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Research Article

Long Time Behavior for a System of Differential Equations with Non-Lipschitzian Nonlinearities

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We consider a general system of nonlinear ordinary differential equations of first order. The nonlinearities involve distributed delays in addition to the states. In turn, the distributed delays involve nonlinear functions of the different variables and states. An explicit bound for solutions is obtained under some rather reasonable conditions. Several special cases of this system may be found in neural network theory. As a direct application of our result it is shown how to obtain global existence and, more importantly, convergence to zero at an exponential rate in a certain norm. All these nonlinearities (including the activation functions) may be non-Lipschitz and unbounded.

1. Introduction

Of concern is the following system:

$$x_i'(t) = -a_i(t)\,x_i(t) + \sum_{j=1}^{m} f_{ij}\left(t, x_j(t), \int_{-\infty}^{t} K_{ij}\left(t, s, x_j(s)\right) ds\right) + c_i(t), \quad (1)$$

with continuous data $x_j(t) = x_{0j}(t)$, $t \in (-\infty, 0]$, coefficients $a_i(t) \geq 0$, and inputs $c_i(t)$, $i = 1, \ldots, m$. The functions f_{ij} and K_{ij} are nonlinear continuous functions. This is a general nonlinear version of several systems that arise in many applications (see [1–9] and Section 4 below).

The literature is very rich of works on the asymptotic behavior of solutions for special cases of system (1) (see for instance [10–19]). Here the integral terms represent some kind of distributed delays but discrete delays may be recovered as well by considering delta Dirac distributions. Different sufficient conditions on the coefficients, the functions, and the kernels have been established ensuring convergence to equilibrium or (uniform, global, and asymptotic) stability. In applications it is important to have global asymptotic stability at a very rapid rate like the exponential rate. Roughly speaking, it has been assumed that the coefficients $a_i(t)$ must dominate the coefficients of some "bad" similar terms that appear in the estimations. For the nonlinearities (activation functions), the first assumptions of boundedness, monotonicity, and differentiability have been all weakened to a Lipschitz condition. According to [8, 20] and other references, even this condition needs to be weakened further. Unfortunately, we can find only few papers on continuous but not Lipschitz continuous activation functions. Assumptions like partially Lipschitz and linear growth, α-inverse Hölder continuous or inverse Lipschitz, non-Lipschitz but bounded were used (see [16, 21, 22]).

For Hölder continuous activation functions we refer the reader to [23], where exponential stability was proved under some boundedness and monotonicity conditions on the activation functions and the coefficients form a Lyapunov diagonally stable matrix (see also [24, 25] for other results without these conditions).

There are, however, a good number of papers dealing with discontinuous activation functions under certain stronger conditions like M-Matrix, the LMI condition (linear matrix inequality) and some extra conditions on the matrices and growth conditions on the activation functions (see [20, 26–37]). Global asymptotic stability of periodic solutions have been investigated, for instance, in [38, 39].

Here we assume that the functions f_{ij} and K_{ij} are (or bounded by) continuous monotone nondecreasing functions that are not necessarily Lipschitz continuous and they may be unbounded (like power type functions with powers bigger than one). We prove that, for sufficiently small initial data, solutions decay to zero exponentially.

The local existence and global existence are standard; see the Gronwall-type Lemma 1 below and the estimation in our theorem. However, the uniqueness of the equilibrium is not an issue here (even in case of constant coefficients) as we are concerned with convergence to zero rather than stability of equilibrium.

After the Preliminaries section, where we present our main hypotheses and the main lemma used in our proof, we state and prove the convergence result in Section 3. The section is ended by some corollaries and important remarks. In the last section we give an application, where this type of systems (or special cases of it) appears in real world problems.

