendus Mathématiques de l'Académie des Sciences*, vol. 25, pp. 105–110, 2005.

[4] G. K. Panda, "Sequence balancing and cobalancing numbers," *The Fibonacci Quarterly*, vol. 45, no. 3, pp. 265–271, 2007.

[5] G. K. Panda and S. S. Rout, "Gap balancing numbers," *Fibonacci Quarterly*, vol. 51, no. 3, pp. 239–248, 2013.

[6] J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, New York, NY, USA, 2006.

[7] O. Hemer, "On some diophantine equations of the type $y^2 - f^2 = x^3$," *Mathematica Scandinavica*, vol. 4, pp. 95–107, 1956.

[8] T. Nagell, "Solution complete de quelques equations cubiques a deux indeterminese," *Journal de Mathématiques, Serie IX*, vol. 4, pp. 209–270, 1925.

[9] M. A. Bennett, "Effective measures of irrationality for certain algebraic numbers," *Australian Mathematical Society. Journal A*, vol. 62, no. 3, pp. 329–344, 1997.

[10] S. Lang, *Algebraic Number Theory*, Springer, New York, NY, USA, 1994.