2. Preliminaries

Our first hypothesis (H1) is

$$\left|f_{ij}\left(t, x_j(t), \int_{-\infty}^{t} K_{ij}\left(t, s, x_j(s)\right) ds\right)\right| \leq b_{ij}(t)\left|x_j(t)\right|^{\alpha_{ij}}\left(\int_{-\infty}^{t} l_{ij}(t-s)\,\psi_{ij}\left(\left|x_j(s)\right|\right) ds\right)^{\beta_{ij}}, \quad i, j = 1, \ldots, m, \tag{2}$$

where b_{ij} are nonnegative continuous functions, l_{ij} are nonnegative continuously differentiable functions, ψ_{ij} are nonnegative nondecreasing continuous functions, and $\alpha_{ij}, \beta_{ij} \geq 0$, $i, j = 1, \ldots, m$. The interesting cases are when α_{ij} and β_{ij} are all nonzero.

Let $I \subset \mathbf{R}$, and let $g_1, g_2 : I \to \mathbf{R} \setminus \{\mathbf{0}\}$. We write $g_1 \propto g_2$ if g_2/g_1 is nondecreasing in I. This ordering as well as the monotonicity condition may be dropped as is mentioned in Remark 8 below.

Lemma 1 (see [40]). *Let $a(t)$ be a positive continuous function in $J := [\alpha, \beta)$, $k_j(t)$, $j = 1, \ldots, n$ nonnegative continuous functions for $\alpha \leq t < \beta$, $g_j(u)$, $j = 1, \ldots, n$ nondecreasing continuous functions in $\mathbf{R}_+$, with $g_j(u) > 0$ for $u > 0$, and $u(t)$ a nonnegative continuous functions in J. If $g_1 \propto g_2 \propto \cdots \propto g_n$ in $(0, \infty)$, then the inequality*

$$u(t) \leq a(t) + \sum_{j=1}^{n} \int_{\alpha}^{t} k_j(s)\, g_j(u(s))\, ds, \quad t \in J, \tag{3}$$

implies that

$$u(t) \leq \omega_n(t), \qquad \alpha \leq t < \beta_0, \tag{4}$$

where $\omega_0(t) := \sup_{0 \leq s \leq t} a(s)$,

$$\omega_j(t) := G_j^{-1}\left[G_j\left(\omega_{j-1}(t)\right) + \int_0^t k_j(s)\, ds\right], \quad j = 1, \ldots, n,$$
$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)}, \quad u > 0 \ \left(u_j > 0, j = 1, \ldots, n\right), \tag{5}$$

and β_0 is chosen so that the functions $\omega_j(t)$, $j = 1, \ldots, n$, are defined for $\alpha \leq t < \beta_0$.

In our case we will need the following notation and hypotheses.

(H2) Assume that $\psi_{ij}(u) > 0$ for $u > 0$ and the set of functions $u(t)^{\alpha_{ij}+\beta_{ij}}$, $\psi_{ij}(u(t))$ may be ordered as $h_1 \propto h_2 \propto \cdots \propto h_n$ (after relabelling). Their corresponding coefficients $\widetilde{b}_{ij}(t) := \exp[\int_0^t a(\sigma) d\sigma] b_{ij}(t)$ $(a(t) := \min_{1 \leq i \leq m} a_i(t))$ and $l_{ij}(0)$ will be renamed λ_k, $k = 1, \ldots, n$.

We define $x(t) := \sum_{i=1}^{m} |x_i(t)|$, $t > 0$, $x_0(t) := \sum_{i=1}^{m} |x_{0i}(t)|$, $t \leq 0$,

$$c(t) := \int_0^t \exp\left[\int_0^s a(\sigma)\, d\sigma\right] \sum_{i=1}^{m} \left|c_i(s)\right| ds, \quad t > 0,$$
$$\omega_0(t) := x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\,\psi_{ij}\left(x_0(\sigma)\right) d\sigma + c(t),$$
$$\omega_j(t) := H_j^{-1}\left[H_j\left(\omega_{j-1}(t)\right) + \int_{\alpha}^{t} \lambda_j(s)\, ds\right], \quad j = 1, \ldots, n,$$
$$H_j(u) := \int_{u_j}^{u} \frac{dx}{h_j(x)}, \quad u > 0 \ \left(u_j > 0, j = 1, \ldots, n\right),$$
$$\widetilde{\omega}_0(t) := \omega_0(0) + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(s)\right| \int_{-s}^{0} \psi_{ij}\left(u_0(\sigma)\right) d\sigma\, ds,$$
$$u_0(\sigma) = \sum_{i,j=1}^{m} \int_{-\infty}^{\sigma} \left|l_{ij}(\sigma - \tau)\right| \psi_{ij}\left(x_0(\tau)\right) d\tau, \quad \sigma < 0,$$
$$\widetilde{\omega}_j(t) := H_j^{-1}\left[H_j\left(\widetilde{\omega}_{j-1}(t)\right) + \int_0^t \widetilde{\lambda}_j(s)\, ds\right], \quad j = 1, \ldots, n, \tag{6}$$

where $\widetilde{\lambda}_j$ are the relabelled coefficients corresponding to $\widetilde{b}_{ij}(t)$ and $l_{ij}(0) + \int_0^{\infty} |l'_{ij}(\sigma)| d\sigma$.

3. Exponential Convergence

In this section it is proved that solutions converge to zero in an exponential manner provided that the initial data are small enough.

Theorem 2. *Assume that the hypotheses (H1) and (H2) hold and* $\int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))d\sigma < \infty$, $i, j = 1, \ldots, m$. *Then, (a) if* $l'_{ij}(t) \leq 0$, $i, j = 1, \ldots, m$, *there exists* $\beta_0 > 0$ *such that*

$$x(t) \leq \omega_n(t) \exp\left[-\int_0^t a(s)\,ds\right], \quad 0 \leq t < \beta_0. \tag{7}$$

(b) If $l'_{ij}(t)$, $i, j = 1, \ldots, m$ *are of arbitrary signs,* $l'_{ij}(t)$ *are summable, and the integral term in* $\widetilde{\omega}_0(t)$ *is convergent then there exists a* $\beta_1 > 0$ *such that the conclusion in (a) is valid on* $0 \leq t < \beta_1$ *with* $\widetilde{\omega}_n$ *instead of* ω_n.

Proof. It is easy to see from (1) and the assumption (H1) that for $t > 0$ and $i = 1, \ldots, m$ we have

$$\begin{aligned} D^+ \left|x_i(t)\right| \leq{}& -a_i(t)\left|x_i(t)\right| \\ &+ \sum_{j=1}^{m}\left|f_{ij}\left(t, x_j(t), \int_{-\infty}^{t} K_{ij}\left(t, s, x_j(s)\right) ds\right)\right| \\ &+ c_i(t), \end{aligned} \tag{8}$$

or, for $t > 0$,

$$\begin{aligned} D^+ x(t) \leq{}& -\min_{1\leq i\leq m}\{a_i(t)\}\, x(t) \\ &+ \sum_{i,j=1}^{m} b_{ij}(t)\left|x_j(t)\right|^{\alpha_{ij}} \\ &\quad\times\left(\int_{-\infty}^{t} l_{ij}(t-s)\psi_{ij}\left(\left|x_j(s)\right|\right) ds\right)^{\beta_{ij}} \\ &+ \sum_{i=1}^{m}\left|c_i(t)\right|, \end{aligned} \tag{9}$$

where D^+ denotes the right Dini derivative. Hence

$$\begin{aligned} &D^+ x(t) \\ &\leq -a(t)\,x(t) \\ &\quad+ \sum_{i,j=1}^{m} b_{ij}(t)\left|x(t)\right|^{\alpha_{ij}}\left(\int_{-\infty}^{t} l_{ij}(t-s)\,\psi_{ij}(x(s))\,ds\right)^{\beta_{ij}} \\ &\quad+ \sum_{i=1}^{m}\left|c_i(t)\right|, \quad t > 0 \end{aligned} \tag{10}$$

and consequently

$$\begin{aligned} &D^+\left\{x(t)\exp\left[\int_0^t a(s)\,ds\right]\right\} \\ &\quad\leq \exp\left[\int_0^t a(s)\,ds\right]\sum_{i,j=1}^{m} b_{ij}(t)\left|x(t)\right|^{\alpha_{ij}} \\ &\qquad\times\left(\int_{-\infty}^{t} l_{ij}(t-s)\psi_{ij}(x(s))\,ds\right)^{\beta_{ij}} \\ &\qquad+ \exp\left[\int_0^t a(s)\,ds\right]\sum_{i=1}^{m}\left|c_i(t)\right|, \\ &\qquad\qquad t > 0. \end{aligned} \tag{11}$$

Thus (by a comparison theorem in [41])

$$\begin{aligned} \widetilde{x}(t) \leq{}& x(0) + c(t) \\ &+ \sum_{j=1}^{m}\int_0^t\left\{\sum_{i=1}^{m}\widetilde{b}_{ij}(s)\left|x(s)\right|^{\alpha_{ij}}\right. \\ &\left.\quad\times\left(\int_{-\infty}^{s} l_{ij}(s-\sigma)\,\psi_{ij}(x(\sigma))\,d\sigma\right)^{\beta_{ij}}\right\} ds, \\ &\qquad t > 0, \end{aligned} \tag{12}$$

where

$$\widetilde{x}(t) := x(t)\exp\left[\int_0^t a(s)\,ds\right]. \tag{13}$$

Let $y(t)$ denote the right hand side of (12). Clearly $\widetilde{x}(t) \leq y(t)$, $t > 0$, and for $t > 0$

$$\begin{aligned} D^+ y(t) ={}& D^+ c(t) \\ &+ \sum_{i,j=1}^{m}\widetilde{b}_{ij}(t)\left|x(t)\right|^{\alpha_{ij}} \\ &\quad\times\left(\int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}(x(\sigma))\,d\sigma\right)^{\beta_{ij}}. \end{aligned} \tag{14}$$

We designate by $z_{ij}(t)$ the integral term in (14); that is,

$$z_{ij}(t) := \int_{-\infty}^{t} l_{ij}(t-\sigma)\,\psi_{ij}(x(\sigma))\,d\sigma \tag{15}$$

and $z(t) := \sum_{i,j=1}^{m} z_{ij}(t)$. A differentiation of $z(t)$ gives

$$\begin{aligned} z'(t) ={}& \sum_{i,j=1}^{m} l_{ij}(0)\,\psi_{ij}(x(t)) \\ &+ \sum_{i,j=1}^{m}\int_{-\infty}^{t} l'_{ij}(t-\sigma)\,\psi_{ij}(x(\sigma))\,d\sigma. \end{aligned} \tag{16}$$

(a) Consider $l'_{ij}(t) \leq 0$, $i, j = 1, \ldots, m$

In this situation (of fading memory) we see from (14) and (16) that if $u(t) := y(t) + z(t)$, then

$$\begin{aligned} D^+ u(t) \leq{}& D^+ c(t) \\ &+ \sum_{i,j=1}^{m}\left[\widetilde{b}_{ij}(t)\,(u(t))^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\,\psi_{ij}(u(t))\right], \\ &\qquad t > 0. \end{aligned} \tag{17}$$

Therefore

$$u(t) \le u(0) + c(t) + \sum_{i,j=1}^{m} \int_0^t \left[\tilde{b}_{ij}(s)(u(s))^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\psi_{ij}(u(s))\right] ds, \quad t > 0, \tag{18}$$

where $u(0) = x(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))d\sigma$. Now we can apply Lemma 1 to obtain

$$\tilde{x}(t) \le u(t) \le \omega_n(t), \quad 0 \le t < \beta_0 \tag{19}$$

with $\omega_0(t) = u(0) + c(t)$ and $\omega_n(t)$ is as in the "Preliminaries" section.

(b) Consider $l'_{ij}(t)$, $i, j = 1, \ldots, m$ of arbitrary signs.

From expressions (14) and (16) we derive that

$$\begin{aligned} D^+u(t) &\le D^+c(t) + \sum_{i,j=1}^{m} \left[\tilde{b}_{ij}(t)(u(t))^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\psi_{ij}(u(t))\right] \\ &\quad + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(\sigma)\right| \psi_{ij}(u(t-\sigma))\, d\sigma, \quad t > 0. \end{aligned} \tag{20}$$

The derivative of the auxiliary function

$$\tilde{u}(t) = u(t) + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(s)\right| \int_{t-s}^{t} \psi_{ij}(u(\sigma))\, d\sigma\, ds, \quad t \ge 0 \tag{21}$$

is equal to (with the help of (20) and (21))

$$\begin{aligned} D^+\tilde{u}(t) &= D^+u(t) + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(s)\right| \left[\psi_{ij}(u(t)) - \psi_{ij}(u(t-s))\right] d\sigma\, ds \\ &\le D^+c(t) + \sum_{i,j=1}^{m} \left[\tilde{b}_{ij}(t)(u(t))^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\psi_{ij}(u(t))\right] \\ &\quad + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(\sigma)\right| \psi_{ij}(u(t-\sigma))\, d\sigma \\ &\quad + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(s)\right| \left[\psi_{ij}(u(t)) - \psi_{ij}(u(t-s))\right] ds \\ &\le D^+c(t) + \sum_{i,j=1}^{m} \left\{\tilde{b}_{ij}(t)(\tilde{u}(t))^{\alpha_{ij}+\beta_{ij}} + \left[l_{ij}(0) + \int_0^{\infty} \left|l'_{ij}(s)\right| ds\right] \psi_{ij}(\tilde{u}(t))\right\}, \quad t > 0. \end{aligned} \tag{22}$$

Therefore

$$\tilde{u}(t) \le \tilde{u}(0) + c(t) + \sum_{i,j=1}^{m} \int_0^t \left\{\tilde{b}_{ij}(s)(\tilde{u}(s))^{\alpha_{ij}+\beta_{ij}} + \left[l_{ij}(0) + \int_0^{\infty} \left|l'_{ij}(\sigma)\right| d\sigma\right] \psi_{ij}(\tilde{u}(s))\right\} ds \tag{23}$$

with

$$\begin{aligned} \tilde{u}(0) &= x(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma + \sum_{i,j=1}^{m} \int_0^{\infty} \left|l'_{ij}(s)\right| \int_{-s}^{0} \psi_{ij}(u_0(\sigma))\, d\sigma\, ds, \\ u_0(\sigma) &= z(\sigma) = \sum_{i,j=1}^{m} z_{ij}(\sigma) = \sum_{i,j=1}^{m} \int_{-\infty}^{\sigma} l_{ij}(\sigma-\tau)\psi_{ij}(x_0(\tau))\, d\tau, \quad \sigma < 0. \end{aligned} \tag{24}$$

Applying Lemma 1 to (23) we obtain

$$\tilde{x}(t) \le \tilde{u}(t) \le \tilde{\omega}_n(t), \quad 0 \le t < \beta_1 \tag{25}$$

and hence

$$\tilde{x}(t) \le \tilde{\omega}_n(t), \quad 0 \le t < \beta_1, \tag{26}$$

where $\tilde{\omega}_0(t) := \tilde{u}(0)$ and

$$\tilde{\omega}_j(t) := H_j^{-1}\left[H_j\left(\tilde{\omega}_{j-1}(t)\right) + \int_0^t \tilde{\lambda}_j(s)\, ds\right], \quad j = 1, \ldots, n, \tag{27}$$

and β_0 is chosen so that the functions $\tilde{\omega}_j(t)$, $j = 1, \ldots, n$, are defined for $0 \le t < \beta_1$. □

Corollary 3. *If, in addition to the hypotheses of the theorem, we assume that*

$$\int_0^{\infty} \chi_k(s)\, ds \le \int_{\omega_{k-1}}^{\infty} \frac{dz}{h_k(z)}, \quad k = 1, \ldots, n, \quad \chi_k(s) = \lambda_k(s), \tilde{\lambda}_k(s) \tag{28}$$

then we have global existence of solutions.

Corollary 4. *If, in addition to the hypotheses of the theorem, we assume that $\omega_n(t)$ ($\tilde{\omega}_n(t)$) grows up at the most polynomially (or just slower than $\exp[\int_0^t a(s)ds]$), then solutions decay at an exponential rate if $\int_0^t a(s)ds \to \infty$ as $t \to \infty$.*

Corollary 5. *In addition to the hypotheses of the theorem, assume that $l'_{ij}(t) \le L_{ij} l_{ij}(t)$, $i, j = 1, \ldots, m$, for some positive* *constants L_{ij} and $\psi_{ij}(t)$ are in the class* $\mathbf{H}$ *(that is $\psi_{ij}(\alpha u) \le \xi_{ij}\psi_{ij}(u)$, $\alpha > 0$, $u > 0$, $i, j = 1, \ldots, m$). Then solutions are bounded by a function of the form* $\exp[-(\int_0^t a(s)ds - Lt)]$, *where* $l = \max\{L_{ij}, i, j = 1, \ldots, m\}$.

Remark 6. We have assumed that α_{ij} and β_{ij} are greater than one but the case when they are smaller than one may be treated similarly. When their sum is smaller than one we have global existence without adding any extra condition.

Remark 7. The decay rate obtained in Corollary 5 is to be compared with the one in the theorem (case (b)). It appears that the estimation in Corollary 5 holds for more general initial data (not as small as the ones in case (b)). However, the decay rate is smaller than the one in (b) besides assuming that $\int_0^t a(s)ds - Lt \to \infty$ as $t \to \infty$.

Remark 8. If we consider the following new functions, then the monotonicity condition and the order imposed in the theorem may be dropped:

$$\begin{aligned} \phi_1(t) &:= \max_{0\le s\le t} g_1(s), \\ \phi_k(t) &:= \max_{0\le s\le t}\left\{\frac{g_k(s)}{\phi_{k-1}(s)}\right\}\phi_{k-1}(t) \end{aligned} \tag{29}$$

and $\psi(t) := \phi_k(t)/\phi_{k-1}(t)$.

4. Application

(Artificial) Neural networks are built in an attempt to perform different tasks just as the nervous system. Typically, a neural network consists of several layers (input layer, hidden layers, and output layer). Each layer contains one or more cells (neurons) with many connections between them. The cells in one layer receive inputs from the previous layer, make some transformations, and send the results to the cells of the subsequent layer.

One may encounter neural networks in many fields such as control, pattern matching, settlement of structures, classification of soil, supply chain management, engineering design, market segmentation, product analysis, market development forecasting, signature verification, bond rating, recognition of diseases, robust pattern detection, text mining, price forecast, botanical classification, and scheduling optimization.

Neural networks not only can perform many of the tasks a traditional computer can do, but also excel in, for instance, classifying incomplete or noisy data, predicting future events, and generalizing.

The system (1) is a general version of simpler systems that appear in neural network theory [1–9] like

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^m f_{ij}(x_j(t)) + c_i(t), \tag{30}$$

or

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) \\ &\quad + \sum_{j=1}^m \int_{-\infty}^t l_{ij}(t-s) f_{ij}(x_j(s))\,ds + c_i(t). \end{aligned} \tag{31}$$

It is well established by now that (for constant coefficients and constant $c_i(t)$) solutions converge in an exponential manner to the equilibrium. Notice that zero in our case is not an equilibrium. This equilibrium exists and is unique in case of Lipschitz continuity of the activation functions. In our case the system is much more general and the activation functions as well as the nonlinearities are not necessarily Lipschitz continuous. However, in case of Lipschitz continuity and existence of a unique equilibrium we expect to have exponential stability using the standard techniques at least when we start away from zero.

For the system

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) \\ &\quad + \sum_{j=1}^m b_{ij}|x_j(t)|^{\alpha_{ij}}\left(\int_{-\infty}^t l_{ij}(t-s)\psi_{ij}(|x_j(s)|)\,ds\right)^{\beta_{ij}} \\ &\quad + c_i(t), \end{aligned} \tag{32}$$

(where ψ_{ij} may be taken as power functions; see also Corollary 5) our theorem gives sufficient conditions guaranteeing the estimation

$$x(t) \le \omega_n(t)\exp\left[-\int_0^t a(s)\,ds\right], \quad 0 \le t < \beta_0. \tag{33}$$

Then, Corollaries 3 and 4 provide practical situations where we have global existence and decay to zero at an exponential rate.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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